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On Slowly Changing Functions Based Growth Analysis of Entire Functions in Terms of Their Relative Orders

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Abstract

The growth properties of entire functions in the light of their relative orders and slowly changing functions are discussed in this paper.

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1 Introduction, Definitions and Notations

Let f and g be any two entire functions defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$, $M_g(r) = \max\{|g(z)| : |z| = r\}$. f and g are said to be asymptotically equivalent if there exists l, $0 < l < \infty$ such that $\frac{M_f(r)}{M_g(r)} \to l$ as $r \to \infty$ and in this case we write $f \sim g$. If $f \sim g$ then clearly $g \sim f$.

The order ρ_f and lower order λ_f of an entire function f are defined in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f{}^{-1}(s) = \infty$.

Bernal [1] introduced the definition of relative order of f with respect to g , denoted by $\rho_g(f)$ as follows :

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [6] if $g(z) = \exp z$.
Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

For an entire function f, the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu_f(r) = \max_{n\geq 0}(|a_n| r^n)$ .Datta and Maji [2] gave an alternative definition of relative order and relative lower order of f with respect to g in the following way:

Definition 1 *[2] The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function f with respect to g are defined as follows:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} \quad \textit{and} \quad \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r}.$$

Now let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [4] defined it in the following way:

Definition 2 *[4] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \;\; for\ r \geq r(\varepsilon) \;\; and$$

uniformly for $k\,(\geq 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \; .$$

Somasundaram and Thamizharasi [5] introduced the notions of L-order and L-lower order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order and L-lower order for entire function are L^*-order and L^*-lower order . Their definitions are as follows:

Definition 3 *[5] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \;\; and \;\; \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

In the line of Somasundaram and Thamizharasi [5], one can define the relative L^*-order of an entire function in the following way :

Definition 4 *The relative L^*-order of an entire function f with respect to another entire function g , denoted by $\rho_g^{L^*}(f)$ is defined in the following way*

$$\begin{aligned} \rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left\{re^{L(r)}\right\}^{\mu} \;\; for\ all\ r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}. \end{aligned}$$

Similarly, one can define the relative L^*-lower order of f with respect to g denoted by $\lambda_g^{L^*}(f)$ as follows :

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Datta, Biswas and Ali [3] also gave an alternative definition of L^*-order and relative L^*-lower order of f with respect to g in the following way:

Definition 5 *The relative L^*-order $\rho_g^{L^*}(f)$ and the relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to g are as follows:*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the paper we study some maximum term and maximum modulus oriented growth properties of composite entire functions on the basis of relative L^*-order and relative L^*-lower order as compared to the relative growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[3] If f, g and h be any three entire functions such that $f \sim h$ then $\rho_g^{L^*}(h) = \rho_g^{L^*}(f)$ and $\lambda_g^{L^*}(h) = \lambda_g^{L^*}(f)$.*

Lemma 2 *[3] If f, g, h and k be any four entire functions with $g \sim h$ and $f \sim k$ then $\rho_f^{L^*}(g) = \rho_k^{L^*}(h) = \rho_f^{L^*}(h) = \rho_k^{L^*}(g)$ and $\lambda_f^{L^*}(g) = \lambda_k^{L^*}(h) = \lambda_f^{L^*}(h) = \lambda_k^{L^*}(g)$.*

3 Main Results

In this section we present the main results of the paper.

Theorem 3 *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f\circ g) \le \rho_h^{L^*}(f\circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$. If $L\left(r^A\right) = o\left\{\log M_h^{-1}M_f\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)} \le \liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A) + L(r^A)} \le \frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}$$
$$\le \limsup_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A) + L(r^A)} \le \frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}.$$

