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The Hyers-Ulam stability of the conformable fractional differential equation

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Abstract

In this paper, we prove a new Gronwall type integral inequality, and then apply it to investigate the Hyer-Ulams stability of the conformable fractional differential equation.

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1 Introduction

The idea of fractional calculus is as old as traditional calculus. The history of fractional calculus dates back to more than 300 years ago, and the original question which led to the name fractional calculus was: what does $\frac{d^n f}{dx^n}$ mean if $n = \frac{1}{2}$. Since then, several mathematicians contributed to the development of fractional calculus, including $Riemann-Liouville$ fractional operator, $Caputo$ fractional operator, $Grunwald-Letniko$ fractional operator, these of the very famous are for these fractional order operator have different properties, the literature has been given in $[1-2]$. Until recently, research on fractional calculus was confined to the filed of mathematics. However in the last two decades, many applications of fractional calculus in various fields of engineering, science mathematics and economics have been found. As a result, fractional calculus has become an important topic for researchers in various fields. However, the authors in [3] define a new well-behaved simple fractional derivative called the conformable fractional derivative, depending just on the basic limit definition of the derivative. Namely, for a function $f : (0, \infty) \to \Re$ the conformable fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

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If f is α-differentiable in some $(0,a), a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$.

As same as the fractional calculus, the development of the Hyers-Ulam stability also has a long history. In 1940, Ulam posed a problem concerning the stability of functional equations: Give conditions in order for a linear function near an approximately linear function to exist. A year later, Hyers[4] gave an affirmative answer. After Hyers's result, many mathematicians make great efforts, they extended results to discuss the differential equations. By the use of the Gronwall-Bellman type inequalities and the technique of weakly Picard operators, I. A. Rus [5−6] investigated the Hyers-Ulam stability of differential and integral equations. The Gronwall type integral inequalities and their applications can be seen in $[7-12]$ and references therein.

Motivated by the above results, in this paper, we prove a new Gronwall type integral inequality, and then, use it to study the Hyers-Ulam stability of the conformable fractional differential equation.

This paper will be divided into several parts as follows: The definitions, notations and related results are presented in section 2. In section 3, we prove a generalized Gronwall type integral inequality for the conformable fractional calculus. In section 4, the Hyers-Ulam stability of the conformable fractional differential equation is discussed.

2 Preliminary Notes

In this section, for completeness, we introduce the definitions of the conformable fractional derivative. Readers can find the detailed properties of the conformable fractional differential operators in $[13-14]$.

Definition 2.1 The (left)fractional derivative staring from a of a function $f:[a,\infty)\to\Re$ of order $0<\alpha\leq 1$ is defined by

$$\begin{aligned} T_\alpha f(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \\ &= \int_a^b f(x)d_\alpha(x,a) \\ &= \int_a^b (x-a)^{\alpha-1}f(x)dx. \end{aligned} \tag{1}$$

Lemma 2.2 Assume $\alpha\in(0,1]$, f is differentiable for $t>a$, then for the all $t>a$ we have

$$T_\alpha f(t)=(t-a)^{1-\alpha}f'(t).$$

Definition 2.3 Let $\alpha\in(0,1]$ and $f:(a,b)\to\Re$ be differentiable, then for all $t>a$, we have

$$I_\alpha^a T_\alpha^a = f(t)-f(a).$$

Lemma 2.4 Let $f, g : [a, b] \to \Re$ be two functions, such that fg is differentiable. then

$$\int_a^b f(x)T_\alpha^a(g)(x)d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)T_\alpha^a(f)(x)d_\alpha(x,a).$$

Let $\alpha \in (0,1]$, $G := \{(t,y) \in \Re, t \in [a,t_0), y \in \Re\}$, consider the following initial value problem of the comfortable differential equation

$$T_\alpha^a y(t) = Ay(t) + f(t, y(t)),\ y(a) = y_0 \tag{2}$$

where $y, f : G :\to \Re$ are real-valued functions and A is constant.
Lemma 2.5 The general solution of the fractional nonhomogeneous equation (2) is expressed by

$$y(t) = y_0 e^{A\frac{(t-a)^\alpha}{\alpha}} + \int_a^t e^{A\frac{(t-a)^\alpha}{\alpha}} e^{-A\frac{(s-a)^\alpha}{\alpha}} (s-a)^{\alpha-1} f(t, y(t))ds.$$

3 A new Gronwall type integral inequality

In this section, we establish a new Gronwall type integral inequality, which generalize previous result in literature.

The main theorem in this section is the following.
Theorem 3.1 If for any $t \in [a, b]$,

$$x(t) \leq h(t) + \int_a^t u(s)x(s)(s-a)^{\alpha-1}ds$$

where $x(t), h(t), u(t)$ are nonnegative and continuous , $u(t)$ is nondecreasing, $h(t)$ is differentiable on$[a, b]$, then

$$x(t) \leq h(t) + \int_a^t u(r)h(r)e^{\int_r^t u(s)(s-a)^{\alpha-1}ds}dr. \tag{3}$$

Proof. Let us consider the function

$$R(t) = h(t) + \int_a^t u(s)x(s)(s-a)^{\alpha-1}ds. \tag{4}$$

Then we have $R(a) = h(a)$ and $x(t) \leq R(t)$,

$$T_\alpha^a R(t) = T_\alpha^a h(t) + u(t)x(t) \leq T_\alpha^a h(t) + u(t)R(t).$$

