

# A few remarks on values of Hurwitz Zeta function at natural and rational arguments

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## Abstract

We exploit some properties of the Hurwitz zeta function  $\zeta(n, x)$  in order to study sums of the form  $\frac{1}{\pi^n} \sum_{j=-\infty}^{\infty} 1/(jk + l)^n$  and  $\frac{1}{\pi^n} \sum_{j=-\infty}^{\infty} (-1)^j/(jk + l)^n$  for  $2 \leq n, k \in \mathbb{N}$ , and integer  $l \leq k/2$ . We show that these sums are algebraic numbers. We also show that  $1 < n \in \mathbb{N}$  and  $p \in \mathbb{Q} \cap (0, 1)$ : the numbers  $(\zeta(n, p) + (-1)^n \zeta(n, 1 - p))/\pi^n$  are algebraic. On the way we find polynomials  $s_m$  and  $c_m$  of order respectively  $2m + 1$  and  $2m + 2$  such that their  $n$ -th coefficients of sine and cosine Fourier transforms are equal to  $(-1)^n/n^{2m+1}$  and  $(-1)^n/n^{2m+2}$  respectively.

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## 1 Introduction

Firstly we find polynomials  $s_m$  and  $c_m$  of orders respectively  $2m + 1$  and  $2m + 2$  such that their  $n$ -th coefficients in respectively sine and cosine Fourier series are of the form  $(-1)^n/n^{2m+1}$  and  $(-1)^n/n^{2m+2}$ . Secondly using these polynomials we study sums of the form:

$$S(n, k, l) = \sum_{j=-\infty}^{\infty} \frac{1}{(jk + l)^n}, \quad \hat{S}(n, k, l) = \sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(jk + l)^n}.$$

for  $n, k, l \in \mathbb{N}$  such that  $n \geq 2, k \geq 2$  generalizing known result of Euler (recently proved differently by Elkies ([6])) stating that the number  $S(n, 4, 1)/\pi^n$

is rational. In particular we show that the numbers  $S(n, k, j)/\pi^n$  and  $\hat{S}(n, k, j)/\pi^n$  for  $\mathbb{N} \ni n, k \geq 2$ , and  $1 \leq j < k$  are algebraic.

The results of the paper are somewhat in the spirit of [5] in the sense that the sums we study are equal to certain combinations of values of the Hurwitz zeta function calculated for natural and rational arguments.

The paper is organized as follows. In the present section we recall basic definitions and a few elementary properties of the Hurwitz and Riemann zeta functions. We present them for completeness of the paper and also to introduce in this way our notation.

In Section 2 are our main results. Longer proof are shifted to Section 3.

To fix notation let us recall that the following function

$$\zeta(s, x) = \sum_{n \geq 0} 1/(n + x)^s,$$

defined for all  $\text{Re}(s) > 1$  and  $1 \geq \text{Re}(x) > 0$  is called Hurwitz zeta function while the function  $\zeta(s, 1) \stackrel{df}{=} \zeta(s)$  is called the Riemann zeta function. One shows that functions  $\zeta(s, x)$  and  $\zeta(s)$  can be extended to meromorphic functions of  $s$  defined for all  $s \neq 1$ . Hurwitz zeta function has been studied from different points of view in e.g. [7], [12], [4].

Let us consider slight generalization of the Hurwitz zeta function namely the function;

$$\hat{\zeta}(s, x) = \sum_{n \geq 0} (-1)^n / (n + x)^s,$$

for the same as before  $s$  and  $x$ .

Using multiplication theorem for  $\zeta(s, x)$  and little algebra we get:

$$\zeta(s, x) = (\zeta(s, x/2) + \zeta(s, (1+x)/2)) / 2^s, \quad \hat{\zeta}(s, x) = (\zeta(s, x/2) - \zeta(s, (1+x)/2)) / 2^s, \tag{1}$$

$$S(n, k, l) = \frac{1}{k^n} (\zeta(n, \frac{l}{k}) + (-1)^n \zeta(n, 1 - \frac{l}{k})), \tag{2}$$

$$\hat{S}(n, k, l) = \frac{1}{k^n} (\hat{\zeta}(n, \frac{l}{k}) + (-1)^{n+1} \hat{\zeta}(n, 1 - \frac{l}{k})),$$

$$S(n, k, l) = S(n, 2k, l) + (-1)^n S(n, 2k, k - l), \tag{3}$$

$$\hat{S}(n, k, l) = S(n, 2k, l) + (-1)^{n+1} S(n, 2k, k - l), \tag{4}$$

$$S(n, k, k - l) = (-1)^n S(n, k, l), \quad \hat{S}(n, k, k - l) = (-1)^{n+1} \hat{S}(n, k, l). \tag{5}$$

Hence there is sense to define sums  $S$  and  $\hat{S}$  for  $l \leq k/2$  only.

