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$\mathbf{BG/BF_1/B/BM}$-algebras are congruence permutable

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Abstract

We show that every pair of congruences on a BG-algebra (also on a BF_1/B/BM-algebra) permutes. This result implies that if A is a BG/BF_1/B/BM-algebra, then the lattice of all congruences on A is modular. Moreover, it is proved that BF-algebras and BCK-algebras (BCI/BCH/BH-algebras, too) are not congruence permutable, in general.

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1 Introduction

In 1966, Y. Imai and K. Iséki [6] introduced the notion of a BCK-algebra. It is well known that BCK-algebras are inspired by some implicational logic. There exist several generalizations of BCK-algebras such as BCI-algebras ([7]), BCH-algebras ([5]), BH-algebras ([8]) and many others. J. Neggers and H. S. Kim [12] introduced the notion of a B-algebra. In [14], A. Walendziak defined BF/BF_1-algebras which are a generalization of B-algebras. C. B. Kim and H. S. Kim introduced BM-algebras ([9]) and BG-algebras ([10]).

In this paper, we prove that every pair of congruences on a BG-algebra (also on a BF_1/B/BM-algebra) permutes. This result implies that if A is a BG/BF_1/B/BM-algebra, then the lattice of all congruences on A is modular. Moreover we show that BF-algebras and BCK-algebras (BCI/BCH/BH-algebras, too) are not congruence permutable, in general.

2 Preliminaries

An algebra $(A;*,0)$ of type $(2,0)$ (i.e., a nonempty set A with a binary operation $*$ and a constant 0) is said to be a *BH-algebra* ([8]) if it satisfies the following axioms:

(B1) $x*x=0$,
(B2) $x*0=x$,
(BH) $x*y=y*x=0 \Longrightarrow x=y$.

A *BCH-algebra* ([5]) is a BH-algebra $(A;*,0)$ verifying the axiom

(BCH) $(x*y)*z=(x*z)*y$.

A BH-algebra $(A;*,0)$ satisfying the identity

(BCI) $((x*y)*(x*z))*(z*y)=0$

is called a *BCI-algebra.* Recall that according to the H. S. Li's axiom system ([11]), an algebra $(A;*,0)$ of type $(2,0)$ is a BCI-algebra if and only if it obeys (B2), (BH), and (BCI).

A *BCK-algebra* is a BCI-algebra $(A;*,0)$ satisfying the following additional axiom:

(BCK) $0*x=0$.

Remark 2.1. We know that every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a BH-algebra.

Let $(A;*,0)$ be an algebra of type $(2,0)$ verifying identities (B1) and (B2). We say that A is a *B-algebra* (resp. *BF/BG-algebra*) if A satisfies axiom (B) (resp., (BF)/(BG)), where:

(B) $(x*y)*z=x*[z*(0*y)]$,
(BF) $0*(x*y)=y*x$,
(BG) $x=(x*y)*(0*y)$.

From Proposition 1.5 (b) of [13] and Proposition 2.2 (ii) of [3] we have

Proposition 2.2. *Every B-algebra satisfies the identities* (BF) *and* (BG).

Lemma 2.4 (ii) of [10] gives

Proposition 2.3. *If* $(A;*,0)$ *is a BG-algebra, then* $0*(0*x)=x$ *for all* $x\in A$.

An algebra $(A;*,0)$ of type $(2,0)$ is called a *BM-algebra* ([9]) if it satisfies (B2) and the following axiom:

(BM) $(x * y) * (x * z) = z * y$.

Remark 2.4. From Theorem 2.6 of [9] it follows that every BM-algebra is a B-algebra. By Proposition 2.8 of [10], every BG-algebra is a BH-algebra. It is easy to see that (BM) implies (BCI). Therefore the class of BM-algebras is a subclass of the class of BCI-algebras.

A *BF$_1$-algebra* ([14]) is a BF-algebra $(A; *, 0)$ such that (BG) holds for all $x, y \in A$.

Proposition 2.5. ([14]) *An algebra* $\mathbf{A} = (A; *, 0)$ *of type* $(2, 0)$ *is a BF$_1$-algebra if and only if it satisfies the laws* (B1), (BF), *and* (BG).

Remark 2.6. Propositions 2.2 and 2.5 show that every B-algebra is a BF$_1$-algebra and every BF$_1$-algebra is a BG-algebra.

We will denote by **BH** (resp., **BCH/BCI/BCK/BM/B/BG/BF/BF$_1$**) the class of all BH-algebras (resp., BCH/BCI/BCK/BM/B/BG/BF/BF$_1$- algebras). We get by Remark 2.1 that

$$\mathbf{BCK} \subset \mathbf{BCI} \subset \mathbf{BCH} \subset \mathbf{BH} \tag{1}$$

and by Remark 2.4 we have

$$\mathbf{BM} \subset \mathbf{B}, \ \ \mathbf{BM} \subset \mathbf{BCI}, \ \text{ and } \ \mathbf{BG} \subset \mathbf{BH}. \tag{2}$$

Remark 2.6 shows that

$$\mathbf{B} \subset \mathbf{BF}_1 \subset \mathbf{BG}. \tag{3}$$

From (1)–(3) we obtain the interrelatioships (see Figure 1) between some of the concepts mentioned above (An arrow indicates proper inclusion, that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \to \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$.).

