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Hermite-Hadamard type inequalities for coordinates convex stochastic processes

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Abstract

The main aim of the present note is to introduce co-ordinates convex stochastic processes . Moreover, we prove Hermite-Hadamard-type inequalities for co-ordinated convex stochastic processes. We also define some mappings about co-ordinates convex stochastic processes and investigate main properties of these mappings.

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1 Introduction

In 1980, Nikodem [6] introduced the convex stochastic processes in his article. Later in 1995, Skowronski [12] presented some further results on convex stochastic processes. Moreover, in 2011, Kotrys [4] derived some Hermite-Hadamard type inequalities for convex stochastic processes. In 2014, Maden *et.al.* [7] introduced the s-convex stochastic processes in the first sense and proved Hermite-Hadamard type inequalities to these processes. Also in 2014, Set *et.al.* [9] presented the s-convex stochastic processes in the second sense and they investigated Hermite-Hadamard type inequalities for these processes. Moreover, in recent papers [1, 5], strongly λ-GA-convex stochastic processes and preinvex stochastic processes has been introduced.

Definition 1.1 *Let (Ω, A, P) be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $A-measurable$.*

Definition 1.2 *Let (Ω, A, P) be an arbitrary probability space and $T \subset \mathbb{R}$ be time. A collection of random variables $X(t, \omega)$, $t \in T$ with values in $\mathbb{R}$ is called a stochastic process. If $X(t, \omega)$ takes values in $S = \mathbb{R}^d$, it is called a $vector-valued$ stochastic process. If the time T can be a discrete subset of $\mathbb{R}$, then $X(t, \omega)$ is called a discrete time stochastic process. If time is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time. For any fixed $\omega \in \Omega$, one can regard $X(t, \omega)$ as a function of t. It is called a sample function of the stochastic process. In the case of a $vector-valued$ process, it is a sample path, a curve in $\mathbb{R}^d$.*

Definition 1.3 *Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : T \times \Omega \to \mathbb{R}$ is*

(1) *convex if*

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(v, \cdot)$$

for all $u, v \in T$ and $\lambda \in [0, 1]$. This class of stochastic process is denoted by C.

(2) *$\lambda-$convex (where λ is a fixed number in $(0, 1)$ if*

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda) X(u, \cdot)$$

for all $u, v \in T$ and $\lambda \in (0, 1)$. This class of stochastic process is denoted by C_λ.

(3) *Wright-convex if*

$$X(\lambda u + (1-\lambda) v, \cdot) + X((1-\lambda)u + \lambda v, \cdot) \leq X(u, \cdot) + X(v, \cdot)$$

for all $u, v \in T$ and $\lambda \in [0, 1]$. This class of stochastic process is denoted by W.

(4) *Jensen-convex if*

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}.$$

(see,[4, 11, 12, 6]).

Clearly, $C \subseteq C_\lambda \subset W$ and $C_{\frac{1}{2}} \subseteq C_\lambda$, for all $\lambda \in (0,1)$. [12]

Definition 1.4 *Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : T \times \Omega \to \mathbb{R}$ is called*

(1) *continuous in probability* in T if

$$P - \lim_{t \to t_0} X(t,\cdot) = X(t_0,\cdot)$$

(where $P-$lim denotes limit in probability) or equivalently

$$\lim_{t \to t_0} P\{|X(t,\cdot) - X(t_0,\cdot)| > \varepsilon\} = 0$$

for any small enough $\varepsilon > 0$ and all $t_0 \in T$.

(2) *mean-square continuous* (or *continuous in quadratic mean*) in T if

$$\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0$$

such that $E[X(t)^2] < \infty$, for all $t_0 \in T$.

(3) *mean-square differentiable* in T if it is mean square continuous and there exists a process $X^{'}(t,\cdot)$ ("speed" of the process) such that

$$\lim_{t \to t_0} E[\left(\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\right)^2] = 0.$$

Different types of continuity concepts can be defined for stochastic processes. Surely (everywhere) and almost surely (almost everywhere or sample path) convergences are rarely used in applications due to the restrictive requirement, that is, as $t \to t_0$, $X(t,\omega)$ has to approach $X(t_0,\omega)$ for each outcome $\omega \in S \subseteq \Omega$. For further reading on stochastic calculus, reader may refer to [10].

Definition 1.5 ([10]) *Let (Ω, A, P) be a probability space and $T \subset \mathbb{R}$ be an interval with $E[X(t)^2] < \infty$ for all $t \in T$. Let $[a, b] \subset T$, $a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t)$ on $[a, b]$ if the following identity holds:*

$$\lim_{n\to\infty} E[(X(\Theta_k (t_k - t_{k-1}) - Y)^2] = 0.$$

Then we can write

$$\int_a^b X(t, \cdot)dt = Y(\cdot) \ (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot)dt \leq \int_a^b Z(t, \cdot)dt(a.e.)$$

where $X(t, \cdot) \leq Z(t, \cdot)$ *(a.e.)* in $[a, b]$.

