Mathematica Aeterna, Vol. 5, 2015, no. 2, 263 - 267

A class of 3-Lie algebras realized by Lie algebras

BAI Ruipu

College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China
email: bairuipu@hbu.edu.cn

GAO Yansha

College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China

GUO Weiwei

College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China

Abstract

The structure of 3-Lie algebra G which is constructed by one dimensional extension of Lie algebra L is studied. Let L be a Lie algebra, then $G = L \oplus Fx_0$ is a 3-Lie algebra with the multiplication (1.1) given below. It is proved that for $I \subseteq L$ is an ideal of L if and only if I is an ideal of 3-Lie algebra G, and G is 2-solvable if and only if L is a solvable Lie algebra. The derivations and inner derivations of G are described by means of derivations of L.

Mathematics Subject Classification: 17B05, 17D99.

Keywords: 3-Lie algebra, derivation, inner derivation.

1 Introduction

In papers [1, 2] Jacobian algebras are constructed, some simple infinite dimensional n-Lie algebras over fields of positive characteristics are obtained. Authors in paper [3] realized 3-Lie algebras by two-dimensional extensions of Lie algebras, γ-matrices, linear functions and commutative associative algebras, and Lie algebras and some linear functions.

In papers [4], authors constructed 3-Lie algebras by commutative associative algebras and derivations and involutions. From papers [2-4], we obtain some methods to realize 3-Lie algebras by relative algebras such as Lie algebras, associative algebras, commutative associative algebras and pre-Lie algebras, etc.

In this paper, we pay our main attention to study the structure of 3-Lie algebras which are constructed by one-dimensional extension of Lie algebras. Let $(L, [,])$ be a Lie algebra over a field F, x_0 be not contained in L. Set $G = L \oplus Fx_0$. Define the linear multiplication $[, ,] : G \wedge G \wedge G \to G$ by

$$[x, y, x_0] = [x, y], \quad \text{for all } x, y \in L, [x, y, z] = 0, \quad \text{for all } x, y, z \in L. \qquad (1.1)$$

Then $(G, [, ,])$ is a 3-Lie algebra.

The organization of the rest of this paper is as follows. Section 2 studies the basic structure of 3-Lie algebra G. Section 3 discusses inner derivations and derivations of the 3-Lie algebra G.

2 Structures of 3-Lie algebra G

In this section we discussion the structure of the 3-Lie algebra G. We first recall some basic definitions on 3-Lie algebras.

A 3-Lie algebra is a vector space A over a field F on which there is a 3-ary skew-symmetric multilinear operation $[, ,]$ satisfying

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3} [x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3].$$

A subspace B of A is called a *subalgebra* (*an ideal*) if $[B, B, B] \subseteq B$ ($[B, A, A] \subseteq B$). If $[B, B, B] = 0$ ($[B, B, A] = 0$) than B is called *an abelian subalgebra* (*an abelian ideal*) . In particular, the subalgebra generated by the vectors $[x_1, x_2, x_3]$ for any $x_1, x_2, x_3 \in A$ is called the *derived algebra* of A, which is denoted by A^1. If $A^1 = 0$, A is called *an abelian algebra.* If an ideal I is an abelian subalgebra of A but it is not an abelian ideal, that is, $[I, I, I] = 0$, but $[I, I, A] \neq 0$, then I is called *a hypo-abelian ideal* .

An ideal I of a 3-Lie algebra A is called *s-solvable*, $2 \leq s \leq 3$, if $I^{(k,s)} = 0$ for some $k \geq 0$, where $I^{(0,s)} = I$ and $I^{(k+1,s)}$ is defined as $I^{(k+1,s)} = [\underbrace{I^{(k,s)}, \cdots, I^{(k,s)}}_{s}, A, \cdots, A]$.

It is clear that if I is s-solvable then I is l-solvable for $l \geq s$. If $I^{(2,s)} = 0$, then I is called 2*-step s-solvable ideal.* The sum of two s-solvable ideals of A is s-solvable. In particular, 2-semisimple and 3-semisimple 3-Lie algebras are simply called *semisimple* and *strong semisimple* 3-Lie algebras, respectively.

