Rank, Mode, Divergence and Spread on Generalized Triangular Fuzzy Numbers

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Abstract

This paper proposes a method for ranking of generalized triangular fuzzy numbers based on rank, mode, divergence and spread.

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1 Introduction

right spreads at some -levels of trapezoidal fuzzy numbers. Chen and Chen [5]
presented a method for fuzzy risk analysis based on ranking generalized fuzzy
numbers with different heights and different spreads. Rezvani [10-16] evaluated
the system of ranking fuzzy numbers and proposed a ranking trapezoidal fuzzy
numbers. Moreover, Pushpinder Singh [17] proposed a method for ranking of
generalized trapezoidal fuzzy numbers based on rank, mode, divergence and
spread.

2 Preliminaries

Generalized fuzzy number $\tilde{N}$ is described as any fuzzy subset of the real line
$R$, whose membership function $\mu_{\tilde{N}}(u)$ satisfies the following conditions,

(i) $\mu_{\tilde{N}}(u)$ is a continuous mapping from $R$ to the closed interval $[0, 1]$,

(ii) $\mu_{\tilde{N}}(u) = 0$, $-\infty < u \leq a$.

(iii) $\mu_L(u) = L(u)$ is strictly increasing on $[a, \bar{a}]$,

(iv) $\mu_{\tilde{N}}(u) = w$, $u = a$,

(v) $\mu_R(u) = R(u)$ is strictly decreasing on $[a, \bar{a}]$,

(vi) $\mu_{\tilde{N}}(u) = 0$, $\bar{a} \leq u < \infty$,

where $0 < w \leq 1$ and $\underline{a}$, $a$ and $\bar{a}$ are real numbers.

We call this type of generalized fuzzy number a triangular fuzzy number, and
it is denoted by

$\tilde{N} = (\underline{a}, a, \bar{a}; w)_{LR}$. \hfill (1)

When $w = 1$, this type of generalized fuzzy number is called normal fuzzy
number and is represented by

$\tilde{N} = (\underline{a}, a, \bar{a})_{LR}$. \hfill (2)

Let $L^{-1}$ and $R^{-1}$ be the inverse function of functions $L$ and $R$ respectively,
then the graded mean $h$-level value of $\tilde{N} = (\underline{a}, a, \bar{a})_{LR}$ is

$h[L^{-1}(h) + R^{-1}(h)]/2$. \hfill (3)

Therefore, the graded mean integration representation of generalized triangular
fuzzy number $\tilde{N}$ with grade $w$ is

$G(\tilde{N}) = \int_{0}^{w} h\left(\frac{L^{-1}(h) + R^{-1}(h)}{2}\right) dh / \int_{0}^{w} h \, dh$, \hfill (4)
where \( h \) lies between 0 and \( w \), \( 0 < w \leq 1 \). So membership function is given by

\[
\mu_{\tilde{N}}(x) = \begin{cases} 
\frac{w(u-a)}{a-a} & a \leq u \leq a, \\
w & w = a, \\
\frac{w(u-a)}{a-a} & a \leq u \leq a.
\end{cases}
\tag{5}
\]

Here,

\[
L(u) = w\left(\frac{u-a}{a-a}\right), \quad a \leq u \leq a,
\tag{6}
\]

and

\[
R(u) = w\left(\frac{u-a}{a-a}\right), \quad a \leq u \leq a,
\tag{7}
\]

then

\[
L^{-1}(h) = \alpha + \frac{(a-a)h}{w}, \quad 0 \leq h \leq w,
\tag{8}
\]

\[
R^{-1}(h) = \overline{\alpha} - \frac{(\overline{\alpha} - a)h}{w}, \quad 0 \leq h \leq w.
\tag{9}
\]