In throughout the paper, we will consider the stochastic processes that is *with continuous time* and *mean-square continuous.*

Now, we present some results proved by Kotrys [4] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1.6 *If $X : I\times\Omega \to \mathbb{R}$ is a stochastic processes of the form $X(t, \cdot) = A(\cdot)\, t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E[A^2] < \infty, E[B^2] < \infty$ and $[a, b] \subset I$, then*

$$\int_a^b X(t, \cdot)\, dt = A(\cdot)\, \frac{b^2 - a^2}{2} + B(\cdot)\,(b - a) \ \ (a.e.).$$

Proposition 1.7 *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic processes and $t_0 \in intI$. Then there exist a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is*

$$X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \ \ (a.e.).$$

for all $t \in I$.

Theorem 1.8 *If stochastic $X : T \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{v - u}\int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}. \qquad (1)$$

The following inequality is well-known in the literature as the Jensen integral inequality for convex stochastic processes (see, [8]):

Theorem 1.9 (Jensen-type inequality) *If $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic processes, for an arbitrary non-negative integrable stochastic processes $\varphi : I \times \Omega \to \mathbb{R}$, we have*

$$X\left(\frac{1}{b-a}\int_a^b \varphi(t,.)\,dt,\cdot\right) \le \frac{1}{b-a}\int_a^b X\circ\varphi(t,.)\,dt.$$

Proof. From Proposition 1.7, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b X\circ\varphi(t,.)\,dt - X\left(\frac{1}{b-a}\int_a^b \varphi(t,.)\,dt,\cdot\right)\\
=\ &\frac{1}{b-a}\int_a^b \left[X\circ\varphi(t,.) - X\left(\frac{1}{b-a}\int_a^b \varphi(t,.)\,dt,\cdot\right)\right]dt\\
\ge\ &A(.)\left\{\frac{1}{b-a}\int_a^b \left[\varphi(t,.) - \frac{1}{b-a}\int_a^b \varphi(t,.)\,dt\right]dt\right\}\\
=\ &A(.)\left\{\frac{1}{b-a}\int_a^b \varphi(t,.) - \frac{1}{b-a}\int_a^b \varphi(t,.)\,dt\right\}\\
=\ &0
\end{aligned}$$

which completes the proof.

Also, note that the related results for convex stochastic processes and various types of convex stochastic processes can be seen in [11, 12, 6, 1, 5, 14, 13].

Definition 1.10 *Let us consider a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:*

$$f(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)w) \le \alpha f(x,y) + (1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\alpha \in [0,1]$. If the inequality reversed then f is said to be concave on Δ. [2]

Definition 1.11 *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$, $y \in [c,d]$.*

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.12 *A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $t, s \in [0,1]$ and $(x,y),(u,w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx+(1-t)y, su+(1-s)w) \\ \leq \; & tsf(x,u)+t(1-s)f(x,w) \\ & +(1-t)sf(y,u)+(1-t)(1-s)+f(y,w). \end{aligned} \tag{2}$$

In [3], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.13 *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq \; & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{a+b}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{c+d}{2}, y\right)dy\right] \\ \leq \; & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy \\ \leq \; & \frac{1}{4}[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \\ & +\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy] \\ \leq \; & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{3}$$

In the present paper, we introduce coordinated convex stochastic processes. Also, we establish Hermite-Hadamard type inequalities for coordinated convex stochastic processes similar to those from [3].

2 Convex stochastic processes on co-ordinates

Definition 2.1 *Let us consider $\Lambda := T_1 \times T_2$, $T_1, T_2 \subset \mathbb{R}$ and $X : \Lambda \times \Omega \to \mathbb{R}$ be a stochastic process. $X : \Lambda \to \mathbb{R}$ is said to be convex in Λ if following inequality holds:*

$$X((\lambda t_1 + (1-\lambda)t_2, \lambda s_1 + (1-\lambda)s_2), .) \leq \lambda X((t_1,s_1), .) + (1-\lambda)X((t_2,s_2), .) \tag{4}$$

for all $(t_1, s_1), (t_2, s_2) \in \Lambda$ *and* $\lambda \in [0,1]$*. If the inequality reversed then* X *is said to be concave on* Λ*.*

Definition 2.2 *A stochastic process* $X : \Lambda \times \Omega \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Λ *if the partial mappings* $X_s : T_1 \times \Omega \to \mathbb{R}$*,* $X_s(u,.) := X((u,s),.)$ *and* $X_t : T_2 \times \Omega \to \mathbb{R}$*,* $X_t(v,.) := X((t,v),.)$ *are convex where defined for all* $u \in T_1$*,* $v \in T_2$

As in (2), a formal definition for co-ordinated convex stochastic processes may be stated as follows:

Definition 2.3 *A stochastic process* $X : \Lambda \times \Omega \to \mathbb{R}$ *will be called co-ordinated convex on* Λ*, for all* $\alpha, \beta \in [0,1]$ *and* $(t_1, s_1), (t_2, s_2) \in \Lambda$*, if the following inequality holds:*

$$\begin{aligned} & X((\alpha t_1 + (1-\alpha)t_2, \beta s_1 + (1-\beta)s_2),.) \\ \leq\ & \alpha\beta X((t_1, s_1),.) + \alpha(1-\beta)X((t_1, s_2),.) \\ & +(1-\alpha)\beta X((t_2, s_1),.) + (1-\alpha)(1-\beta)X((t_2, s_2),.) \end{aligned}$$

Lemma 2.4 *Every convex stochastic process* $X : \Lambda \times \Omega \to \mathbb{R}$ *on* Λ *is convex on the co-ordinates.*

Proof. Suppose that stochastic process$X : \Lambda \times \Omega \to \mathbb{R}$ is convex on Λ. Consider $X_t : T_2 \times \Omega \to \mathbb{R}$, $X_t(v,.) := X((t,v),.)$. Then for all $\lambda \in [0,1]$ and $s_1, s_2 \in T_2$, one has:

$$\begin{aligned} & X_t((\lambda s_1 + (1-\lambda)s_2),.) \\ =\ & X((t, \lambda s_1 + (1-\lambda)s_2),.) \\ =\ & X((\lambda t + (1-\lambda)t, \lambda s_1 + (1-\lambda)s_2),.) \\ \leq\ & \lambda X((t, s_1),.) + (1-\lambda)X((t, s_2),.) \\ =\ & \lambda X_t(s_1,.) + (1-\lambda)X_t(s_2,.) \end{aligned}$$

which shows the convexity of X_t.

The fact that $X_s : T_1 \times \Omega \to \mathbb{R}$, $X_s(u,.) := X((u,s),.)$ is also convex on T_1 for all $s \in T_2$ goes likewise, and we shall omit the details.

The following inequalties of Hadamard type hold.

Theorem 2.5 *Suppose that stochastic process $X:\Lambda\times\Omega\to\mathbb{R}$ is convex on the co-ordinates on Λ. Then one has the inequalities:*

$$\begin{aligned}
& X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right) \qquad (5)\\
\leq\ & \frac{1}{2}\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2}X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt\right.\\
& \left.+\frac{1}{s_2-s_1}\int_{s_1}^{s_2}X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds\right]\\
\leq\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)\,dtds\\
\leq\ & \frac{1}{4}\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2}X((t,s_1),.)\,dt+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}X((t,s_2),.)\,dt\right.\\
& \left.+\frac{1}{s_2-s_1}\int_{s_1}^{s_2}X((t_1,s),.)\,ds+\frac{1}{s_2-s_1}\int_{s_1}^{s_2}X((t_2,s),.)\,ds\right]\\
\leq\ & \frac{X((t_1,s_1),.)+X((t_1,s_2),.)+X((t_2,s_1),.)+X((t_2,s_2),.)}{4}.
\end{aligned}$$

Proof. Since stochastic process $X:\Lambda\times\Omega\to\mathbb{R}$ is convex on the co-ordinates on Λ, it follows that the mapping $X_t:T_2\times\Omega\to\mathbb{R}$, $X_t(s,.):=X((t,s),.)$ is convex on T_2 for all $t\in T_1$. Then, by the inequality (1), one has:

$$\begin{aligned}
X_t\left(\frac{s_1+s_2}{2},.\right) &\leq \frac{1}{s_2-s_1}\int_{s_1}^{s_2}X_t(s,.)\,ds\\
&\leq \frac{X_t(s_1,.)+X_t(s_2,.)}{2},\ t\in T_1.
\end{aligned}$$

Namely,

$$\begin{aligned}
X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right) &\leq \frac{1}{s_2-s_1}\int_{s_1}^{s_2}X((t,s),.)\,ds\\
&\leq \frac{X((t,s_1),.)+X((t,s_2),.)}{2},\ t\in T_1.
\end{aligned}$$

Integrating this inequality on $[t_1,t_2]$ and multiplying each side of the inequality by $\frac{1}{t_2-t_1}$, we have:

$$\begin{aligned}
& \frac{1}{t_2-t_1}\int_{t_1}^{t_2}X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt \qquad (6)\\
\leq\ & \frac{1}{(t_2-t_1)(w_2-w_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)\,dsdt\\
\leq\ & \frac{1}{2}\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{X((t,s_1),.)}{2}dt+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{X((t,s_2),.)}{2}dt\right].
\end{aligned}$$

By a similar argument applied for the mapping $X_s : T_1 \times \Omega \to \mathbb{R}$, $X_s(u,.) := X((u,s),.)$ we get

$$\begin{aligned} & \frac{1}{s_2-s_1}\int_{s_1}^{s_2} X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds \qquad (7) \\ \leq\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2} X((t,s),.)\,dsdt \\ \leq\ & \frac{1}{2}\left[\frac{1}{s_2-s_1}\int_{s_1}^{s_2}\frac{X((t_1,s),.)}{2}ds+\frac{1}{s_2-s_1}\int_{s_1}^{s_2}\frac{X((t_2,s),.)}{2}ds\right]. \end{aligned}$$

Summing the inequalities (6) and (7), we get the second and third inequality in (5).

