Mathematica Aeterna, Vol. 5, 2015, no. 2, 357 - 361

Local Regularity for Minimizers of Integral Functionals

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Abstract

We prove local regularity for minimizers of integral functionals of the form

$$\int_\Omega f(x,u,Du)dx$$

where the integrand $f(x,s,z)=f_0(x,s,z)+f_1(x,s,z):\Omega\times \mathrm{R}\times \mathrm{R}^n\to \mathrm{R}$ is a Carathéodory function, $f_0(x,s,z)$ grows like $|z|^p$ with $p>1$, and

$$|f_1(x,s,z)|\le \varphi_1(x)|z|,\quad \varphi_1(x)\in L^{p'r}_{loc}(\Omega),\ 1<r<\frac{n}{p}.$$

Mathematics Subject Classification: 49N60, 35J60.

Keywords: Local regularity, Minimizer, Integral functional.

1 Introduction and Statement of Result.

Let Ω be a bounded open subset of R^n, $n\ge 2$. We consider integral functionals of the type

$$\mathcal{F}(u;\Omega)=\int_\Omega f(x,u,Du)dx, \tag{1.1}$$

where the integrand $f(x,s,z)$ satisfies the following assumptions:

(i) $f(x,s,z):\Omega\times \mathrm{R}\times \mathrm{R}^n\to \mathrm{R}$ is a Carathéodory function which can be written as

$$f(x,s,z)=f_0(x,s,z)+f_1(x,s,z),$$

(ii) there exists $\varphi_0(x)\in L^r_{loc}(\Omega)$, $1<r<\frac{n}{p}$, such that

$$L^{-1}|z|^p\le f_0(x,s,z)\le L(|z|^p+\varphi_0(x)),$$

(iii) there exists $\varphi_1(x) \in L^{p'r}_{loc}(\Omega)$, $\varphi_1(x) \geq 0$, such that

$$|f_1(x, s, z)| \leq \varphi_1(x)|z|.$$

We point out that no differentiability assumption is made on $\mathcal{F}(u; \Omega)$.

Definition 1.1 *By a minimizer of the functional $\mathcal{F}$ we mean functions $u \in W^{1,p}_{loc}(\Omega)$, such that for every function $\psi \in W^{1,p}(\Omega)$ with $supp\psi \subset\subset \Omega$ it results in*

$$\mathcal{F}(u; supp\psi) \leq \mathcal{F}(u + \psi; supp\psi). \tag{1.2}$$

Continuity properties of minimizers of integral functionals (1.1) with the integrand $f(x, s, z)$ satisfies the assumptions (i), (ii) and (iii) have been studied in [1] by Ferone and Fusco. In this paper we obtain a local regularity result for minimizers of integral functionals (1.1). Local regularity properties are important among the regularity theories of nonlinear elliptic PDEs and systems, see the monograph [2] by Bensoussan and Frehse. For some local regularity results related to (1.1), we refer the readers to [3-6].

The main result of the present paper is the following theorem.

Theorem 1.1 *Under the previous assumptions (i)-(iii), if $u \in W^{1,p}_{loc}(\Omega)$, $1 < p < n$, is a minimizer of the integral functional (1.1), then it belongs to $L^{(pr)^*}_{loc}(\Omega)$.*

2 Preliminaries.

For $x_0 \in \Omega$ and $t > 0$, we denote by $B_t(x_0)$, or simply B_t, the ball of radius t centered at x_0. For $k > 0$, let

$$A_k = \{x \in \Omega : |u(x)| > k\}, \quad A_{k,t} = A_k \cap B_t.$$

Moreover, for $m < n$, m^* is always the real number satisfying $\frac{1}{m^*} = \frac{1}{m} - \frac{1}{n}$.

The following lemma can be found in [7].