**Theorem 1.** Let \( \tilde{N} = (\underline{a}, a, \overline{a}; w)_{LR} \), be generalized triangular fuzzy number with \( 0 < w \leq 1 \) and \( \underline{a}, a \) and \( \overline{a} \) are real numbers. Then the graded mean integration representation of \( \tilde{N} \) is

\[
G(\tilde{N}) = \frac{\alpha + 4a + \overline{\alpha}}{6}.
\tag{10}
\]

**Proof:**

\[
P(N) = \int_{0}^{w} \frac{h}{2} \left[ \alpha + \frac{(a-a)h}{w} + \overline{\alpha} - \frac{(\overline{\alpha} - a)h}{w} \right] dh / \int_{0}^{w} h dh
\]

\[
= \int_{0}^{w} \frac{h}{2} \left[ (\alpha + \overline{\alpha}) + \frac{(2a-a-\overline{\alpha})h}{w} \right] dh / \int_{0}^{w} h dh
\]

\[
= w^2 \frac{\alpha + \overline{\alpha}}{4} + w^2 \frac{2a-a-\overline{\alpha}}{6} / \frac{w^2}{2} = \frac{\alpha + \overline{\alpha}}{4} + \frac{2a-a-\overline{\alpha}}{6} / \frac{1}{2}
\]

\[
= \frac{\alpha + 4a + \overline{\alpha}}{12} / \frac{1}{2} = \frac{\alpha + 4a + \overline{\alpha}}{6}.
\]
3 Find $R(\tilde{N})$, $\text{mode}(\tilde{N})$, $\text{divergence}(\tilde{N})$, Left and Right spread($\tilde{N}$)

Let $\tilde{N} = (a, a, \overline{a}; w)_{LR}$ be generalized triangular fuzzy number, then we find the values of $R(\tilde{N})$, $\text{mode}(\tilde{N})$, $\text{divergence}(\tilde{N})$, Left and Right spread.

$$R(\tilde{N}) = \frac{1}{2} \int_{0}^{w} [L^{-1}(h) + R^{-1}(h)] \, dh$$

$$= \frac{1}{2} \int_{0}^{w} \left[ \frac{a + (a - \overline{a})h}{w} + \overline{a} - \frac{(\overline{a} - a)h}{w} \right] \, dh = \frac{1}{2} \int_{0}^{w} \left[ \left(\frac{a + \overline{a}}{w}\right) + \frac{(2a - a - \overline{a})h}{w} \right] \, dh$$

$$= \frac{w(a + \overline{a})}{2} + \frac{w(2a - a - \overline{a})}{4} = \frac{w(a + 2a + \overline{a})}{4}$$

$$\Rightarrow R(\tilde{N}) = \frac{w(a + 2a + \overline{a})}{4}. \quad (11)$$

Now, for $\text{mode}(\tilde{N})$

$$\text{mode}(\tilde{N}) = \frac{1}{2} \int_{0}^{w} a \, dh = \frac{aw}{2}, \quad (12)$$

For $\text{divergence}(\tilde{N})$

$$\text{divergence}(\tilde{N}) = w(\overline{a} - a), \quad (13)$$

For Left and right spread($\tilde{N}$)

$$\text{Left spread}(\tilde{N}) = w(a - \overline{a}), \quad (14)$$

$$\text{Right spread}(\tilde{N}) = w(\overline{a} - a). \quad (15)$$

4 Some property for generalized triangular fuzzy numbers

**Theorem 2.** Let $\tilde{N} = (a, a, \overline{a}; w)_{LR}$ and $\tilde{M} = (b, b, \overline{b}; w)_{LR}$ be two generalized triangular fuzzy numbers, then

(i) If $R(\tilde{N}) = R(\tilde{M})$ then $w_{a}(a + 2a + \overline{a}) = w_{b}(b + 2b + \overline{b}), \quad (16)$

(ii) If $\text{mode}(\tilde{N}) = \text{mode}(\tilde{M})$ then $aw_{a} = bw_{b}, \quad (17)$