Proof. From the definition of relative L^*-order and relative L^*-lower order in terms of maximum modulus of entire function we have for arbitrary positive ε and for all sufficiently large values of r that

$$\begin{aligned} \log M_h^{-1} M_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\log\left\{re^{L(r)}\right\} \\ i.e.,\ \log M_h^{-1} M_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\{\log r+L(r)\} \\ i.e.,\ \log M_h^{-1} M_{f\circ g}(r) &\geq \left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\log r+\frac{1}{A}L\left(r^A\right)\right\} \\ &\quad +\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\} \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \log M_h^{-1} M_f\left(r^A\right) &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e.,\ \log M_h^{-1} M_f\left(r^A\right) &\leq \left(\rho_h^{L^*}(f)+\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\} \\ i.e.,\ \frac{\log M_h^{-1} M_f\left(r^A\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)} &\leq \log r+\frac{1}{A}L\left(r^A\right)\ . \qquad (2) \end{aligned}$$

Now from (1) and (2) it follows for all sufficiently large values of r that

$$\begin{aligned} &\log M_h^{-1} M_{f\circ g}(r) \\ &\geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\log M_h^{-1} M_f\left(r^A\right)+\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f(r^A)+L(r^A)} &\geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\cdot\frac{\log M_h^{-1} M_f\left(r^A\right)}{\log M_h^{-1} M_f(r^A)+L(r^A)} \\ &\quad +\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log M_h^{-1} M_f(r^A)+L(r^A)} \end{aligned}$$

$$\begin{aligned} &i.e.,\ \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f(r^A)+L(r^A)} \\ &\qquad \geq \frac{\frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log M_h^{-1} M_f(r^A)}}+\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log M_h^{-1} M_f(r^A)}{L(r^A)}}\ . \qquad (3) \end{aligned}$$

Since $L\left(r^A\right)=o\left\{\log M_h^{-1} M_f\left(r^A\right)\right\}$ as $r\to\infty$, it follows from (3) that

$$\liminf_{r\to\infty}\frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f(r^A)+L(r^A)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}\ . \qquad (4)$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (4) that

$$\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\geq\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\ . \tag{5}$$

Again for a sequence of values of r tending to infinity ,

$$\begin{aligned}\log M_h^{-1}M_{f\circ g}(r) &\leq \left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ i.e.,\ \log M_h^{-1}M_{f\circ g}(r) &\leq \left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\log r+\frac{1}{A}L\left(r^A\right)\right\}\\ &\quad+\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}\end{aligned} \tag{6}$$

and for all sufficiently large values of r ,

$$\begin{aligned}\log M_h^{-1}M_f\left(r^A\right) &\geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\log\left\{r^Ae^{L\left(r^A\right)}\right\}\\ i.e.,\ \log M_h^{-1}M_f\left(r^A\right) &\geq \left(\lambda_h^{L^*}(f)-\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\}\\ i.e.,\ \frac{\log M_h^{-1}M_f\left(r^A\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)} &\geq \log r+\frac{1}{A}L\left(r^A\right)\ .\end{aligned} \tag{7}$$

Combining (6) and (7) we get for a sequence of values of r tending to infinity that

$$\begin{aligned}&\log M_h^{-1}M_{f\circ g}(r)\\ &\leq\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\log M_h^{-1}M_f\left(r^A\right)+\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}\end{aligned}$$

$$\begin{aligned}i.e.,\ \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)} &\leq\frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\cdot\frac{\log M_h^{-1}M_f\left(r^A\right)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\\ &\quad+\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log M_h^{-1}M_f(r^A)+L(r^A)}\end{aligned}$$

$$\begin{aligned}i.e.,\ &\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\\ &\leq\frac{\frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L(r^A)}{\log M_h^{-1}M_f(r^A)}}+\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log M_h^{-1}M_f(r^A)}{L(r^A)}}\ .\end{aligned} \tag{8}$$

As $L\left(r^A\right)=o\left\{\log M_h^{-1}M_f\left(r^A\right)\right\}$ as $r\to\infty$ we get from (8) that

$$\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\leq\frac{\lambda_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\ . \tag{9}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (9) that

$$\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\le\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}\ . \tag{10}$$