In the sequel  $\mathbb{N}$  and  $\mathbb{N}_0$  will denote respectively sets of natural and natural plus  $\{0\}$  numbers while  $\mathbb{Z}$  and  $\mathbb{Q}$  will denote respectively sets of integer and

rational numbers.  $B_k$  will denote  $k - th$  Bernoulli number. It is known that  $B_{2n+1} = 0$  for  $n \geq 1$  and  $B_1 = 1/2$ . One knows also that

$$\zeta(2l) = (-1)^{l+1} B_{2l} \frac{(2\pi)^{2l}}{2(2l)!} \tag{6}$$

for all  $l \in \mathbb{N}$ .

For more properties of the Hurwitz zeta function and the Bernoulli numbers see [9], [3], [11] and [2].

Having this and (3) and applying multiplication theorem for  $\zeta(s, x)$  we can easily generalize result of Euler reminded by Elkies in [6]). Let us recall that since Euler times it is known that.

$$\frac{1}{\pi^n} S(n, 4, 1) = \begin{cases} \frac{(2^{2l}-1)}{2(2l)!} (-1)^{l+1} B_{2l} & \text{if } n = 2l, \\ \frac{(-1)^l E_{2l}}{2^{2l+2}(2l)!} & \text{if } n = 2l + 1. \end{cases}$$

Apart from this result one can easily show (manipulating multiplication theorem for Hurwitz zeta function) that for  $n \in \mathbb{N}$ :

$$\begin{aligned} \frac{(2l)!}{\pi^{2l}} S(2l, 2, 1) &= \frac{2(2l)!}{(2\pi)^{2l}} \zeta(2l, \frac{1}{2}) = (2^{2l} - 1)(-1)^{l+1} B_{2l}, \\ \frac{2(2l)!}{\pi^{2l}} S(2l, 3, 1) &= (\frac{2}{3})^{2l} (-1)^{l+1} (3^{2l} - 1) B_{2l}, \\ \frac{2(2l)!}{\pi^{2l}} S(2l, 6, 1) &= (2^{2l} - 1)(1 - 3^{-2l})(-1)^{l+1} B_{2l}. \end{aligned}$$

Notice also that from (1) it follows that  $S(2l + 1, 2, 1) = 0$  for  $l \in \mathbb{N}$  and from (1) and (2) we get:  $\hat{S}(2l, 2, 1) = 0$  and  $\hat{S}(2l + 1, 2, 1) = 2S(2l + 1, 4, 1)$ . Results concerning  $S(2l, 6, 1)$  were also obtained in [5](16b).

## 2 Main results

To get relationships between particular values of  $\zeta(n, q)$  for  $n \in \mathbb{N}$  and  $q \in \mathbb{Q}$  we will use some elementary properties of the Fourier transforms.

Let us denote

$$\mathcal{F}_n^s(f(\cdot)) = \mathcal{F}_n^s = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \quad \mathcal{F}_n^c(f(\cdot)) = \mathcal{F}_n^c = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

for  $n \in \mathbb{N}_0$ . We will abbreviate  $\mathcal{F}_n^s(f(\cdot))$  and  $\mathcal{F}_n^c(f(\cdot))$  to  $\mathcal{F}_n^s$  and  $\mathcal{F}_n^c$  if the function  $f$  is given. We have:

$$f(x) = \frac{1}{2} \mathcal{F}_0^c + \sum_{n \geq 1} \mathcal{F}_n^c \cos(nx) + \sum_{n \geq 1} \mathcal{F}_n^s \sin(nx)$$

for all points of continuity of  $f$  and  $(f(x^+) + f(x^-))/2$  if at  $x$   $f$  has jump discontinuity as a result of well known properties of Fourier series.