By the inequality (1), we also have:

$$X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right) \leq \frac{1}{t_2-t_1}\int_{t_1}^{t_2} X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt$$

and

$$X\left(\left(\frac{t_1+t_2}{2},\frac{\omega_1+\omega_2}{2}\right),.\right) \leq \frac{1}{s_2-s_1}\int_{s_1}^{s_2} X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds$$

which give, by addition, the first inequality in (5).

Lastly, by the same inequality, we can also write:

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} X((t,s_1),.)dt \leq \frac{X((t_1,s_1),.)+X((t_2,s_1),.)}{2}$$

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} X((t,s_2),.)dt \leq \frac{X((t_1,s_2),.)+X((t_2,s_2),.)}{2}$$

$$\frac{1}{s_2-s_1}\int_{s_1}^{s_2} X((t_1,s),.)ds \leq \frac{X((t_1,s_1),.)+X((t_1,s_2),.)}{2}$$

and

$$\frac{1}{s_2-s_1}\int_{s_1}^{s_2} X((t_2,s),.)ds \leq \frac{X((t_2,s_1),.)+X((t_2,s_2),.)}{2}$$

which give, by the addition, the last inequality (5).

3 Some mappings associated to Hermte-Hadamard inequality

Now, for stochastic process $X : \Lambda \times \Omega \to \mathbb{R}$, $[t_1, t_2] \times [s_1, s_2] \subset \Lambda$, we can define the following mapping $H : [0,1]^2 \times \Omega \to \mathbb{R}$,

$$\begin{aligned} H\left((\alpha,\beta),.\right) &: \\ = \frac{1}{(t_2-t_1)(s_2-s_1)} & \int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right.\right. \\ & \left.\left., \beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right) dsdt. \end{aligned}$$

The properties of this mapping are embodied in the following theorem.

Theorem 3.1 *Suppose that stochastic process $X : \Lambda \times \Omega \to \mathbb{R}$ is convex on the co-ordinates on Λ. Then:*

(i) The mapping H is convex on the co-ordinates on $[0,1]^2$.

(ii) We have the bounds:

$$\sup_{(\alpha,\beta)\in[0,1]^2} H((\alpha,\beta),.) = \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2} X((t,s),.)\,dsdt = H(1,1);$$

$$\inf_{(\alpha,\beta)\in[0,1]^2} H((\alpha,\beta),.) = X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right) = H(0,0);$$

(iii) The mapping H is monotonic nondecreasing on the co-ordinates.

Proof. (i) Let s be a fixed number in $[0,1]$. Then for all $\gamma, \theta \geq 0$ with $\gamma + \theta = 1$ and $k_1, k_2 \in [0,1]$.

$$\begin{aligned} & H((\gamma k_1+\theta k_2, s),.) \\ = & \frac{1}{(t_2-t_1)(s_2-s_1)} \\ & \times \int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left((\gamma k_1+\theta k_2)t+(1-(\gamma k_1+\theta k_2))\frac{t_1+t_2}{2}\right.\right., \\ & \left.\left.\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right) dsdt \end{aligned}$$

$$\begin{aligned} = & \frac{1}{(t_2-t_1)(s_2-s_1)} \\ & \times \int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left(\gamma\left(k_1 t+(1-k_1)\frac{t_1+t_2}{2}\right)\right.\right. \\ & \left.\left.+\theta\left(k_2 t+(1-k_2)\frac{t_1+t_2}{2}\right), \beta\omega+(1-\beta)\frac{\omega_1+\omega_2}{2}\right),.\right) dsdt \end{aligned}$$

$$\begin{aligned}
&\leq \gamma\frac{1}{(t_2-t_1)(s_2-s_1)} \\
&\quad\times\int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left(k_1 t+(1-k_1)\frac{t_1+t_2}{2},\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right)dsdt \\
&\quad+\theta\frac{1}{(t_2-t_1)(s_2-s_1)} \\
&\quad\times\int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left(k_2 t+(1-k_2)\frac{t_1+t_2}{2},\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right)dsdt \\
&= \gamma H((k_1,s),.)+\theta H((k_2,s),.).
\end{aligned}$$

If $t\in[0,1]$ is fixed, then for all $\gamma,\theta\geq 0$ with $\gamma+\theta=1$ and $l_1,l_2\in[0,1]$, we also have:

$$H((t,\gamma l_1+\theta l_2),.)\leq \gamma H((t,l_1),.)+\theta H((t,l_2),.)$$

and the statement is proved.