Lemma 2.1 *Let $u \in W^{1,p}_{loc}(\Omega)$, $g \in L^r_{loc}(\Omega)$, where $1 < p < n$ and r satisfies*

$$1 < r < \frac{n}{p}.$$

Assume that the following integral estimate holds:

$$\int_{A_{k,\tau}} |Du|^p dx \leq c_0 \left[\int_{A_{k,t}} g dx + (t - \tau)^{-\alpha} \int_{A_{k,t}} |u|^p dx \right], \tag{2.1}$$

for every $k \in N$ and $R_0 \leq \tau < t \leq R_1$, where c_0 is a positive constant that depends only on N, p, r, R_0, R_1 and $|\Omega|$, and α is a real positive constant. Then $u \in L^{(pr)^*}_{loc}(\Omega)$.

The following lemma comes from [8], and will be used in the proof of Theorem 1.1.

Lemma 2.2 *Let $f(\tau)$ be a non-negative bounded function defined for $0 \le R_0 \le t \le R_1$. Suppose that for $R_0 \le \tau < t \le R_1$ we have*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \theta f(t),$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant c, depending only on α and θ such that for every ρ, R, $R_0 \le \rho < R \le R_1$ we have

$$f(\rho) \le c[A(R-\rho)^{-\alpha} + B].$$

3 Proof of Theorem 2.1.

Owing to Lemma 2.1, it is sufficient to prove that u satisfies the integral estimate (2.1) with $\alpha = p$ and $g = \varphi_0 + \varphi_1^{p'}$. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily fixed. Choose $\psi = -\eta(u - T_k(u))$ in (1.2), where η is a cut-off function such that

$$\text{supp}\eta \subset B_t,\ 0 \le \eta \le 1,\ \eta = 1 \text{ in } B_\tau,\ |D\eta| \le 2(t-\tau)^{-1},$$

and

$$T_k(u) = \max\{-k, \min\{u, k\}\}$$

is the usual truncation of u at level $k > 0$. We obtain by Definition 1.1 that

$$\int_{B_t} f(x, u, Du)dx \le \int_{B_t} f(x, u + \psi, Du + D\psi)dx. \tag{3.1}$$

Since $\psi = 0$ on $\{x \in B_t : |u| \le k\}$, then (3.1) yields

$$\int_{A_{k,t}} f(x, u, Du)dx \le \int_{A_{k,t}} f(x, u + \psi, Du + D\psi)dx. \tag{3.2}$$

Thus, by using (ii) and (3.2) we obtain

$$\begin{aligned} & L^{-1}\int_{A_{k,t}} |Du|^p dx \le \int_{A_{k,t}} f_0(x, u, Du)dx \\ \le\ & -\int_{A_{k,t}} f_1(x, u, Du)dx + \int_{A_{k,t}} f_0(x, u + \psi, Du + D\psi)dx \\ & + \int_{A_{k,t}} f_1(x, u + \psi, Du + D\psi)dx \\ =\ & I_1 + I_2 + I_3. \end{aligned} \tag{3.3}$$

Our nearest goal is to estimate $|I_i|$, $i = 1, 2, 3$. Condition (iii) together with Young inequality yields

$$\begin{aligned} |I_1| &\le \int_{A_{k,t}} |f_1(x,u,Du)|\,dx \le \int_{A_{k,t}} \varphi_1 |Du| dx \\ &\le C(\varepsilon)\|\varphi_1\|^{p'}_{L^{p'}(A_{k,t})} + \varepsilon \|Du\|^p_{L^p(A_{k,t})}. \end{aligned} \tag{3.4}$$

(ii) implies

$$\begin{aligned} |I_2| &\le \int_{A_{k,t}} |f_0(x,u+\psi, Du + D\psi|\,dx \\ &\le L\int_{A_{k,t}} |Du + D\psi|^p dx + L\int_{A_{k,t}} \varphi_0 dx = L(J_1 + J_2). \end{aligned} \tag{3.5}$$