Let us define two families of polynomials:

$$c_m(x) = \frac{x^2(-1)^{m+1}}{(2m+2)!} \sum_{j=0}^m \binom{2m+2}{2j} x^{2(m-j)} \pi^{2j} B_{2j}(2^{2j-1} - 1), \quad (7)$$

$$s_m(x) = \frac{x(-1)^m}{(2m+1)!} \sum_{j=0}^m \binom{2m+1}{2j} x^{2(m-j)} \pi^{2j} B_{2j}(2^{2j-1} - 1), \quad (8)$$

for all  $m \in \mathbb{N}_0$ . Let us consider also the following polynomials:

$$ch_m(x) = (-1)^m \frac{2(2\pi)^{2m+1}}{(2m+2)!} B_{2m+2}\left(\frac{x}{2\pi}\right),$$

$$sh_m(x) = (-1)^{m-1} \frac{2(2\pi)^{2m}}{(2m+1)!} B_{2m+1}\left(\frac{x}{2\pi}\right).$$

Basing partially on [8] we have the following auxiliary lemma.

**Lemma 1** For  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$

$$\mathcal{F}_n^s(s_m(\cdot)) = \frac{(-1)^n}{n^{2m+1}}, \quad \mathcal{F}_n^c(c_m(\cdot)) = \frac{(-1)^n}{n^{2m+2}}, \quad (9)$$

$$\mathcal{F}_n^s(sh_m(\cdot)) = \frac{1}{n^{2m+1}}, \quad \mathcal{F}_n^c(ch_m(\cdot)) = \frac{1}{n^{2m+2}}, \quad (10)$$

$$\mathcal{F}_0^c(c_m(\cdot)) = \pi^{2m+2} (-1)^m B_{2m+2} \frac{2(2^{2m+1} - 1)}{(2m+2)!} = 2\zeta(2m+2)(1 - 2^{-2m-1}),$$

$$\mathcal{F}_0^c(ch_m(\cdot)) = 0. \quad (11)$$

**Proof.** Is shifted to Section 3. ■

**Remark 1** Notice that we have just summed the two following Dirichlet series:  $DC(s, x) = \sum_{j \geq 1} (-1)^j \frac{\cos(jx)}{j^s}$  and  $DS(s, x) = \sum_{j \geq 1} (-1)^j \frac{\sin(jx)}{j^s}$ , for particular values of  $s$ . Namely we have showed that

$$DC(2m+2, x) = c_m(x) + \pi^{2m+2} (-1)^{m+1} B_{2m+2} \frac{(2^{2m+1} - 1)}{(2m+2)!},$$

$$DS(2m+3, x) = s_{m+1}(x)$$

and for  $|x| \leq \pi$ ,  $m \in \mathbb{N}_0$ . What are the values of these series for  $s \neq 1, 2, 3, \dots$ ?

**Remark 2** Notice that  $c_m(\frac{\pi l}{k})/\pi^{2m+2}$  and  $s_m(\frac{\pi l}{k})/\pi^{2m+1}$  for all  $k \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$  and  $l \leq k$ , are rational numbers.

As a consequence we get the following theorem that allows calculating values of  $S(n, k, l)$  and  $\hat{S}(n, k, l)$  for  $\mathbb{N} \ni n, k \geq 2$  and  $l \leq k$ .

**Theorem 2** For  $m \in \mathbb{N}_0, i \in \mathbb{N}$ ,

i)

$$c_m\left(\frac{(2l-1)\pi}{2i+1}\right) = \frac{(-1)^m B_{2m+2} \pi^{2m+2}}{(2m+2)!} \left(2^{2m+1} \left(1 + \frac{1}{(2i+1)^{2m+2}}\right) - 1\right) \quad (12)$$

$$+ \sum_{j=1}^i (-1)^j \cos\left(\frac{j(2l-1)\pi}{2i+1}\right) S(2m+2, 2i+1, j),$$

for  $l = 1, \dots, i-1$ .

ii)

$$c_m\left(\frac{(2l-1)\pi}{2i}\right) = \frac{\pi^{2m+2} (-1)^m B_{2m+2} (2^{2m+1} - 1)}{(2m+2)!} (1 - (2i)^{-2m-2}) \quad (13)$$

$$+ \sum_{j=1}^{i-1} (-1)^j \cos\left(\frac{(2l-1)j\pi}{2i}\right) \hat{S}(2m+2, 2i, j),$$

for  $l = 1, \dots, i$ .

iii)

$$c_m\left(\frac{2l\pi}{2i+1}\right) = \frac{\pi^{2m+2} (-1)^m B_{2m+2} (2^{2m+1} - 1)}{(2m+2)!} (1 - (2i+1)^{-2m-2}) \quad (14)$$

$$+ \sum_{j=1}^i (-1)^j \cos\left(\frac{2lj\pi}{2i+1}\right) \hat{S}(2m+2, 2i+1, j),$$

for  $l = 1, \dots, i$ .

