Mathematica Aeterna, Vol. 5, 2015, no. 2, 257 - 261

HYERS-ULAM STABILITY OF FREDHOLM INTEGRAL EQUATION

Zhaohui Gu

School of Economics & Trade,
Guangdong University of Foreign Studies,
Guangzhou, 510006, P. R. China
E-mail : zhgugz@163.com

Jinghao Huang

Department of Mathematics,
Sun Yat-Sen University,
Guangzhou, 510275 P. R. China
E-mail : hjinghao@mail2.sysu.edu.cn

Abstract

We prove the Hyers-Ulam stability of Fredholm integral equation. That is, if x is an approximate solution of $x(t) = f(t)+\lambda \int_a^b K(t,s)x(s)ds$, then there exists an exact solution of the differential equation near to x.

Mathematics Subject Classification: 34K20; 26D10.

Keywords: Hyers-Ulam stability, integral equation, Fredholm equation.

1 Introduction

S.M. Ulam[14] gave a wide-ranging talk about a series of important unsolved problems in 1940. The question concerning the stability of group homomorphisms is one of them. A year later, D.H. Hyers[1] proved the stability for the case of approximately additive mappings under the assumption that G_1 and G_2 are Banach spaces. After then, the Hyers-Ulam stability of function equation(see [10, 11, 12, 2]) and differential function(see[8, 9, 13, 5, 6, 7, 3, 4]) was investigated by several mathematicians.

In this paper, we will investigate the Hyers-Ulam stability of integral equation:

$$x(t) = f(t) + \lambda \int_a^b K(t,s)x(s)ds \qquad \text{(Fredholm equation)} \qquad (1.1)$$

by fixed point theorem.

The following theorems are the key in proving our main theorem.

Theorem 1.1 (fixed point theorem)**.** *Let (X, d) be a completed metric space. Assume that $T : X \to X$ is a strictly contractive operator with $d(Tx, Ty) \leq \theta d(x, y)(0 < \theta < 1)$. Then*

(a)there exists an unique fixed point x^* of $T(Tx^* = x^*)$;

(b)the sequence $\{T^n x\}$ converges to x^*.

Theorem 1.2 (Hölder inequality)**.** *Assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, x \in L^p(E), y \in L^q(E)$, then $xy \in L(E)$ and*

$$\int_E |x(t)y(t)|dt \leq (\int_E |x(t)^p|dt)^{\frac{1}{p}} (\int_E |y(t)|^q)^{\frac{1}{q}}.$$

2 Main Results

The following theorem is the main result of this paper.

Theorem 2.1. *Suppose that $z : [a, b] \to \mathbb{R}$, $f \in L^2([a, b])$ and $K(t, s) \in L^2([a, b] \times [a, b])$. If $z(t)$ satisfies the following inequality*

$$|z(t) - f(t) - \lambda \int_a^b K(t, s)z(s)ds| \leq \varepsilon \qquad (\varepsilon \geq 0), \tag{2.1}$$

where $|\lambda \int_a^b K(t, s)ds| \leq M < 1$ for every $t \in [a, b]$ and $|\lambda[\int_a^b \int_a^b K^2(t, s)dsdt]^{\frac{1}{2}}| \leq M < 1$, then there exists a solution x satisfies Eq.(1.1) and

$$|x(t) - z(t)| < \frac{1}{1 - M}\varepsilon$$

for every $t \in [a, b]$.

Proof. Define an operator T by:

$$(Tx)(t) = f(t) + g(t) + \lambda \int_a^b K(t, s)x(s)ds, \quad x \in L^2([a, b]). \tag{2.2}$$

Then, by using the Hölder inequality(theorem 1.2), we have

$$\begin{aligned} \int_a^b \Big| \int_a^b K(t, s)x(s)ds \Big|^2 dt &\leq \int_a^b [\int_a^b K^2(t, s)ds \int_a^b x^2(s)ds]dt \\ &\leq \int_a^b x^2(s)ds \cdot \int_a^b \int_a^b K^2(t, s)dsdt \leq \infty, \end{aligned}$$

which implies that $Tx \in L^2([a,b])$ and T is a self-mapping of $L^2([a,b])$. Thus, the solution of Eq.(2.2) is the fixed point of T.

Moreover,

$$\begin{aligned} d_2(Tx,Ty) &= [\int_a^b |(Tx)(t) - (Ty)(t)|^2 dt]^{\frac{1}{2}} \\ &= [\int_a^b |\lambda \int_a^b K(t,s)\{x(s) - y(s)\}ds|^2 dt]^{\frac{1}{2}} \\ &\leq |\lambda|[\int_a^b \{\int_a^b K^2(t,s)ds \int_a^b |x(s) - y(s)|^2 ds\}dt]^{\frac{1}{2}} \\ &= |\lambda|[\int_a^b \int_a^b K^2(t,s)dsdt]^{\frac{1}{2}} d(x,y). \end{aligned}$$

And we note that

$$\left|\lambda[\int_a^b \int_a^b K^2(t,s)dsdt]^{\frac{1}{2}}\right| \leq M < 1.$$

Thus, T is a contractive operator.