(iii) If $\text{divergence}(\tilde{N}) = \text{divergence}(\tilde{M})$ then $w_{a}(\overline{a} - a) = w_{b}(\overline{b} - b). \quad (18)$
Proof:

(i) We have \( R(\widetilde{N}) = \frac{w_a(a + 2a + \bar{a})}{4} = \frac{w_b(b + 2b + \bar{b})}{4} \)

\[ \Rightarrow w_a(a + 2a + \bar{a}) = w_b(b + 2b + \bar{b}) = R(\widetilde{M}) . \]

(ii) We have mode(\(\widetilde{N}\)) = \(\frac{aw_a}{2} = \frac{bw_b}{2}\) \Rightarrow aw_a = bw_b = mode(\(\widetilde{M}\)) .

(iii) We have divergence(\(\widetilde{N}\)) = \(w_a(\bar{a} - a) = w_b(\bar{b} - b) = \text{divergence}(\widetilde{M}) .\)

Also of theorem 2. we can say

\[ w_a a = w_b b , \quad (19) \]
\[ w_a \bar{a} = w_b \bar{b} , \quad (20) \]
\[ w_a a = w_b b . \quad (21) \]

Theorem 3. Let \(\widetilde{N} = (a, a, a; w_a)_{LR}\) and \(\widetilde{M} = (b, b, b; w_b)_{LR}\) be two generalized triangular fuzzy numbers, then

(i) Left spread(\(\widetilde{N}\)) > Left spread(\(\widetilde{M}\)) \iff w_a a > w_b b ,

(ii) Left spread(\(\widetilde{N}\)) < Left spread(\(\widetilde{M}\)) \iff w_a a < w_b b ,

(iii) Left spread(\(\widetilde{N}\)) = Left spread(\(\widetilde{M}\)) \iff w_a a = w_b b .

Proof:

(i) Let Left spread(\(\widetilde{N}\)) > Left spread(\(\widetilde{M}\)) \Rightarrow w_a(a - a) > w_b(b - b) with use equation (19), we have

\[ w_a a > w_b b . \]

Contrary

Let \(w_a a > w_b b\), with use equation (19), we have

\[ w_a(a - a) > w_b(b - b) \Rightarrow \text{Left spread}(\widetilde{N}) > \text{Left spread}(\widetilde{M}) . \]

(ii) Let Left spread(\(\widetilde{N}\)) < Left spread(\(\widetilde{M}\)) \Rightarrow w_a(a - a) < w_b(b - b) with use
equation (19), we have

\[ w_a < w_b \]

Contrary

Let \( w_a < w_b \), with use equation (19), we have

\[ w_a(a - a) < w_b(b - b) \Rightarrow \text{Left spread}(\tilde{N}) < \text{Left spread}(\tilde{M}). \]

(iii) Let \( \text{Left spread}(\tilde{N}) = \text{Left spread}(\tilde{M}) \Rightarrow w_a(a - a) = w_b(b - b) \) with use equation (19), we have

\[ w_a = w_b. \]

Contrary

Let \( w_a = w_b \), with use equation (19), we have

\[ w_a(a - a) = w_b(b - b) \Rightarrow \text{Left spread}(\tilde{N}) = \text{Left spread}(\tilde{M}). \]

**Theorem 4.** Let \( \tilde{N} = (a, a, \pi, w_a)_{LR} \) and \( \tilde{M} = (b, b, \nu, w_b)_{LR} \) be two generalized triangular fuzzy number, then

(i) Right spread(\( \tilde{N} \)) > Right spread(\( \tilde{M} \)) iff \( -w_a a > -w_b b \),

(ii) Right spread(\( \tilde{N} \)) < Right spread(\( \tilde{M} \)) iff \( -w_a a < -w_b b \),

(iii) Right spread(\( \tilde{N} \)) = Right spread(\( \tilde{M} \)) iff \( -w_a a = -w_b b \).