iv) For  $m, k \in \mathbb{N}$ :

$$s_m\left(\frac{(2l-1)\pi}{2i+1}\right) = \sum_{j=1}^i (-1)^j \sin\left(\frac{(2l-1)j\pi}{2i+1}\right) S(2m+1, 2i+1, j), \quad (15)$$

for  $l = 1, \dots, i-1$ .

v)

$$s_m\left(\frac{(2l-1)\pi}{2i}\right) = (-1)^{i+l+1} \frac{1}{i^{2m+1}} S(2m+1, 4, 1) \quad (16)$$

$$+ \sum_{j=1}^{i-1} (-1)^j \sin\left(\frac{(2l-1)j\pi}{2i}\right) \hat{S}(2m+1, 2i, j),$$

for  $l = 1, \dots, i$ .

vi)

$$s_m\left(\frac{2l\pi}{2i+1}\right) = \sum_{j=1}^i (-1)^j \sin\left(\frac{2lj\pi}{2i+1}\right) \hat{S}(2m+1, 2i+1, j), \quad (17)$$

for  $l = 1, \dots, i$ .

**Proof.** Is shifted to Section 3. ■

As a simple corollary we get some particular values  $S(n, k, l)$

**Corollary 3** i) For  $m = 0, 1, 2, \dots$ , such that  $2m + 1 > 1$ :

$$\frac{1}{\pi^{2m+2}} \hat{S}(2m+2, 4, 1) = -\frac{\sqrt{2}}{\pi^{2m+2}} (c_m(\pi/4) - \zeta(2m+2)(1-2^{-2m-1})(1-4^{-2m-2})),$$

$$\frac{1}{\pi^{2m+1}} \hat{S}(2m+1, 4, 1) = -\sqrt{2} \left( \frac{1}{\pi^{2m+1}} s_m(\pi/4) - (-1)^m E_{2m} / (2^{4m+3} (2m)! \right),$$

in particular  $\hat{S}(2, 4, 1)/\pi^2 = \sqrt{2}/16$ ,  $\hat{S}(3, 4, 1)/\pi^3 = 3\sqrt{2}/128$

ii)

$$\frac{2}{\pi^{2m+2}} S(2m+2, 8, 1) = -\frac{\sqrt{2}}{\pi^{2m+2}} (c_m(\pi/4) - \zeta(2m+2)(1-2^{-2m-1})(1-4^{-2m-2}))$$

$$+ \frac{1}{\pi^{2m+2}} \zeta(2m+2)(1-2^{-2m-2}),$$

$$\frac{2}{\pi^{2m+2}} S(2m+2, 8, 3) = \frac{\sqrt{2}}{\pi^{2m+2}} (c_m(\pi/4) - \zeta(2m+2)(1-2^{-2m-1})(1-4^{-2m-2}))$$

$$+ \frac{1}{\pi^{2m+2}} \zeta(2m+2)(1-2^{-2m-2}).$$

In particular  $S(2, 8, 1) = \pi^2(1 + \sqrt{2}/2)/16$ ,  $S(2, 8, 3) = \pi^2(1 - \sqrt{2}/2)/16$ ,  
 $S(4, 8, 1) = \pi^4(1 + 11\sqrt{2}/16)/192$ ,  $S(4, 8, 3) = \pi^4(1 - 11\sqrt{2}/16)/192$ .

iii)  $S(3, 8, 1) = \pi^3(1 - 3\sqrt{2}/4)/32$ ,  $S(3, 8, 3) = \pi^3(1 + 3\sqrt{2}/4)/32$ ,  $S(5, 8, 1) = 5\pi^5(1 - 57\sqrt{2}/16)/1536$ ,  $S(5, 8, 3) = 5\pi^5(1 + 57\sqrt{2}/16)/1536$ .

iv) For  $m \in \mathbb{N}_0$

$$\sum_{j=0}^m \binom{2m+2}{2j} B_{2j} (2^{2j-1} - 1) / 9^{(m-j)+1} = -B_{2m+2} (4^m - 1 + 4^m / 3^{2m+1}),$$

$$\sum_{j=0}^m \binom{2m+2}{2j} B_{2j} (2^{2j-1} - 1) 2^{2j} = -B_{2m+2} (2^{2m+1} - 1) (2^{2m+2} - 1).$$

v) For  $m \in \mathbb{N}$  :

$$\frac{2(2m + 1)!}{\pi^{2m+1}} S(2m+1, 3, 1) = (-1)^{m+1} \frac{4\sqrt{3}}{3} \sum_{j=0}^m \binom{2m + 1}{2j + 1} B_{2(m-j)} (2^{2(m-j)-1} - 1) / 3^{2j+1}. \tag{18}$$