(ii) Since X is convex on the co-ordinates on Λ, we have, by Jensen's inequality for integrals, that:

$$\begin{aligned}
&H((\alpha,\beta),.) \\
&= \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2} X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right.\right. \\
&\quad\left.\left.,\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right)dsdt
\end{aligned}$$

$$\begin{aligned}
\geq\ & \frac{1}{(t_2-t_1)}\int_{t_1}^{t_2} X\left[\alpha t+(1-\alpha)\frac{t_1+t_2}{2},\right. \\
&\left.\frac{1}{s_2-s_1}\int_{s_1}^{s_2}\left(\beta\omega+(1-\beta)\frac{s_1+s_2}{2}\right)ds\right),\cdot\Big]dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(t_2-t_1)}\int_{t_1}^{t_2} X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right)dt \\
&\geq X\left[\left(\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}\left[\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right]dt,\frac{s_1+s_2}{2}\right),.\right] \\
&= X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right) \\
&= X((0,0),.).
\end{aligned}$$

By the convexity of H on the co-ordinates, we have:

$$\begin{aligned}
& H((\alpha,\beta),.) \\
\leq\ & \beta\frac{1}{(t_2-t_1)}\times\int_{t_1}^{t_2}\left[\frac{1}{(s_2-s_1)}\int_{s_1}^{s_2}X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2},s\right),.\right)ds\right. \\
& \left.+(1-\beta)\frac{1}{(s_2-s_1)}\times\int_{s_1}^{s_2}X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right)ds\right]dt
\end{aligned}$$

$$\begin{aligned}
\leq\ & \beta\frac{1}{(s_2-s_1)}\int_{s_1}^{s_2}\left[\alpha\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}X((t,s),.)\,ds\right. \\
& \left.+(1-\alpha)\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds\right]dt \\
& +(1-\beta)\frac{1}{(s_2-s_1)}\int_{s_1}^{s_2}\left[\alpha\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt\right. \\
& \left.+(1-\alpha)X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right)\right]ds
\end{aligned}$$

$$\begin{aligned}
=\ & \alpha\beta\frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2}X((t,s),.)\,dtds \\
& +(1-\alpha)\beta\frac{1}{(s_2-s_1)}\int_{t_1}^{t_2}X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds \\
& +\alpha(1-\beta)\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt \\
& +(1-\alpha)(1-\beta)X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right).
\end{aligned}$$

By the inequality (1), we also have:

$$X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)\leq\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}X((t,s),.)\,dt,\ s\in[s_1,s_2]$$

and

$$X\left(t,\frac{s_1+s_2}{2}\right)\leq\frac{1}{(s_2-s_1)}\int_{s_1}^{s_2}X((t,s),.)\,ds,\ t\in[t_1,t_2].$$

So, by the integration, we get that:

$$\begin{aligned}
& \frac{1}{(s_2-s_1)}\int_{s_1}^{s_2}X\left(\left(\frac{t_1+t_2}{2},s\right),.\right)ds \\
\leq\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2}X((t,s),.)\,dtds
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2} X\left(\left(t,\frac{s_1+s_2}{2}\right),.\right)dt \\
\le\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2} X\left((t,s),.\right)dtds
\end{aligned}$$

Using the above inequality, we obtain that

$$\begin{aligned}
& H((\alpha,\beta),.) \\
\le\ & [\alpha\beta+(1-\alpha)\beta+\alpha(1-\beta)+(1-\alpha)(1-\beta)] \\
& \times\frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2} X\left((t,s),.\right)dtds \\
=\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2} X\left((t,s),.\right)dtds \\
=\ & H((1,1),.)
\end{aligned}$$

and the second bound in (ii) is proved.

(iii) Firstly, we will show that

$$H((\alpha,\beta),.)\ge H((0,\beta),.),\ \text{for all } (\alpha,\beta)\in[0,1]^2. \tag{8}$$

By Jensen's inequality for integrals,

$$\begin{aligned}
& H((\alpha,\beta),.) \\
\ge\ & \frac{1}{s_2-s_1}\int_{s_1}^{s_2} X\left(\left(\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}\left[\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right]dt\right.\right. \\
& \left.\left.,\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right)ds \\
=\ & \frac{1}{s_2-s_1}\int_{s_1}^{s_2} X\left(\left(\frac{t_1+t_2}{2},\beta s+(1-\beta)\frac{s_1+s_2}{2}\right),.\right)ds \\
=\ & H((0,\beta),.),
\end{aligned}$$

for all $(\alpha,\beta)\in[0,1]^2$.

Now let $0\le\alpha_1\le\alpha_2\le 1$. By the convexity of mapping $H((\cdot,\beta),.)$ for all $\beta\in[0,1]$, we have

$$\frac{H((\alpha_2,\beta),.)-H((\alpha_1,\beta),.)}{\alpha_2-\alpha_1}\ge\frac{H((\alpha_1,\beta),.)-H((0,\beta).)}{\beta}\ge 0.$$

For the last inequality, we use (8).