**Proof:**

(i) Let Right spread(\( \tilde{N} \)) > Right spread(\( \tilde{M} \)) \( \Rightarrow w_a(\pi - a) > w_b(\nu - b) \) with use equation (20), we have

\[ -w_a a > -w_b b \]

Contrary

Let \( -w_a a > -w_b b \), with use equation (20), we have
\(w_a(\overline{\alpha} - \alpha) > w_b(\overline{\beta} - \beta) \Rightarrow \text{Right spread}(\tilde{N}) > \text{Right spread}(\tilde{M}).\)

(ii) Let \(\text{Right spread}(\tilde{N}) < \text{Right spread}(\tilde{M}) \Rightarrow w_a(\overline{\alpha} - \alpha) < w_b(\overline{\beta} - \beta)\) with use equation (20), we have

\(-w_a a < -w_b b.\)

Contrary

Let \(-w_a a < -w_b b,\) with use equation (20), we have

\(w_a(\overline{\alpha} - \alpha) < w_b(\overline{\beta} - \beta) \Rightarrow \text{Right spread}(\tilde{N}) < \text{Right spread}(\tilde{M}).\)

(iii) Let \(\text{Right spread}(\tilde{N}) = \text{Right spread}(\tilde{M}) \Rightarrow w_a(\overline{\alpha} - \alpha) = w_b(\overline{\beta} - \beta)\) with use equation (20), we have

\(-w_a a = -w_b b.\)

Contrary

Let \(-w_a a = -w_b b,\) with use equation (20), we have

\(w_a(\overline{\alpha} - \alpha) = w_b(\overline{\beta} - \beta) \Rightarrow \text{Right spread}(\tilde{N}) = \text{Right spread}(\tilde{M}).\)

**Theorem 5.** Let \(\tilde{N} = (a, \overline{\alpha}; w_a)_{LR}\) and \(\tilde{M} = (b, \overline{\beta}; w_b)_{LR}\) be two generalized triangular fuzzy number, such that

(a) \(R(\tilde{N}) = R(\tilde{M}),\)

(b) \(\text{mode}(\tilde{N}) = \text{mode}(\tilde{M}),\)

(c) \(\text{divergence}(\tilde{N}) = \text{divergence}(\tilde{M}),\)

then

(i) \(\text{Left spread}(\tilde{N}) > \text{Left spread}(\tilde{M})\) iff \(\text{Right spread}(\tilde{N}) > \text{Right spread}(\tilde{M}),\)

(ii) \(\text{Left spread}(\tilde{N}) < \text{Left spread}(\tilde{M})\) iff \(\text{Right spread}(\tilde{N}) < \text{Right spread}(\tilde{M}),\)

(iii) \(\text{Left spread}(\tilde{N}) = \text{Left spread}(\tilde{M})\) iff \(\text{Right spread}(\tilde{N}) = \text{Right spread}(\tilde{M}).\)
Proof:

(i) Let $a = a_1 + a_2$ and $b = b_1 + b_2$, so $Left\ spread(\tilde{N}) > Left\ spread(\tilde{M}) \iff w_a(a_1 - a) > w_b(b_1 - b)$, with use (19) iff $w_a a_1 > w_b b_1$. Now of (21)($w_a(a_1 + a_2) = w_b(b_1 + b_2)$), we have iff $w_a a_2 < w_b b_2 \iff -w_a a_2 > -w_b b_2$. Then with (20) iff $w_a(a - a_2) > w_b(b - b_2) \iff Right\ spread(\tilde{N}) > Right\ spread(\tilde{M})$.

(ii) Let $a = a_1 + a_2$ and $b = b_1 + b_2$, so $Left\ spread(\tilde{N}) < Left\ spread(\tilde{M}) \iff w_a(a_1 - a) < w_b(b_1 - b)$, with use (19) iff $w_a a_1 < w_b b_1$. Now of (21)($w_a(a_1 + a_2) = w_b(b_1 + b_2)$), we have iff $w_a a_2 > w_b b_2 \iff -w_a a_2 < -w_b b_2$. Then with (20) iff $w_a(a - a_2) < w_b(b - b_2) \iff Right\ spread(\tilde{N}) < Right\ spread(\tilde{M})$.