**Proof.** i) We apply Theorem 2, (13) and (16) with  $i = 2$ . ii) We use (3) and (4). iii) We argue in a similar manner. iv) We set  $i = 1$  and  $l = 1$  in (12) and use the fact that  $S(2m + 2, 3, 1) = \frac{\pi^{2m+2}}{2(2m+2)!} (\frac{2}{3})^{2m+2} (-1)^{m+1} (3^{2m+2} - 1) B_{2m+2}$ . Then we set  $i = 1$  and  $l = 1$  in (13) and then cancel out  $\pi^{2m+2} / (2m + 2)!$ . v) We set  $i = 1$ , and  $l = 1$  in (15) . ■

Based on the above mentioned theorem we can deduce our main result:

**Theorem 4** For every  $\mathbb{N} \ni k, n \geq 2$ , and  $1 \leq l \leq k$  the numbers  $S(n, k, l) / \pi^n$  and  $\hat{S}(n, k, l) / \pi^n$  are algebraic.

**Proof.** First of all recall that in the Introduction we have found  $S(n, 2, 1)$  and  $\hat{S}(n, 2, 1)$  (compare equations (16) and (13)). Further analyzing equations (12)-(17) we notice that quantities  $S(2m + 2, 2i + 1, j)$ ,  $\hat{S}(2m + 2, 2i, j)$ ,  $\hat{S}(2m + 2, 2i + 1, j)$ ,  $S(2m + 1, 2i + 1, j)$ ,  $\hat{S}(2m + 1, 2i, j)$  satisfy certain systems of linear equations with matrices  $[(-1)^j \cos \frac{(2l-1)j\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}$ ,  $[(-1)^j \sin \frac{(2l-1)j\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}$  and  $[(-1)^j \cos \frac{2lj\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}$ ,  $[(-1)^j \sin \frac{2lj\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}$ . These matrices are non-singular since their determinants differ from the determinants of matrices  $C_n, S_n, C_n^*$  and  $S_n^*$  (defined by (21) and (22), below) only possibly in sign. As shown in Lemma 6 matrices  $C_n, S_n, C_n^*$  and  $S_n^*$  are nonsingular. Moreover notice that all entries of these matrices are algebraic numbers. On the other hand as one can see the numbers  $c_m(l\pi/k) / \pi^{2m+2}$  and  $s_m(l\pi/k) / \pi^{2m+1}$  for all  $\mathbb{N} \ni m, k, l < \lfloor k/2 \rfloor$  are rational. Hence solutions of these equations are algebraic numbers. Now taking into account that for  $k = 2$  we do not get in fact any equation for  $\hat{S}(n, 2, 1)$  . Thus we deduce that for  $n, k, l \in \mathbb{N}$  such that  $n \geq 2, k > 2, l \leq k - 1$  the numbers  $\hat{S}(n, l, k) / \pi^n$  and for  $n \geq 2$ , odd  $k$  and  $l \leq k - 1$  the numbers  $S(n, k, l) / \pi^n$  are algebraic.

To show that numbers  $S(n, l, k) / \pi^n$  are algebraic also for even  $l$  we refer to (3) and (4). First taking  $l$  odd we see that  $S(n, 2l, k) / \pi^n$  for all  $k \leq l$  (the case  $k = l$  leads to  $S(n, 2, 1)$  considered by the end of Introduction) these numbers are algebraic. Then taking  $l = 2m$  with  $m$  even we deduce in the similar way that  $S(n, 4m, k) / \pi^n, k \leq 2m$  are algebraic. And so on by induction we deduce that indeed for all even  $l$  and  $k \leq l/2$  numbers  $S(n, l, k)$  are algebraic. ■

As an immediate corollary of Theorem 4 and (3) and (4) we have the following result:

**Proposition 5** For  $1 < n \in \mathbb{N}$  and  $p \in \mathbb{Q} \cap (0, 1)$  :  
 the numbers  $(\zeta(n, p) + (-1)^n \zeta(n, 1-p))/\pi^n$  and  $(\hat{\zeta}(n, p) + (-1)^{n-1} \hat{\zeta}(n, 1-p))/\pi^n$   
 are algebraic.