Also for a sequence of values of r tending to infinity ,

$$\begin{aligned}\log M_h^{-1}M_f\left(r^A\right) &\le \left(\lambda_h^{L^*}(f)+\varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\}\\ i.e.,\ \log M_h^{-1}M_f\left(r^A\right) &\le \left(\lambda_h^{L^*}(f)+\varepsilon\right)\left\{A\log r+L\left(r^A\right)\right\}\\ i.e.,\ \frac{\log M_h^{-1}M_f\left(r^A\right)}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)} &\le \log r+\frac{1}{A}L\left(r^A\right)\ . \end{aligned}\tag{11}$$

Now from (1) and (11) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned}\log M_h^{-1}M_{f\circ g}(r) \ge \frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)}\log M_h^{-1}M_f\left(r^A\right)&\\ +\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}&\end{aligned}$$

$$\begin{aligned}i.e.,\ \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)} \ge \frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)}\cdot\frac{\log M_h^{-1}M_f\left(r^A\right)}{\log M_h^{-1}M_f(r^A)+L(r^A)}&\\ +\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log M_h^{-1}M_f(r^A)+L(r^A)}&\end{aligned}$$

$$\begin{aligned}i.e.,\ &\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\\ &\ge\frac{\frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log M_h^{-1}M_f(r^A)}}+\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L(r^A)}-\frac{1}{A}\right\}}{1+\frac{\log M_h^{-1}M_f(r^A)}{L(r^A)}}\ .\end{aligned}\tag{12}$$

In view of the condition $L\left(r^A\right)=o\left\{\log M_h^{-1}M_f\left(r^A\right)\right\}$ as $r\to\infty$ we obtain from (12) that

$$\limsup_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\ge\frac{\lambda_h^{L^*}(f\circ g)-\varepsilon}{A\left(\lambda_h^{L^*}(f)+\varepsilon\right)}\ . \tag{13}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (13) that

$$\limsup_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\ge\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}\ . \tag{14}$$

Also for all sufficiently large values of r ,

$$\begin{aligned} \log M_h^{-1} M_{f\circ g}(r) &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\log\left\{re^{L(r)}\right\} \\ i.e.,\ \log M_h^{-1} M_{f\circ g}(r) &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\log r + L(r)\right\} \\ i.e.,\ \log M_h^{-1} M_{f\circ g}(r) &\leq \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\log r + \frac{1}{A}L\left(r^A\right)\right\} \\ &\quad + \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}\ . \quad (15) \end{aligned}$$

So from (7) and (15) it follows for all sufficiently large values of r that

$$\begin{aligned} \log M_h^{-1} M_{f\circ g}(r) &\leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\log^{[n]}\mu\left(r^A, f\right) \\ &\quad + \left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} &\leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)} \cdot \frac{\log^{[n]}\mu\left(r^A, f\right)}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} \\ &\quad + \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{L(r) - \frac{1}{A}L\left(r^A\right)\right\}}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} \end{aligned}$$

$$\begin{aligned} i.e.,\ &\frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} \\ &\leq \frac{\frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}{1 + \frac{L\left(r^A\right)}{\log M_h^{-1} M_f\left(r^A\right)}} + \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)} - \frac{1}{A}\right\}}{1 + \frac{\log M_h^{-1} M_f\left(r^A\right)}{L\left(r^A\right)}}\ . \quad (16) \end{aligned}$$

Using $L\left(r^A\right) = o\left\{\log M_h^{-1} M_f\left(r^A\right)\right\}$ as $r \to \infty$ we obtain from (16) that

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{A\left(\lambda_h^{L^*}(f)-\varepsilon\right)}\ . \quad (17)$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (17) that

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right) + L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}\ . \quad (18)$$

Thus the theorem follows from (5), (10), (14) and (18). ■

Similarly in view of Theorem 1 , we may state the following theorem without its proof for the right factor g of the composite function $f\circ g$:

Theorem 4 *Let f, g and h be any three entire functions with $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. If $L(r^A) = o\left\{\log M_h^{-1} M_g(r^A)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(g)} \leq \liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g(r^A) + L(r^A)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)}$$
$$\leq \limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)} .$$

We shall use the technique of Theorem 1, Theorem 2 and Definition 5 to get the parallel results on the maximum term of composite entire functions in the next two theorems.