It follows from theorem 1.1 that Eq.(2.2) has a unique solution $x^* \in L^2([a,b])$, where $x^* = \lim_{n\to\infty} x_n$ for $x_n(t) = f(t) + g(t) + \lambda \int_a^b K(t,s)x_{n-1}(s)ds$ and $x_0 \in L^2([a,b])$ is an arbitrary function.

Now, assume that $g(t) = 0$ in Eq.(2.2)(equivalent to Eq.(1.1)), then we can know that there exists an unique solution $x^* \in L^2([a,b])$ of

$$x(t) = f(t) + \lambda \int_a^b K(t,s)x(s)ds, \tag{2.3}$$

where $x^* = \lim_{n\to\infty} x_n$ for $x_n(t) = f(t) + \lambda \int_a^b K(t,s)x_{n-1}(s)ds$ and x_0 is an arbitrary function in $L^2([a,b])$.

Then, let $z \in L^2([a,b])$ be a solution of Ineq.(2.1) and

$$z(t) - f(t) - \lambda \int_a^b K(t,s)z(s)ds := h(t). \tag{2.4}$$

Obviously, $|h(t)| \leq \varepsilon$ for all $t \in [a,b]$. Then we can know that the solution of Eq.(2.4) is $z^* = \lim_{n\to\infty} z_n$, where $z^* \in L^2([a,b])$ and $z_n(t) = f(t) + h(t) + \lambda \int_a^b K(t,s)z_{n-1}(s)ds$ and z_0 is an arbitrary function in $L^2([a,b])$.

At last, let $x_0(t) = z_0(t) = 0$, then we have

$$
\begin{aligned}
|x_1(t) - z_1(t)| &= |h(t)| \leq \varepsilon; \\
|x_2(t) - z_2(t)| &= |h(t) + \lambda \int_a^b K(t,s)(x_1(s) - z_1(x))ds| \leq \varepsilon(1 + \lambda \int_a^b |K(t,s)|ds); \\
|x_3(t) - z_3(t)| &= |h(t) + \lambda \int_a^b K(t,s)(x_2(s) - z_2(x))ds| \\
&\leq \varepsilon + \varepsilon\lambda \int_a^b |K(t,s_2)|(1 + \lambda \int_a^b |K(s_2,s_1)|ds_1)ds_2 \\
&\leq \varepsilon(1 + \lambda \int_a^b |K(t,s)|ds + \lambda^2 \int_a^b |K(t,s_2)| \int_a^b |K(s_2,s_1)|ds_1 ds_2); \\
\cdots\cdots\cdots &\cdots \cdots\cdots \\
|x_n(t) - z_n(t)| &\leq \varepsilon(1 + \lambda \int_a^b |K(t,s)|ds + \lambda^2 \int_a^b |K(t,s_2)| \int_a^b |K(s_2,s_1)|ds_1 ds_2 \\
&\quad + \lambda^n \int_a^b \cdots \int_a^b |K(t,s_n)K(s_n,s_{n-1}) \cdots K(s_2,s_1)|ds_n \cdots ds_2 ds_1) \\
&\leq \varepsilon(1 + M + M^2 + \cdots + M^n) \qquad (|\lambda \int_a^b K(t,s)ds| \leq M < 1) \\
&= \varepsilon \frac{1 - M^{n+1}}{1 - M}; \\
|x^*(t) - z^*(t)| &\leq \frac{1}{1 - M}\varepsilon, \qquad (\text{as} \quad n \to \infty),
\end{aligned}
$$

which completes our proof.

□

ACKNOWLEDGEMENTS. This is a text of acknowledgements.

References

[1] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27 (1941) 222-224.

[2] K.-W. Jun and Y.-H. Lee, A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation, J. Math. Anal. Appl. 238 (1999) 305-315.

[3] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett. 17 (2004) 1135-1140.

[4] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order (II), Appl. Math. Lett. 19 (2006) 854-858.

[5] T. Miura, On the Hyers-Ulam stability of a differentiable map, Sci. Math. Japan 55 (2002) 17-24.

[6] T. Miura, S.-M. Jung, S.-E. Takahasi, Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$, J. Korean Math. Soc. 41 (2004) 995-1005.

[7] Y. Li, Y. Shen, Hyers-Ulam stability of linear differential equations of second order, Appl. Math. Lett. 23 (2010) 306-309.

[8] M. Obłoza, Hyers stability of the linear differential equation, Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993) 259-270.

[9] M. Obłoza, Connections between Hyers and Lyapunov stability of the ordinary differential equations, Rocznik Nauk.-Dydakt. Prace Mat. 14 (1997) 141-146.

[10] C.-G. Park, On the stability of the linear mapping in Banach modules, J. Math. Anal. Appl. 275 (2002) 711-720.

[11] Th.M. Rassias, On the stability of linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297-300.

[12] Th.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000) 23-130.

[13] S.-E. Takahasi, T. Miura, S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309-315.

[14] S.M. Ulam, A Collection of the Mathematical Problems, Interscience, New York, 1960.

Received: March, 2015