### 3 Proofs

**Proof.** [Proof of Lemma 1](10) and (11) were basically proved in [8]. Only small modification connected with the change of variables was needed. We will need the following observations: For all  $m \geq 0, n \geq 1$ :

$$\mathcal{F}_n^s(x^{2m+1}) = (-1)^{n+1} \frac{2(2m+1)!}{n^{2m+1}} \sum_{k=0}^m (-1)^k \pi^{2(m-k)} \frac{n^{2(m-k)}}{(2m+1-2k)!}, \tag{19}$$

$$\mathcal{F}_n^c(x^{2m}) = (-1)^n \frac{2(2m)!}{n^{2m}} \sum_{k=0}^{m-1} (-1)^k \pi^{2(m-1-k)} \frac{n^{2(m-1-k)}}{(2m-1-2k)!}. \tag{20}$$

Let us denote:  $b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2m+1} \sin(nx) dx$  and  $a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2m} \cos(nx) dx$ . Integrating by parts twice we obtain the following recursions

$$b_m = 2(-1)^{n+1} \frac{\pi^{2m-1}}{n} - \frac{(2m+1)2m}{n^2} b_{m-1}, \quad a_m = (-1)^n \frac{4m}{n^2} \pi^{2m-1} + \frac{2m(2m-1)}{n^2} a_{m-1}$$

with  $b_0 = (-1)^{n+1} \frac{2}{n}$ , and with  $a_0 = 0$  for  $m \geq 1$ . Iterating them we get (19) and (20).

We show (7) and (8) by straightforward computation. First let us consider (7). Making use of (20) and after some algebra including changing the order of summation and setting  $l = m - k - j$  we get:

$$\mathcal{F}_n^c(c_m(\cdot)) = (-1)^{m+n+1} \sum_{k=0}^m \frac{(-1)^k n^{-2(k+1)}}{(2m-2k+1)!} \pi^{2(m-k)} \sum_{l=0}^{m-k} B_{2l}(2^{2l-1}-1) \binom{2m-2k+1}{2l}$$

Now we use [15](12) with  $x = 0$  and [15](4) to show that

$$\sum_{l=0}^{m-k} \binom{2m-2k+1}{2l} B_{2l}(2^{2l-1}-1) = \begin{cases} 0 & \text{if } k < m \\ -1/2 & \text{if } k = m \end{cases}$$

Hence indeed  $\mathcal{F}_n^c(c_m(\cdot)) = \frac{(-1)^n}{n^{2m+2}}$ . To get (8) we proceed similarly and after some algebra we get

$$\begin{aligned} \mathcal{F}_n^s(s_m(\cdot)) &= (-1)^{m+n} \sum_{k=0}^m \frac{(-1)^k \pi^{2(m-k)}}{n^{2k+1} (2m-2k+1)!} \sum_{l=0}^{m-k} \binom{2m-2k+1}{2l} B_{2l}(2^{2j-1}-1) \\ &= \frac{(-1)^{n+1}}{n^{2m+1}}. \end{aligned}$$



To get  $\frac{1}{\pi} \int_{-\pi}^{\pi} c_m(x) dx$  we proceed as follows:

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} c_m(x) dx &= \frac{2(-1)^{m+1} \pi^{2m+2}}{(2m+2)!} \sum_{j=0}^m \binom{2m+2}{2(m-j)} \frac{1}{(2j+3)} B_{2(m-j)} (2^{2(m-j)-1} - 1) \\ &= \frac{2(-1)^{m+1} \pi^{2m+2}}{(2m+3)!} \sum_{k=0}^m \binom{2m+3}{2k} B_{2k} (2^{2k-1} - 1). \end{aligned}$$

Now we again manipulate [15] (4) and (12) to get the desired formula. ■

The proof of Theorem 4 is based on the Lemma concerning the following matrices

$$C_n = [\cos \frac{(2l-1)j\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}, \quad S_n = [\sin \frac{(2l-1)j\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}, \tag{21}$$

$$C_n^* = [\cos \frac{2lj\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}, \quad S_n^* = [\sin \frac{2lj\pi}{n}]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}. \tag{22}$$