Theorem 5 *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. If $L(r^A) = o\left\{\log \mu_h^{-1} \mu_f(r^A)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(f)} \leq \liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_f(r^A) + L(r^A)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(f)}$$
$$\leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_f(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(f)} .$$

Theorem 6 *Let f, g and h be any three entire functions with $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. If $L(r^A) = o\left\{\log \mu_h^{-1} \mu_g(r^A)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(g)} \leq \liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_g(r^A) + L(r^A)} \leq \frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)}$$
$$\leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_g(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)} .$$

The proofs of Theorem 3 and Theorem 4 are omitted.

Theorem 7 *Let f, g and h be any three entire functions such that $0 < \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \rho_h^{L^*}(f) < \infty$. If $L(r^A) = o\left\{\log M_h^{-1} M_f(r^A)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(f)} \leq \limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f(r^A) + L(r^A)} .$$

Proof. From the definition of $\rho_h^{L^*}(f)$ in terms of maximum modulus, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log M_h^{-1} M_f\left(r^A\right) &\geq \left(\rho_h^{L^*}(f)-\varepsilon\right) \log \left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e., \ \log M_h^{-1} M_f\left(r^A\right) &\geq \left(\rho_h^{L^*}(f)-\varepsilon\right)\left\{A \log r+L\left(r^A\right)\right\} \\ i.e., \ \frac{\log M_h^{-1} M_f\left(r^A\right)}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)} &\geq \log r+\frac{1}{A} L\left(r^A\right) . \end{aligned} \tag{19}$$

Now from (15) and (19) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} \log M_h^{-1} M_{f \circ g}(r) \leq \frac{\left(\rho_h^{L^*}(f \circ g)+\varepsilon\right)}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)} & \log M_h^{-1} M_f\left(r^A\right) \\ &+\left(\rho_h^{L^*}(f \circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A} L\left(r^A\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e., \ \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \leq & \frac{\rho_h^{L^*}(f \circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)} \cdot \frac{\log M_h^{-1} M_f\left(r^A\right)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \\ &+\frac{\left(\rho_h^{L^*}(f \circ g)+\varepsilon\right)\left\{L(r)-\frac{1}{A} L\left(r^A\right)\right\}}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \end{aligned}$$

$$\begin{aligned} i.e., \ & \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \\ & \leq \frac{\frac{\rho_h^{L^*}(f \circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L\left(r^A\right)}{\log M_h^{-1} M_f\left(r^A\right)}}+\frac{\left(\rho_h^{L^*}(f \circ g)+\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)}-\frac{1}{A}\right\}}{1+\frac{\log M_h^{-1} M_f\left(r^A\right)}{L\left(r^A\right)}} . \end{aligned} \tag{20}$$

Using $L\left(r^A\right)=o\left\{\log M_h^{-1} M_f\left(r^A\right)\right\}$ as $r \rightarrow \infty$ we obtain from (20) that

$$\liminf_{r \rightarrow \infty} \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f \circ g)+\varepsilon}{A\left(\rho_h^{L^*}(f)-\varepsilon\right)} . \tag{21}$$

As $\varepsilon(>0)$ is arbitrary, it follows from (21) that

$$\liminf_{r \rightarrow \infty} \frac{\log M_h^{-1} M_{f \circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A \rho_h^{L^*}(f)} . \tag{22}$$