An ideal I of a 3-Lie algebra A is called *nilpotent*, if $I^s = 0$ for some $s \geq 0$, where $I^0 = I$ and I^s is defined as $I^s = [I^{s-1}, I, A]$, for $s \geq 1$. If A is a nilpotent ideal, then A is called a nilpotent 3-Lie algebra.

The center of A is $Z(A) = \{x \mid x \in A, [x, A, A] = 0\}$. It is clear that $Z(A)$ is an abelian ideal of A.

Lemma 2.1 *Let L be a Lie algebra, x_0 be not contained in L, $G = L \oplus Fx_0$ (the direct sum of vector spaces), then G is a 3-Lie algebra in the multiplication (1.1), and for all positive integer m, $G^m = L^m$, $G^{(m,2)} = L^{(m,2)} = L^{(m)}$, $G^{(2,3)} = 0$.*

Proof. By the multiplication (1.1) and a direct computation we know that G is a 3-Lie algebra. And $G^1 = [G, G, G] = [L, L, L] + [L, L, Fx_0] = L^1$, $G^2 = [G^1, G, G] = [L^1, L, Fx_0] = L^2$, by induction, suppose $G^{m-1} = L^{m-1}$ holds, then

$$G^m = [G^{m-1}, G, G] = [L^{m-1}, L + Fx_0, L + Fx_0] = [L^{m-1}, L] = L^m.$$

Similarly, we have $G^{(m,2)} = L^{(m,2)} = L^{(m)}$ and $G^{(2,3)} = 0$.

Theorem 2.2 *Let L be a Lie algebra over a field F, $G = L \oplus Fx_0$ be the 3-Lie algebra with the multiplication (1.1). Then we have 1) G is 2-solvable if and only if L is a solvable Lie algebra. 2) G is nilpotent if and only if L is nilpotent. 3) G is 3-solvable. 4) $Z(G) = Z(L)$.*

Proof The results 1), 2) and 3) follow from Lemma 2.1, directly. Now in the case $L^1 = 0$, then $G^1 = L^1 = 0$. $Z(G) = Z(L)$. If $L^1 \neq 0$, there exist $y, z \in L$ such that $[y, z] \neq 0$. For every $x + \lambda x_0 \in Z(G)$, where $x \in L, \lambda \in F$, from $[x + \lambda x_0, y, z] = \lambda[y, z] = 0$, we obtain $\lambda = 0$. Again by $[x + \lambda x_0, G, x_0] = [x, L] = 0$, it follows that $x \in Z(L)$. Therefore, $Z(G) \subseteq Z(L)$. It is cleat that $Z(L) \subseteq Z(G)$. The result follows.

In the following, let L be a Lie algebra over a field F, $G = L \oplus Fx_0$ be the 3-Lie algebra with the multiplication (1.1).

Theorem 2.3 *Let I be a subspace of L. Then we have*

1) I is an ideal of G if and only if I is an ideal of L.

2) Let $J = I \oplus Fx_0$. Then J is an ideal of G if and only if $L^1 \subseteq I$.

3) If I is an ideal of L and $L^1 \subseteq I$, then $J^{(m,2)} \subseteq I^{(m-1)}$ for every positive m. Therefore, if I is a solvable ideal of Lie algebra L, then J is a 3-solvable ideal of G.

4) If L is a simple Lie algebra, then L is a Hypo-nilpotent ideal of 3-Lie algebra G .

Proof Follows from $[I, G, G] = [I, L, x_0] = [I, L]$, the result 1) holds.

By the multiplication (1.1), $[J, G, G] = [I, L, x_0] + [x_0, L, L] = [I, L] + [L, L]$, we have $[J, G, G] \subseteq J$ if and only if $[L, L] \subseteq I$. It follows the result 2).