Substituting

$$D\psi = -(u - T_k(u))D\eta - \eta Du,$$

into J_1, and use the fact $|u - T_k(u)| \le |u|$, we can derive

$$\begin{aligned} |J_1| &\le \int_{A_{k,t}} |(1-\eta)Du - (u - T_k(u))D\eta|^p dx \\ &\le 2^{p-1}\int_{A_{k,t}\setminus A_{k,\tau}} |Du|^p dx + 2^{2p-1}\int_{A_{k,t}} \frac{|u - T_k(u)|^p}{(t-\tau)^p} dx \\ &\le 2^{p-1}\int_{A_{k,t}\setminus A_{k,\tau}} |Du|^p dx + 2^{2p-1}\int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx. \end{aligned} \tag{3.6}$$

(iii), Young inequality and (3.6) yield

$$\begin{aligned} |I_3| &\le \int_{A_{k,t}} |f_1(x,u+\psi, Du + D\psi)| dx \le \int_{A_{k,t}} \varphi_1 |Du + D\psi| dx \\ &\le C(\varepsilon)\|\varphi_1\|^{p'}_{L^{p'}(A_{k,t})} + \varepsilon\|Du + D\psi\|^p_{L^p(A_{k,t})} \\ &\le C(\varepsilon)\|\varphi_1\|^{p'}_{L^{p'}(A_{k,t})} + \varepsilon\left[2^{p-1}\int_{A_{k,t}\setminus A_{k,\tau}} |Du|^p dx + 2^{2p-1}\int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx\right]. \end{aligned} \tag{3.7}$$

It is no loss of generality to assume $\varepsilon < 1$. Substituting (3.4)-(3.7) into (3.3), we have

$$\begin{aligned} &\int_{A_{k,\tau}} |Du|^p dx \le \int_{A_{k,t}} |Du|^p dx \\ \le\ & 2^{p-1}(L+1)\int_{A_{k,t}\setminus A_{k,\tau}} |Du|^p dx + L\varepsilon\int_{A_{k,t}} |Du|^p dx \\ & + 2^{2p-1}(L+1)\int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx + L^2\int_{A_{k,t}} \varphi_0 dx \\ & + 2C(\varepsilon)L\int_{A_{k,t}} \varphi_1^{p'} dx \end{aligned} \tag{3.8}$$

Adding both sides $2^{p-1}(L+1)$ times the left hand side and divided both sides by $2^{p-1}(L+1)+1$, one has

$$\begin{aligned}
&\int_{A_{k,\tau}} |Du|^p dx \\
\leq\ & \left(\theta + \frac{L\varepsilon}{2^{p-1}(L+1)}\right) \int_{A_{k,t}} |Du|^p dx \\
&+ \frac{2^{2p-1}(L+1)}{2^{p-1}(L+1)+1} \int_{A_{k,t}} \frac{|u|^p}{(t-\tau)^p} dx + \frac{L^2}{2^{p-1}(L+1)+1} \int_{A_{k,t}} \varphi_0 dx \\
&+ \frac{2C(\varepsilon)L}{2^{p-1}(L+1)+1} \int_{A_{k,t}} \varphi_1^{p'} dx,
\end{aligned} \tag{3.9}$$

where $\theta = \frac{2^{p-1}(L+1)}{2^{p-1}(L+1)+1} < 1$, c_0 is a constant depends only on p and L. Taking ε small enough such that $\theta + \frac{L\varepsilon}{2^p L+1} < 1$. Lemma 2.2 implies that for any $0 \leq R_0 \leq \rho < R \leq R_1$,

$$\int_{A_{k,\rho}} |Du|^p dx \leq c_0 \left[(R-\rho)^{-p} \int_{A_{k,R}} |u|^p dx + \int_{A_{k,R}} (\varphi_0 + \varphi_1^{p'}) dx \right],$$

where c_0 is a constant depending only on p and L. Theorem 1.1 follows from Lemma 2.1.

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Received: April, 2015