Again for a sequence of values of r tending to infinity ,

$$\begin{aligned} \log M_h^{-1} M_{f \circ g}(r) &\geq \left(\rho_h^{L^*}(f \circ g)-\varepsilon\right) \log \left\{r e^{L(r)}\right\} \\ i.e., \ \log M_h^{-1} M_{f \circ g}(r) &\geq \left(\rho_h^{L^*}(f \circ g)-\varepsilon\right)\{\log r+L(r)\} \\ i.e., \ \log M_h^{-1} M_{f \circ g}(r) &\geq \left(\rho_h^{L^*}(f \circ g)-\varepsilon\right)\left\{\log r+\frac{1}{A} L\left(r^A\right)\right\} \\ &\quad +\left(\rho_h^{L^*}(f \circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A} L\left(r^A\right)\right\} . \end{aligned} \tag{23}$$

So combining (2) and (23) we get for a sequence of values of r tending to infinity that

$$\log M_h^{-1} M_{f\circ g}(r) \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)} \log M_h^{-1} M_f\left(r^A\right) + \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}$$

$$i.e., \ \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)} \cdot \frac{\log M_h^{-1} M_f\left(r^A\right)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} + \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{L(r)-\frac{1}{A}L\left(r^A\right)\right\}}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)}$$

$$i.e., \ \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\frac{\rho_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L\left(r^A\right)}{\log M_h^{-1} M_f\left(r^A\right)}} + \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)\left\{\frac{L(r)}{L\left(r^A\right)}-\frac{1}{A}\right\}}{1+\frac{\log M_h^{-1} M_f\left(r^A\right)}{L\left(r^A\right)}} . \quad (24)$$

Since $L\left(r^A\right)=o\left\{\log M_h^{-1} M_f\left(r^A\right)\right\}$ as $r\to\infty$, it follows from (24) that

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\rho_h^{L^*}(f\circ g)-\varepsilon}{A\left(\rho_h^{L^*}(f)+\varepsilon\right)} . \quad (25)$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (25) that

$$\limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_f\left(r^A\right)+L\left(r^A\right)} \geq \frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)} . \quad (26)$$

Thus the theorem follows from (22) and (26). ■

Theorem 8 *Let f, g and h be any three entire functions with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$. If $L\left(r^A\right)=o\left\{\log M_h^{-1} M_g\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)} \leq \limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g\left(r^A\right)+L\left(r^A\right)} .$$

The proof is omitted .

Similarly, using Definition 5 one may easily establish the following two theorems:

Theorem 9 *Let f, g and h be any three entire functions such that $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(f)<\infty$. If $L\left(r^A\right)=o\left\{\log\mu_h^{-1}\mu_f\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r^A)+L(r^A)}\le\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\le\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_f(r^A)+L(r^A)}.$$

Theorem 10 *Let f, g and h be any three entire functions with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$. If $L\left(r^A\right)=o\left\{\log\mu_h^{-1}\mu_g\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_g(r^A)+L(r^A)}\le\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\le\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log\mu_h^{-1}\mu_g(r^A)+L(r^A)}.$$

The following theorem is a natural consequence of Theorem 1 and Theorem 5 :

Theorem 11 *Let f, g and h be any three entire functions such that $0<\lambda_h^{L^*}(f\circ g)\le\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(f)\le\rho_h^{L^*}(f)<\infty$. If $L\left(r^A\right)=o\left\{\log M_h^{-1}M_f\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\le\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\le\min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\right\}$$
$$\le\max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(f)}\right\}\le\limsup_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_f(r^A)+L(r^A)}\le\frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(f)}.$$

The proof is omitted.

Combining Theorem 2 and Theorem 6, we may state the following theorem:

Theorem 12 *Let f, g and h be any three entire functions with $0<\lambda_h^{L^*}(f\circ g)\le\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(g)\le\rho_h^{L^*}(g)<\infty$. If $L\left(r^A\right)=o\left\{\log M_h^{-1}M_g\left(r^A\right)\right\}$ as $r\to\infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\le\liminf_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_g(r^A)+L(r^A)}\le\min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\right\}$$
$$\le\max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{A\rho_h^{L^*}(g)}\right\}\le\limsup_{r\to\infty}\frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_g(r^A)+L(r^A)}\le\frac{\rho_h^{L^*}(f\circ g)}{A\lambda_h^{L^*}(g)}.$$

Analogously, one may prove the following two theorems with the help of Theorem 3, Theorem 7 and Theorem 4, Theorem 8 respectively. Hence their proofs are omitted.