If I is an ideal of L and $L^1 \subseteq I$, then

$$J^{(1,2)} = [J, J, G] = [I, I] + [I, L] = I^{(1)} + [I, L] \subseteq I^{(1)} + I \subseteq I = I^{(0)},$$

$$J^{(2,2)} = [J^{(1,2)}, J^{(1,2)}, G] \subseteq [I, I, L + Fx_0] \subseteq I^{(1)},$$

suppose that $J^{(m-1,2)} \subseteq I^{(m-2)}$, then

$$J^{(m,2)} = [J^{(m-1,2)}, J^{(m-1,2)}, G] \subseteq [I^{(m-2)}, I^{(m-2)}, L + Fx_0] \subseteq I^{(m-1)}.$$

the result 3) follows.

By the above discussion, if L is a simple Lie algebra, then L is a proper ideal of G, and $[L, L, L] = 0$, $[L, L, G] = [L, L, x_0] = L^1 \neq 0$. The proof is completed.

3 Derivations of 3-Lie algebra G

We study the inner derivations and derivations of the 3-Lie algebra G. First we study the inner derivation algebra $ad(G)$.

Theorem 3.1 *Let L be an Lie algebra, and $G = L \oplus Fx_0$ be the 3-Lie algebra with the multiplication (1.1), then the inner derivation algebra $ad(G)$ has the semi-direct summation $ad(G) = ad(L) \oplus K$, where the inner derivation algebra $ad(L)$ is a subalgebra of $ad(G)$ and $K = \{ad(x, y)|$ for all $x, y \in L\}$ is an abelian ideal of $ad(G)$.*

Proof By the multiplication (1.1), for all $x, y, z \in L$, the inner derivations $ad(x_0, x)$, $ad(x, y) : G \to G$, satisfy $ad(x_0, x)(z) = [x_0, x, z] = adx(z), ad(x, y)(z) = 0$. Let $K = \{ad(x, y)|$ for all $x, y \in L\}$. Then for all $x, y, u.v \in L$, we have $[ad(x, y), ad(u, v)] = 0$, $[ad(x_0, x), ad(u, v)] = ad([x, u], v) + ad(u, [x, v])$, $[ad(x_0, x), ad(x_0, v)] = ad(x_0, [x_0, x, v]) = ad(x_0, [x, v])$.

Therefore, K is an abelian ideal of $ad(G)$, and $ad(x_0, L)$ is a subalgebra. Define linear isomorphism $\phi : ad(x_0, L) \to ad(L)$ by $\phi(ad(x_0, x)) = ad(x)$ for all $x \in L$. Then we have $[\phi(ad(x_0, x), \phi(ad(x_0, y)] = [ad(x), ad(y)] = ad([x, y]) = \phi(ad(x_0, [x, y])) = \phi([ad(x_0, x), ad(x_0, y)])$. Therefore, ϕ is a Lie algebra isomorphism.

Let $p_1 : G \to L$ and $p_0 : G \to Fx_0$ be projections, that is, for all $z = x + \lambda x_0 \in G, p_1(z) = x, p_0(z) = \lambda x_0$, where $x \in L, \lambda \in F$. Then for all linear map $D \in Der(G)$, $D_1 : G \to L \hookrightarrow G$, $D_0 : G \to Fx_0 \hookrightarrow G$ are linear maps, and for all $z \in G$, $D(z) = D_1(z) + D_0(z)$. Suppose $D_0(x) = \lambda_x x_0$, for all $x \in G$, where $\lambda_x \in F$. Denote $D_L = D_1|_L : L \to L$.

By the above notations we have the following result.

Theorem 3.2 *For all derivations D of the 3-Lie algebra G satisfy*

1) $D_0(G^1) = 0$, therefore, if L is perfect, that is, $L^1 = L$, then $D_0(L) = 0$;

2) D_L is a quasiderivation of the Lie algebra L, that is, $D_L \in QDer(L)$;

3) for all $x, y, z \in L$, set $D_0(x) = \lambda_x x_0$, then

$$\lambda_x[y, z] + \lambda_y[z, x] + \lambda_z[x, y] = 0; \tag{3.1}$$

4) $D(L^1) \subseteq L$.