**Lemma 6** For every  $\mathbb{N} \ni n \geq 2$  matrices  $C_n, S_n, C_n^*, S_n^*$  are nonsingular.

**Proof.** The proof is based on the following observations. Let  $i$  denote unitary unit. Since  $\sin x = (\exp(ix) - \exp(-ix))/2i$  one can easily notice that matrices  $S_n$  and  $S_n^*$  are of the type  $[x_l^j - x_l^j]_{l,j=1,\dots, \lfloor (n-1)/2 \rfloor}$  where  $x_l$  equals respectively  $\exp(\frac{(2l-1)i\pi}{n})$  and  $\exp(\frac{2li\pi}{n})$  ( $n$  is odd in this case). Now we apply formula (2.3) of [10] and deduce that these matrices are nonsingular provided that quantities  $x_l$  are nonzero, are all different, and  $x_l x_k \neq 1, k, l = 1, \dots, \lfloor (n-1)/2 \rfloor$  which is the case in both situations. Now notice that  $\forall n \geq 0 : \cos(\alpha) = (-1)^{n-1} \sin((2n-1)\pi/2 + \alpha)$ . Hence  $\cos \frac{(2l-1)j\pi}{n} = (-1)^{l-1} \sin((2l-1)\pi/2 + \frac{(2l-1)j\pi}{n}) = (-1)^{l-1} \sin(\frac{\pi(j-1/2)(2l-1)}{n}) = (-1)^{l-1} (x_l^{(j-1/2)} - x_l^{-(j-1/2)})/(2i)$  and we use formula (2.4) of [10] to deduce that matrix  $C_n$  is nonsingular. Now let us consider matrix  $C_n^*$ . Notice that then  $n = 2m + 1$  for some natural  $m$ .  $\cos(\frac{2lj\pi}{2m+1}) = (-1)^j \cos(j\pi - \frac{2lj\pi}{2m+1}) = (-1)^j \cos(\frac{j\pi(2m-l+1)-1}{2m+1})$ . And thus we have reduced this case to the previous one. Hence  $C_n^*$  is also nonsingular. ■

**Proof.** [Proof of Theorem 2] In the proofs of all assertions we use basically the same tricks concerning properties of trigonometric functions and changing order of summation. That is why we will present proof of only one (out of 6) in detail. All remaining proofs are similar. We start with an obvious identity:

$$c_m(\frac{\pi l}{k}) = \pi^{2m+2} (-1)^m B_{2m+2} \frac{(2^{2m+1} - 1)}{(2m+2)!} + \sum_{j=1}^k \cos(\frac{j l \pi}{k}) (-1)^j \sum_{r=0}^{\infty} \frac{(-1)^{r(k+l)}}{(rk+j)^{2m+2}} \tag{23}$$

that is true since by  $\cos(n\pi + \alpha) = (-1)^n \cos \alpha$  we have:

$$\sum_{n \geq 1}^{\infty} (-1)^n \cos\left(\frac{n\pi l}{k}\right) / n^{2m+2} = \sum_{j=1}^k \sum_{r=0}^{\infty} (-1)^{r(k+l)+j} \cos\left(\frac{j\pi l}{k}\right) / (rk + j)^{2m+2},$$

Hence we have to consider two cases  $k + l$  odd and  $k + l$  even.

If  $k + l$  is even then we have

$$\sum_{n \geq 1}^{\infty} (-1)^n \cos\left(\frac{n\pi l}{k}\right) / n^{2m+2} = \frac{1}{k^{2m+2}} \sum_{j=1}^k (-1)^j \cos\left(\frac{\pi j l}{k}\right) \zeta(2m + 2, j/k).$$

For  $k = 2i + 1$  and  $l$  odd we have:  $\sum_{j=1}^{2i+1} (-1)^j \cos(j\pi/(2i + 1)) \zeta(2m + 2, j/(2i + 1))$ . Now recall that for  $j = 2i + 1$  we get  $\cos(\pi) = -1$  and that for  $j = 1, \dots, i$  we have  $\cos((2i + 1 - j)l\pi/(2i + 1)) = -\cos(jl\pi/(2i + 1))$  and  $(-1)^{2i+1-j} = -(-1)^j$ . Thus we get

$$\begin{aligned} & \sum_{j=1}^{2i+1} (-1)^j \cos\left(\frac{jl\pi}{2i+1}\right) \zeta\left(2m+2, \frac{j}{2i+1}\right) = \zeta(2m+2) \\ & + \sum_{j=1}^i (-1)^j \cos\left(\frac{jl\pi}{2i+1}\right) \left(\zeta\left(2m+2, \frac{j}{2i+1}\right) + \zeta\left(2m+2, 1 - \frac{j}{2i+1}\right)\right). \end{aligned}$$