(iii) Let $a = a_1 + a_2$ and $b = b_1 + b_2$, so $Left\ spread(\tilde{N}) = Left\ spread(\tilde{M}) \iff w_a(a_1 - a) = w_b(b_1 - b)$, with use (19) iff $w_a a_1 = w_b b_1$. Now of (21)($w_a(a_1 + a_2) = w_b(b_1 + b_2)$), we have iff $w_a a_2 = w_b b_2 \iff -w_a a_2 = -w_b b_2$. Then with (20) iff $w_a(a - a_2) = w_b(b - b_2) \iff Right\ spread(\tilde{N}) = Right\ spread(\tilde{M})$.

5 Proposed approach for ranking of generalized triangular fuzzy numbers

Let $\tilde{N} = (\underline{a}, a, \overline{a}; w_a)_{LR}$ and $\tilde{M} = (\underline{b}, b, \overline{b}; w_b)_{LR}$ be two generalized triangular fuzzy number, then use the following steps to compare $\tilde{N}, \tilde{M}$

* step 1: Find $R(\tilde{N})$ and $R(\tilde{M})$

Case (i) If $R(\tilde{N}) > R(\tilde{M})$ then $\tilde{N} > \tilde{M}$

Case (ii) If $R(\tilde{N}) < R(\tilde{M})$ then $\tilde{N} < \tilde{M}$

Case (iii) If $R(\tilde{N}) = R(\tilde{M})$ then go to step 2.

* step 2: Find $mode(\tilde{N})$ and $mode(\tilde{M})$

Case (i) If $mode(\tilde{N}) > mode(\tilde{M})$ then $\tilde{N} > \tilde{M}$

Case (ii) If $mode(\tilde{N}) < mode(\tilde{M})$ then $\tilde{N} < \tilde{M}$

Case (iii) If $mode(\tilde{N}) = mode(\tilde{M})$ then go to step 3.

* step 3: Find $divergence(\tilde{N})$ and $divergence(\tilde{M})$
Case (i) If \( \text{divergence}(\tilde{N}) > \text{divergence}(\tilde{M}) \) then \( \tilde{N} > \tilde{M} \)

Case (ii) If \( \text{divergence}(\tilde{N}) < \text{divergence}(\tilde{M}) \) then \( \tilde{N} < \tilde{M} \)

Case (iii) If \( \text{divergence}(\tilde{N}) = \text{divergence}(\tilde{M}) \) then go to step 4.

* step 4: Find \( \text{Left spread}(\tilde{N}) \) and \( \text{Left spread}(\tilde{M}) \)

Case (i) If \( \text{Left spread}(\tilde{N}) > \text{Left spread}(\tilde{M}) \)

i.e., \( w_a > w_b \) then \( \tilde{N} > \tilde{M} \) from theorem 3.

Case (ii) If \( \text{Left spread}(\tilde{N}) < \text{Left spread}(\tilde{M}) \)

i.e., \( w_a < w_b \) then \( \tilde{N} < \tilde{M} \) from theorem 3.

Case (iii) If \( \text{Left spread}(\tilde{N}) = \text{Left spread}(\tilde{M}) \)

i.e., \( w_a = w_b \) then go to step 5 from theorem 3.

* step 5: Find \( w_a \) and \( w_b \) Case (i) If \( w_a > w_b \) then \( \tilde{N} > \tilde{M} \)

Case (ii) If \( w_a < w_b \) then \( \tilde{N} < \tilde{M} \)

Case (iii) If \( w_a = w_b \) then \( \tilde{N} = \tilde{M} \)

6 Examples

In this section, examples of fuzzy numbers are compared using the proposed approach.