Theorem 13 *Let f, g and h be any three entire functions such that $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \leq \rho_h^{L^*}(f) < \infty$. If $L\left(r^A\right) = o\left\{\log \mu_h^{-1}\mu_f\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(f)} \leq \liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f(r^A) + L(r^A)} \leq \min\left\{\frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(f)}, \frac{\rho_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(f)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(f)}, \frac{\rho_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(f)}\right\} \leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_f(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(f)} .$$

Theorem 14 *Let f, g and h be any three entire functions with $0 < \lambda_h^{L^*}(f \circ g) \leq \rho_h^{L^*}(f \circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. If $L\left(r^A\right) = o\left\{\log \mu_h^{-1}\mu_g\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(g)} \leq \liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r^A) + L(r^A)} \leq \min\left\{\frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)}, \frac{\rho_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)}, \frac{\rho_h^{L^*}(f \circ g)}{A\rho_h^{L^*}(g)}\right\} \leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_g(r^A) + L(r^A)} \leq \frac{\rho_h^{L^*}(f \circ g)}{A\lambda_h^{L^*}(g)} .$$

Theorem 15 *Let f, g, h and k be any four entire functions such that $f\circ g \sim k$ and $0 < \rho_h^{L^*}(f \circ g) < \infty$. If $L\left(r^A\right) = o\left\{\log M_h^{-1}M_k\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_k(r^A) + L(r^A)} \leq \frac{1}{A} \leq \limsup_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_k(r^A) + L(r^A)} .$$

Proof. Since $f \circ g \sim k$, in view of Lemma 1 and the inequalities (22) and (26) we obtain that

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_k(r^A) + L(r^A)} \leq \frac{1}{A} \tag{27}$$

and

$$\limsup_{r\to\infty} \frac{\log M_h^{-1}M_{f\circ g}(r)}{\log M_h^{-1}M_k(r^A) + L(r^A)} \geq \frac{1}{A}. \tag{28}$$

Thus the theorem follows from (27) and (28) . ■

Theorem 16 *Let f, g, h and k be any four entire functions with $f \circ g \sim k$ and $0 < \rho_h^{L^*}(f \circ g) < \infty$. If $L\left(r^A\right) = o\left\{\log \mu_h^{-1}\mu_k\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_k(r^A) + L(r^A)} \leq \frac{1}{A} \leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log \mu_h^{-1}\mu_k(r^A) + L(r^A)} .$$

The proof is omitted as it can be carried out in the line of Theorem 13.

Theorem 17 *Let f, g, h,k and l be any five entire functions such that $f \circ g \sim k$ and $h \sim l$. Also let $0 < \rho_h^{L^*}(f \circ g) < \infty$. If $L\left(r^A\right) = o\left\{\log M_l^{-1} M_k\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_l^{-1} M_k(r^A) + L(r^A)} \leq \frac{1}{A} \leq \limsup_{r\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r)}{\log M_h^{-1} M_g(r^A) + L(r^A)} .$$

Theorem 18 *Let f, g, h,k and l be any five entire functions with $f \circ g \sim k$ and $h \sim l$. Also let $0 < \rho_h^{L^*}(f \circ g) < \infty$. If $L\left(r^A\right) = o\left\{\log \mu_l^{-1} \mu_k\left(r^A\right)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_l^{-1} \mu_k(r^A) + L(r^A)} \leq \frac{1}{A} \leq \limsup_{r\to\infty} \frac{\log \mu_h^{-1} \mu_{f\circ g}(r)}{\log \mu_h^{-1} \mu_g(r^A) + L(r^A)} .$$

The proofs of Theorems 15 and Theorem 16 are omitted as those can be carried out in the line of Theorem 6 and Theorem 7 respectively and with the help of Lemma 2.

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