Theorem 3.2 *Suppose that stochastic process* $X:\Lambda\times\Omega\to\mathbb{R}$ *is convex on* $[t_1,t_2]\times[s_1,s_2]\subset\Lambda$. *Then*

(i) The mapping H is convex on Λ;

(ii) Define the mapping $h:[0,1]\to\mathbb{R}$, $h(\alpha):=H((\alpha,\alpha),.)$. Then h is convex, monotonic nondecreasing on $[0,1]$ and one has the bounds:

$$\sup_{(\alpha,\beta)\in[0,1]^2} h(\alpha)=h(1)=\frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)\,dsdt$$

and

$$\inf_{(\alpha,\beta)\in[0,1]^2} h(\alpha)=h(0)=X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right)$$

Proof. (i) Let $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in[0,1]^2$ and $\gamma,\theta\geq 0$ with $\gamma+\theta=1$. Since $X:\Lambda\times\Omega\to\mathbb{R}$ is convex on $[t_1,t_2]\times[s_1,s_2]$, we have

$$\begin{aligned}
& H(\gamma(\alpha_1,\beta_1)+\theta(\alpha_2,\beta_2),.)\\
=\ & H((\gamma\alpha_1+\theta\alpha_2,\gamma\beta_1+\theta\beta_2),.)\\
=\ & \frac{1}{(t_2-t_1)(s_2-s_1)}\\
& \times\int_{t_1}^{t_2}\int_{s_1}^{s_2}X\left[\left(\gamma\left(\alpha_1t+(1-\alpha_1)\frac{t_1+t_2}{2},\beta_1s+(1-\beta_1)\frac{s_1+s_2}{2}\right)\right.\right.\\
& \left.\left.+\theta\left(\alpha_2t+(1-\alpha_2)\frac{t_1+t_2}{2},\beta_2s+(1-\beta_2)\frac{s_1+s_2}{2}\right)\right),.\right]dsdt\\
\leq\ & \gamma.\frac{1}{(t_2-t_1)(s_2-s_1)}\\
& \times\int_{t_1}^{t_2}\int_{s_1}^{s_2}X\left(\left(\alpha_1t+(1-\alpha_1)\frac{t_1+t_2}{2},\beta_1s+(1-\beta_1)\frac{s_1+s_2}{2}\right),.\right)dsdt\\
& +\theta.\frac{1}{(t_2-t_1)(s_2-s_1)}\\
& \times\int_{t_1}^{t_2}\int_{s_1}^{s_2}X\left(\left(\alpha_2t+(1-\alpha_2)\frac{t_1+t_2}{2},\beta_2s+(1-\beta_2)\frac{s_1+s_2}{2}\right),.\right)dsdt\\
=\ & \gamma H((\alpha_1,\beta_1),.)+\theta H((\alpha_2,\beta_2),.),
\end{aligned}$$

which shows that H is convex on $[0,1]^2$.

(ii) Let $\alpha_1,\alpha_2\in[0,1]$ and $\gamma,\theta\geq 0$ with $\gamma+\theta=1$. Then

$$\begin{aligned}
h(\gamma\alpha_1+\theta\alpha_2) &= H((\gamma\alpha_1+\theta\alpha_2,\gamma\alpha_1+\theta\alpha_2),.)\\
&= H((\gamma(\alpha_1,\alpha_1)+\theta(\alpha_2,\alpha_2)),.)\\
&\leq \gamma H((\alpha_1,\alpha_1),.)+\theta H((\alpha_2,\alpha_2),.)\\
&= \gamma h(\alpha_1)+\theta h(\alpha_2)
\end{aligned}$$

which shows the convexity of h on $[0,1]$.

We have, by the above theorem, that

$$h(\alpha) := H((\alpha,\alpha),.) \geq H((0,0),.) = X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right),\ \alpha \in [0,1]$$

and

$$\begin{aligned} h(\alpha) &= H((\alpha,\alpha),.) \leq H((1,1),.) \\ &= \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{s_1}^{s_2}\int_{t_1}^{t_2} X(t,s)\,dtds,\ \alpha \in [0,1] \end{aligned}$$

which prove the required bounds.

Now let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Then, by the conveity of h we have that

$$\frac{h(\alpha_2)-h(\alpha_1)}{\alpha_2-\alpha_1} \geq \frac{h(\alpha_1)-h(0)}{\alpha_1} \geq 0$$

and the theorem is proved.

Next, we will consider the following mapping which is closely connected with Hadamard's inequality: $\widetilde{H} : [0,1]^2 \times \Omega \to [0,\infty)$ the stochastic processes

$$\begin{aligned} &\widetilde{H}((\alpha,\beta),.) = \frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\ &\times \int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X((\alpha t+(1-\alpha)u, \beta s+(1-\beta)v),.)dtdudsdv. \end{aligned}$$

The following theorem contains the main properties of these stochastic processes.