Now we use (3). Since  $k + l$  even and  $k$  even leads to triviality we consider two cases leading to  $k + l$  odd i.e.  $k = 2i$  and  $l = 2m - 1$  odd  $m < i$  which leads to:

$$\begin{aligned} & \sum_{j=1}^{2i} \cos\left(\frac{j(2m-1)\pi}{2i}\right) (-1)^j \sum_{r=0}^{\infty} (-1)^{r(2i+2m+1)} / (r(2i+1) + j)^{2m+2} \\ & = (-1)^m B_{2m+2} \frac{\pi^{2m+2}}{(2m+2)!} \frac{2^{2m+1} - 1}{2^{2m+2} i^{2m+2}} \\ & + \sum_{j=1}^{i-1} (-1)^j \cos\left(\frac{jl\pi}{2i}\right) (S(2m+2, 4i, j) - S(2m+2, 4i, 2i-j)). \end{aligned}$$

We used here (6) (3) and (23).

Similarly we consider the case  $k = 2i + 1$  and  $l = 2m$  which leads to (14). Note that this case is very similar to the case  $k = 2i + 1$   $l = 2m - 1$ .

Assertions iv)-vi) are proved in the similar way. The only difference is that we do not have to remember about  $\frac{1}{2}\mathcal{F}_0^c$  and that we utilize properties of sin function not of cos. ■

## References

- [1] Apostol, T. M. *Some series involving the Riemann zeta function*. Proc. Amer. Math. Soc. **5**, (1954). 239–243. MR0060534 (15,685c)
- [2] Carlitz, L. *Bernoulli numbers*. Fibonacci Quart **6** 1968 no. 3 71–85. MR0232748 (38 #1071)
- [3] Chandrasekharan, K., *Lectures on The Riemann Zeta-Function*, Tata Institute of Fundamental Research, Bombay, 1953
- [4] Coffey, Mark W. An efficient algorithm for the Hurwitz zeta and related functions. *J. Comput. Appl. Math.* **225** (2009), no. 2, 338–346. MR2494704 (2010d:11147)
- [5] Cvijović, Djurdje; Klinowski, Jacek. *Values of the Legendre chi and Hurwitz zeta functions at rational arguments*. Math. Comp. **68** (1999), no. 228, 1623–1630. MR1648375 (2000c:11144)
- [6] Elkies, Noam D. *On the sums  $\sum_{k=-\infty}^{\infty} (4k+1) x^{-n}$* . Amer. Math. Monthly **110** (2003), no. 7, 561–573. MR2001148 (2004f:11152), arXiv:math.CA/0101168
- [7] Hessami Pilehrood, Kh.; Hessami Pilehrood, T. *Bivariate identities for values of the Hurwitz zeta function and supercongruences*. Electron. J. Combin. **18** (2011), no. 2, Paper 35, 30 pp. MR2900448 (Reviewed)
- [8] Navas, Luis M.; Ruiz, Francisco J.; Varona, Juan L. *Old and new identities for Bernoulli polynomials via Fourier series*. Int. J. Math. Math. Sci. 2012, Art. ID 129126, 14 pp. MR2947312 (Reviewed)
- [9] Karatsuba, A. A.; Voronin, S. M. *The Riemann zeta-function*. Translated from the Russian by Neal Koblitz. de Gruyter Expositions in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1992. xii+396 pp. ISBN: 3-11-013170-6 MR1183467 (93h:11096)
- [10] Krattenthaler, C. *Advanced determinant calculus*. The Andrews Festschrift (Maratea, 1998). Sem. Lothar. Combin. 42 (1999), Art. B42q, 67 pp. (electronic). MR1701596 (2002i:05013)
- [11] Magnus, Wilhelm; Oberhettinger, Fritz; Soni, Raj Pal. *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52 Springer-Verlag New York, Inc., New York 1966 viii+508 pp. MR0232968 (38 #1291)

- [12] Miller, Jeff; Adamchik, Victor S. *Derivatives of the Hurwitz zeta function for rational arguments*. J. Comput. Appl. Math. **100** (1998), no. 2, 201–206. MR1659109 (2000g:11085)
- [13] Ramaswami, V. *Notes on Riemann's {zeta}-Function*. J. London Math. Soc. **S1-9** no. 3, 165. MR1574175
- [14] Srivastava, H. M. *Sums of certain series of the Riemann zeta function*. J. Math. Anal. Appl. (1988), no. 1, 129–140. MR0958860 (89i:11009)
- [15] Paweł J. Szablowski, *A few remarks on Euler and Bernoulli polynomials and their connections with binomial coefficients and modified Pascal matrices*, Mathem Aeterna, **4**(1)(2014), 83-88, <http://arxiv.org/abs/1307.0300>

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