3 Results

We shall say that an algebra A has *permuting congruences*, or that A is *congruence permutable*, if every pair of congruences on A permutes, that is, $\alpha \circ \beta = \beta \circ \alpha$ for every $\alpha, \beta \in \mathrm{Con}A$ (where $\mathrm{Con}A$ denotes the set of all congruences on A). A variety $\mathbf{V}$ of algebras is said to be *congruence permutable* if all the algebras in $\mathbf{V}$ have permuting congruences.

Lemma 3.1 (see e.g. [2]) *Let* $\mathbf{V}$ *be a variety of algebras. The variety* $\mathbf{V}$ *is congruence permutable if and only if there is a 3-ary term* t *such that the identities* $t(x, y, y) = x$ and $t(x, x, y) = y$ *are valid in* $\mathbf{V}$.

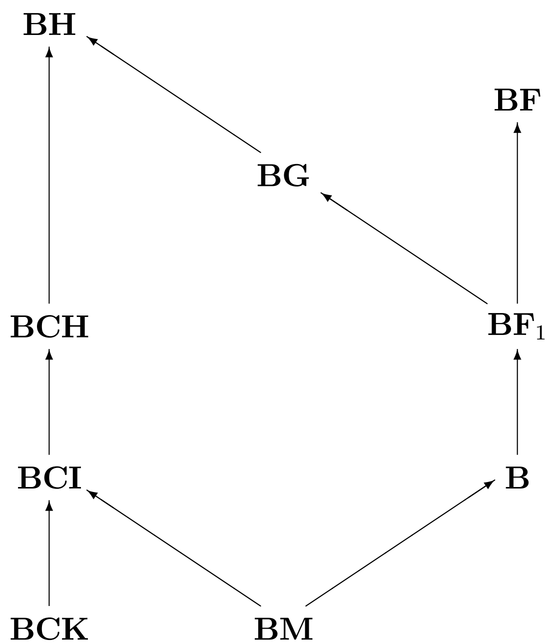

Figure 1

The class **BM** of all BM-algebras is a variety. Similarly, the classes **B**, **BG**, **BF** and $\mathbf{BF}_1$ are varieties.

Theorem 3.2. *The variety* **BG** *is congruence permutable.*

Proof. Let $(A;*,0)$ be a BG-algebra and let $t(x,y,z) = (x*y)*(0*z)$. By (BG),

$$t(x,y,y) = (x*y)*(0*y) = x.$$

From (B1) and Proposition 2.3 we have

$$t(x,x,y) = 0*(0*y) = y.$$

Applying Lemma 3.1 we conclude that the variety **BG** is congruence permutable. □

Corollary 3.3. *The varieties* $\mathbf{BF}_1$, **B** *and* **BM** *are congruence permutable.*

Let A be an algebra. With respect to the set inclusion, $\mathrm{Con}(A)$ forms a lattice. The least and largest congruences of A are denoted by 0_A and 1_A, that is, $0_A = \{(a,a) : a \in A\}$ and $1_A = A^2$. It is known (see for an example [1]) that if an algebra A has permuting congruences, then $\mathrm{Con}(A)$ is a modular lattice. From this we have

Theorem 3.4. *Let A be a* $\mathrm{BG}/\mathrm{BF}_1/\mathrm{B}/\mathrm{BM}$*-algebra. Then the lattice* $\mathrm{Con}(A)$ *is modular.*

Example 3.5. Let $A = \{0, 1, 2, 3\}$ and $*$ be defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

From [4] it follows that $(A, *, 0)$ is a BCK-algebra. Let $\alpha = 0_A \cup \{(0, 1), (1, 0)\}$ and $\beta = 0_A \cup \{(0, 2), (2, 0)\}$. It is easy to check that $\alpha, \beta \in \mathrm{Con}A$. We have $(1, 2) \in \alpha \circ \beta$ but $(1, 2) \notin \beta \circ \alpha$. Therefore $\alpha \circ \beta \neq \beta \circ \alpha$.

Remark 3.6. From the above example we conclude that there is a BCK-algebra which is not congruence permutable. Hence BCI/BCH/BH-algebras are not congruence permutable, in general.

Proposition 3.7. *There is a BF-algebra which is not congruence permutable.*

Proof. Let $A = \{0, 1, 2, 3\}$ and $*$ be defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	0	0
2	2	0	0	0
3	3	0	0	0

It is easy to see that $(A, *, 0)$ is a BF-algebra. Set $\alpha = 0_A \cup \{(1, 2), (2, 1)\}$ and $\beta = 0_A \cup \{(2, 3), (3, 2)\}$. Obviously, $\alpha, \beta \in \mathrm{Con}A$. We get $(1, 3) \in \alpha \circ \beta$ but $(1, 3) \notin \beta \circ \alpha$. Then $\alpha \circ \beta \neq \beta \circ \alpha$. Thus A is not congruence permutable. □

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