Theorem 3.3 *Suppose that stochastic process $X : \Lambda \times \Omega \to \mathbb{R}$ is convex on the co-ordinates on $[t_1,t_2] \times [\omega_1,\omega_2] \subset \Lambda$. Then:*

(i) We have the equalities:

$$\widetilde{H}\left(\left(\alpha+\frac{1}{2},\beta\right),.\right) = \widetilde{H}\left(\left(\frac{1}{2}-\alpha,\beta\right),.\right)$$

for all $\alpha \in \left[0,\frac{1}{2}\right],\ \beta \in [0,1]$;

$$\widetilde{H}\left(\left(\alpha+\frac{1}{2},\beta\right),.\right) = \widetilde{H}\left(\left(\frac{1}{2}-\alpha,\beta\right),.\right)$$

for all $\alpha \in \left[0,\frac{1}{2}\right]$, $\beta \in [0,1]$;

$$\widetilde{H}((1-\alpha,\beta),.) = \widetilde{H}((\alpha,\beta),.)$$

and

$$\widetilde{H}((\alpha,1-\beta),.)=\widetilde{H}((\alpha,\beta),.),$$

for all $(\alpha,\beta)\in[0,1]^2$.

(ii) The stochastic process $\widetilde{H}$ is convex on the co-ordinates.

(iii) We have the bounds

$$\begin{aligned}\inf_{(\alpha,\beta)\in[0,1]^2}\widetilde{H}((\alpha,\beta),.) &= \widetilde{H}\left(\left(\frac{1}{2},\frac{1}{2}\right),.\right)=\frac{1}{(t_2-t_1)^2(s_2-s_1)^2}\\ &\times\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2}X\left(\left(\frac{t_1+t_2}{2},\frac{s_1+s_2}{2}\right),.\right)dtdudsdv\end{aligned}$$

and

$$\begin{aligned}\sup_{(\alpha,\beta)\in[0,1]^2}\widetilde{H}((\alpha,\beta),.) &= \widetilde{H}((0,0),)\\ &= \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)\,dsdt.\end{aligned}$$

(iv) The stochastic process $\widetilde{H}((\cdot,\beta),.)$ is monoton nonincreasing on $\left[0,\frac{1}{2}\right)$ and nondecreasing on $\left[\frac{1}{2},1\right]$ for all $\alpha\in[0,1]$. A similar property has the stochastic process $\widetilde{H}((\alpha,\cdot),.)$ for all $\beta\in[0,1]$.

(v) We have the inequality

$$\begin{aligned}\widetilde{H}((\alpha,\beta),.) \geq\ & \max\{H((\alpha,\beta),.),H((1-\alpha,\beta),.)\\ &,H((\alpha,1-\beta),.),H((1-\alpha,1-\beta),.)\}\end{aligned} \tag{9}$$

for all $\alpha\in[0,1]$.

Proof. (i),(ii) are obvious from definition of $\widetilde{H}$.

(iii) By the convexity of X in the first variable, we get that

$$\begin{aligned}&X\left(\left(\frac{t+u}{2},\beta s+(1-\beta)v\right),.\right)\\ =\ & X\left(\left(\frac{\alpha t+(1-\alpha)u+(1-\alpha)t+\alpha u}{2},\beta v+(1-\beta)s\right),.\right)\\ \leq\ & \frac{1}{2}[X((\alpha t+(1-\alpha)u,\beta s+(1-\beta)v),.)\\ &+X(((1-\alpha)t+\alpha u,\beta v+(1-\beta)s),.)]\end{aligned}$$

for all $(t,u)\in[t_1,t_2]^2$,$(v,s)\in[s_1,s_2]^2$ and $(\alpha,\beta)\in[0,1]^2$.

Integrating on $[t_1, t_2]^2$, we get

$$\begin{aligned} & \frac{1}{(t_2-t_1)^2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} X((\alpha t+(1-\alpha)u, \beta s+(1-\beta)v), .)dtdu \quad (10) \\ \geq\ & \frac{1}{(t_2-t_1)^2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} X\left(\left(\frac{t+u}{2}, \beta s+(1-\beta)v\right), .\right)dtdu \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{1}{(s_2-s_1)^2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X\left(\left(\frac{t+u}{2}, \beta\omega+(1-\beta)v\right), .\right)dtdu \quad (11) \\ \geq\ & \frac{1}{(s_2-s_1)^2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X\left(\left(\frac{t+u}{2}, \frac{s+v}{2}\right), .\right)dtdu. \end{aligned}$$

Now, integrating 10 inequality integrating on $[s_1, s_2]^2$ and inequality 11 on $[t_1, t_2]^2$ we deduce,

$$\begin{aligned} & \widetilde{H}((\alpha, \beta), .) \\ \geq\ & \frac{1}{(t_2-t_1)^2(s_2-s_1)^2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X\left(\left(\frac{t+u}{2}, \frac{s+v}{2}\right), .\right)dtdudsdv \end{aligned}$$

for $(\alpha, \beta) \in [0,1]^2$. Therefore, the first bound in (iii) is proved.