Proof. For all $x, y \in L, D \in Der(G)$, since
$D([x, y, x_0]) = D_1([x, y, x_0]) + D_0([x, y, x_0]) = D_1([x, y]) + D_0([x, y])$,
$[D(x), y, x_0]+[x, D(y), x_0]+[x, y, D(x_0)] = [D_1(x), y]+[x, D_1(y)]+\lambda_{x_0}[x, y] \in L$,
we have $D_0([x, y]) = 0$, it follows $D_0(G^1) = 0$. Since

$D_L([x, y]) = D_1([x, y, x_0]) = D([x, y, x_0]) = [D_L(x), y, x_0] +[x, D_L(y), x_0]+ \lambda_{x_0}[x, y, x_0]$. Therefore, $(D_L - \lambda_{x_0})([x, y]) = [D_L(x), y] + [x, D_L(y)]$. The result 2) follows.

For all $x, y, z \in L$, by the multiplication (1.1) $[x, y, z] = 0$, then $D([x, y, z]) = [D(x), y, z]+ [x, D(y), z]+ [x, y, D(z)] = \lambda_x[y, z]+ \lambda_y[z, x]+ \lambda_z[x, y] = 0$. It follows the result 3).

The result 4) follows from $D([x,y,x_0]) = [D_1(x),y] + [x,D_1(y)] + \lambda_{x_0}[x,y]$.

Theorem 3.3 *Let G be the 3-Lie algebra with the multiplication (1.1) and $L^1 \neq L$. Let $G = L^1 \oplus U \oplus Fx_0$. Then*

$Q_0 = \{D : G \to Fx_0 | D(L^1 + Fx_0) = 0, D$ *satisfies Eq.(3.1)*$\}$

is an abelian subalgebra of $Der(G)$, and $ad(G) \cap Q_0 = 0$.

Proof Since for all $x,y,z \in L$, $D([x,y,x_0]) = 0$ and $[D(x),y,x_0] + [x,D(y),x_0] + [x,y,D(x_0)] = [\lambda_x x_0,y,x_0] + [x,\lambda_y x_0,x_0] + [x,y,D(x_0)] = 0$, $[D(x),y,z] + [x,D(y),z] + [x,y,D(z)] = \lambda_x[y,z] + \lambda_y[z,x] + \lambda_z[x,y] = 0 = D([x,y,z])$, we obtain $Q_0 \subseteq Der(G)$, and it is clear that Q_0 is an abelian subalgebra. Thanks to $D(G) \subseteq Fx_0$ for all $D \in Q_0$, $Q_0 \cap ad(G) = 0$.

Theorem 3.4 *Let G be the 3-Lie algebra with the multiplication (1.1), $D : G \to G$ be a linear map. Suppose $D = D_1 + D_0$, and $D_0(x) = \lambda_x x_0$. Then D is a derivation if and only if D satisfies Eq.(3.1), and D_1 satisfies $D_1([x,y]) = [D_1(x),y] + [x,D_1(y)] + \lambda_0[x,y]$.*

Proof The result follows from Theorem 3.2 and Theorem 3.3, directly.

Remark From above discussion, we have $Der(L) \subseteq Der(G)$ and for every $D \in Der(L)$, D can be seen as $D(x_0) = 0$.

Acknowledgements

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

References

[1] V. FILIPPOV, n-Lie algebras, *Sib. Mat. Zh.,* 1985, 26 (6), 126-140.

[2] A. POZHIDAEV, Simple quotient algebras and subalgebras of Jacobian algebras, Sib. Math. J., 1998, 39(3), 512-517

[3] R. BAI, C. BAI, J. WANG, Realizations of 3-Lie algebras, J. Mathe. Phys., 2010, 51, 063505.

[4] R. BAIY. WU Constructions of 3-Lie algebrasLinear and Multilinear Algebra, 2014, http://dx.doi.org/10.1080/03081087.2014.986121, arXiv: 1306.1994.

Received: April, 2015