**Example 1.** Let \( \tilde{N} = (0.2, 0.3, 0.5; 0.21) \) and \( \tilde{M} = (0.1, 0.15, 0.25; 0.42) \) be two generalized triangular fuzzy number

* step 1

\( R(\tilde{N}) = 0.06825 \) and \( R(\tilde{M}) = 0.06825 \). Since \( R(\tilde{N}) = R(\tilde{M}) \), so go to step 2

* step 2

\( \text{mode}(\tilde{N}) = 0.0315 \) and \( \text{mode}(\tilde{M}) = 0.0315 \). Since \( \text{mode}(\tilde{N}) = \text{mode}(\tilde{M}) \),
so go to step 3

* step 3

divergence(\(\tilde{N}\)) = 0.063 and divergence(\(\tilde{M}\)) = 0.063 . Since divergence(\(\tilde{N}\)) = divergence(\(\tilde{M}\)), so go to step 4

* step 4

Left spread(\(\tilde{N}\)) = 0.021 and Left spread(\(\tilde{M}\)) = 0.021 . Since Left spread(\(\tilde{N}\))=Left spread(\(\tilde{M}\)), so go to step 5

* step 5

\(w_a = 0.21, w_b = 0.42\) Since \(w_a < w_b \Rightarrow \tilde{N} < \tilde{M}\).

**Example 2.** Let \(\tilde{N} = (0.4, 0.6, 0.8; 0.35)\) and \(\tilde{M} = (0.3, 0.5, 0.7; 0.42)\) be two generalized triangular fuzzy number

* step 1

\(R(\tilde{N}) = 0.21\) and \(R(\tilde{M}) = 0.21\) . Since \(R(\tilde{N}) = R(\tilde{M})\), so go to step 2

* step 2

\(mode(\tilde{N}) = 0.105\) and \(mode(\tilde{M}) = 0.105\) .Since \(mode(\tilde{N}) = mode(\tilde{M})\), so go to step 3

* step 3

\(\text{divergence}(\tilde{N}) = 0.14\) and \(\text{divergence}(\tilde{M}) = 0.168\) .

So \(\text{divergence}(\tilde{N}) < \text{divergence}(\tilde{M})\) then \(N < M\).

**Example 3.** Let \(\tilde{N} = (0.1, 0.3, 0.5; 1)\) and \(\tilde{M} = (0.2, 0.2, 0.6; 1)\) be two generalized triangular fuzzy number

* step 1

\(R(\tilde{N}) = 0.3\) and \(R(\tilde{M}) = 0.3\) . Since \(R(\tilde{N}) = R(\tilde{M})\), so go to step 2

* step 2
mode(\(\tilde{N}\)) = 0.15 and mode(\(\tilde{M}\)) = 0.1. So mode(\(\tilde{N}\)) > mode(\(\tilde{M}\)) then \(N > M\).

**Example 4.** Let \(\tilde{N} = (0.2, 0.3, 0.5; 0.21)\) and \(\tilde{M} = (0.1, 0.4, 0.4; 0.21)\) be two generalized triangular fuzzy number

* step 1

\[ R(\tilde{N}) = 0.06825 \text{ and } R(\tilde{M}) = 0.06825. \]  
Since \(R(\tilde{N}) = R(\tilde{M})\), so go to step 2

* step 2

mode(\(\tilde{N}\)) = 0.0315 and mode(\(\tilde{M}\)) = 0.042. So mode(\(\tilde{N}\)) < mode(\(\tilde{M}\)), then \(A < B\).

**Example 5.** Let \(\tilde{N} = (0.2, 0.3, 0.4; 0.26)\) and \(\tilde{M} = (0.1, 0.2, 0.3; 0.26)\) be two generalized triangular fuzzy number

* step 1

\[ R(\tilde{N}) = 0.078 \text{ and } R(\tilde{M}) = 0.052. \]  
So \(R(\tilde{N}) > R(\tilde{N})\) then \(N > M\).

**References**


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