Since X is convex on the co-ordinates on Δ

$$\begin{aligned} & \widetilde{H}((\alpha, \beta), .) = \frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\ & \times \int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X((\alpha t+(1-\alpha)u, \beta s+(1-\beta)v), .)dtdudsdv \end{aligned}$$

$$\begin{aligned} \leq\ & \alpha\beta\frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\ & \times \int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X((t,s), .)dtdudsdv \\ & +\alpha(1-\beta)\frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\ & \times \int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2} X((t,v), .)dtdudsdv \end{aligned}$$

$$
\begin{aligned}
&+(1-\alpha)\beta\frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\
&\times\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2}X((u,s),.)dtdudsdv \\
&+(1-\alpha)(1-\beta)\frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \\
&\times\int_{t_1}^{t_2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2}X((u,v),.)dtdudsdv
\end{aligned}
$$

$$
\begin{aligned}
= &\; [\alpha\beta+\alpha(1-\beta)+(1-\alpha)\beta+(1-\alpha)(1-\beta)] \\
&\times\frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)dsdt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X((t,s),.)dsdt \\
&= \widetilde{H}((0,0),.).
\end{aligned}
$$

Therefore, the second bound in (iii) is proved.

(iv) The monotonicity of $\widetilde{H}(\alpha,\beta)$ follows by a similar argument as in the proof of theorem 2.5, and we shall omit the details.

(v) By Jensen's inequality we have successively for all $(\alpha,\beta)\in[0,1]^2$ that

$$
\begin{aligned}
&\widetilde{H}((\alpha,\beta),.) \\
\geq &\; \frac{1}{(t_2-t_1)(s_2-s_1)^2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2}X\left(\left(\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2}[\alpha t+(1-\alpha)u]\,du\right.\right. \\
&\left.\left.,\beta s+(1-\beta)v\right),.\right)dtdsdv
\end{aligned}
$$

$$
\begin{aligned}
= &\; \frac{1}{(t_2-t_1)(s_2-s_1)^2}\int_{t_1}^{t_2}\int_{s_1}^{s_2}\int_{s_1}^{s_2}X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right.\right. \\
&\left.\left.,\beta s+(1-\beta)v\right),.\right)dtd\omega dv
\end{aligned}
$$

$$
\begin{aligned}
\geq &\; \frac{1}{(t_2-t_1)(s_2-s_1)}\int_{t_1}^{t_2}\int_{s_1}^{s_2}X\left(\left(\alpha t+(1-\alpha)\frac{t_1+t_2}{2}\right.\right. \\
&\left.\left.,\beta s+(1-\beta)\frac{v_1+v_2}{2}\right),.\right)dtd\omega \\
&= H((\alpha,\beta),.).
\end{aligned}
$$

In addition, as

$$\tilde{H}((\alpha,\beta),.) = \tilde{H}((1-\alpha,\beta),.) = \tilde{H}((\alpha,1-\beta),.) = \tilde{H}((1-\alpha,1-\beta),.)$$

for all $(\alpha,\beta) \in [0,1]^2$, then by the above inequality we deduce (9).

Finally we can also state the following theorem which can be proved in a similar fashion to Theorem 3.1 and we will omit the details.

Theorem 3.4 *Suppose that stochastic process* $X : \Lambda \times \Omega \to \mathbb{R}$ *is convex on the co-ordinates on* $[t_1,t_2] \times [s_1,s_2] \subset \Lambda$. *Then we have:*

(i) The mapping $\tilde{H}$ is convex on Δ.

(ii) Define the mapping $\tilde{h} : [0,1] \to \mathbb{R}$, $\tilde{h}(t) := \tilde{H}((t,t),.)$. Then $\tilde{h}$ is convex, monotonic nonincreasing on $\left[0,\frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2},1\right]$ and one has the bounds:

$$\sup_{\alpha,\beta\in[0,1]} \tilde{h}(\alpha) = \tilde{h}(1) = \tilde{h}(0) = \frac{1}{(t_2-t_1)(s_2-s_1)} \int_{t_1}^{t_2} \int_{s_1}^{s_2} X((t,s),.)\,dsdt$$

and

$$\inf_{\alpha,\beta\in[0,1]} \tilde{h}(\alpha,\beta) = \tilde{h}(\frac{1}{2})$$

$$= \frac{1}{(t_2-t_1)^2(s_2-s_1)^2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{s_1}^{s_2} X\left(\left(\frac{t_1+t_2}{2}, \frac{s_1+s_2}{2}\right),.\right) dtdudsdv.$$

(iii) One has the inequality

$$\tilde{h}(\alpha) \geq \max\left\{\tilde{h}(\alpha), \tilde{h}(1-\alpha)\right\} \text{ for all } \alpha \in [0,1].$$

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