By multiplication with $K(t) = e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds}$, we obtain

$$T_\alpha^a(e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds}R(t)) \leq e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds}T_\alpha^a h(t).$$

With the help of Definition 2.3, Lemma 2.4 and $R(a) = h(a)$, since $R(t)K(t)$ is differentiable on $[a, b]$, then

$$R(t)e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds} - h(a) \leq \int_a^t e^{-\int_a^r u(s)(s-a)^{\alpha-1}ds}T_\alpha^a h(r)d_\alpha(r, a).$$

$$\begin{aligned} R(t)e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds} &\leq h(t)e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds} \\ &\quad - \int_a^t h(r)T_\alpha^a e^{-\int_a^r u(s)(s-a)^{\alpha-1}ds}d_\alpha(r, a) \\ &= h(t)e^{-\int_a^t u(s)(s-a)^{\alpha-1}ds} \\ &\quad + \int_a^t h(r)u(r)e^{-\int_a^r u(s)(s-a)^{\alpha-1}ds}d_\alpha(r, a). \end{aligned}$$

Hence

$$R(t) \leq h(t) + e^{\int_a^t u(s)(s-a)^{\alpha-1}ds}\int_a^t h(r)u(r)e^{-\int_a^r u(s)(s-a)^{\alpha-1}ds}d_\alpha(r, a). \quad (5)$$

Using (3) and (5), we then find

$$x(t) \leq h(t) + e^{\int_a^t u(s)(s-a)^{\alpha-1}ds}\int_a^t h(r)u(r)e^{-\int_a^r u(s)(s-a)^{\alpha-1}ds}d_\alpha(r, a)$$

$$x(t) \leq h(t) + \int_a^t u(r)h(r)e^{\int_r^t u(s)(s-a)^{\alpha-1}ds}dr$$

This completes the proof. . □

Corollary 3.2 [14] Let $x(t) : [a, b] \to \Re$ be nonegative and continuous, $M, k > 0$. If

$$x(t) \leq M + \int_a^t kx(x)(s-a)^{\alpha-1}ds$$

then for all the $t \in [a, b]$

$$x(t) \leq Me^{k\frac{(t-a)^\alpha}{\alpha}}.$$

Corollary 3.3 If for any $t \in [a, b]$,

$$x(t) \leq M + \int_a^t u(s)x(s)(s-a)^{\alpha-1}ds$$

where $u(t), x(t)$ are nonnegative and continuous, $u(t)$ is nondecreasing, $M > 0$ is a constant. Then

$$x(t) \leq Me^{\int_a^t u(s)(s-a)^{\alpha-1}ds}.$$

Corollary 3.4 Let $x(t) : [a, b] \to \Re$ is a nonnegative continuous function satisfying:

$$x(t) \leq h(t) + \int_a^t kx(s)(s-a)^{\alpha-1}ds$$

where $k > 0$ is a constant and $h(t)$ is a nonnegative differentiable function on $[a, b]$,then

$$x(t) \leq h(t) + ke^{k\frac{(t-a)^\alpha}{\alpha}} \int_a^t h(s)e^{-k\frac{(s-a)^\alpha}{\alpha}} d_\alpha(s, a) \qquad t \in [a, b].$$

4 The Hyers-Ulam stability of a conformable fractional differential equation

In this section, we consider the Hyers-Ulam stability of a comfortable fractional differential equation as follows

$$T^a_\alpha y(t) = Ay(t) + f(t, y(t)).$$

The main result is

Theorem 4.1 Assume the hypotheses of Lemma 2.5 hold. If $f : G \to \Re$ is continuous and satisfying Lipschitz condition with respect to the second variable, i,e. there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|$$

for (x, y_1) and $(x, y_1) \in G$. Then, for every $\varepsilon > 0$ and $y_\varepsilon(x) : [a, t_0] \to \Re$ satisfying

$$|T^a_\alpha y_\varepsilon(t) - Ay_\varepsilon(t) - f(t, y_\varepsilon(t))| \leq \varepsilon,\ t \in [a, t_0] \tag{6}$$

there exist a solution $y(t) : [a, t_0] \to \Re$ of equation(2.1) and a constant $K > 0$ such that

$$|y(x) - y_\varepsilon(x)| \leq K\varepsilon \qquad t \in [a, t_0]$$

where $G := \{(t, y) \in \Re, t \in [a, t_0], y \in \Re\}, K(\varepsilon) = \frac{1}{A+L}(e^{(L+A)\frac{(t-a)^\alpha}{\alpha}} - 1)$

Proof. By (6), we have

$$\varepsilon \leq T^a_\alpha y_\varepsilon(t) - Ay_\varepsilon(t) - f(t, y_\varepsilon(t)) \leq \varepsilon. \tag{7}$$

Definition 2.3 and the Laplace transform give

$$y_\varepsilon(t) \le e^{A\frac{(t-a)^\alpha}{\alpha}}(\frac{\varepsilon}{A} + y_0) - \frac{\varepsilon}{A} + e^{A\frac{(t-a)^\alpha}{\alpha}} \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} f(s, y(s)) d_\alpha(s, a). \quad (8)$$

By Lemma 2.5,

$$y(t) = y_0 e^{A\frac{(t-a)^\alpha}{\alpha}} + \int_a^t e^{A\frac{(t-a)^\alpha}{\alpha}} e^{-A\frac{(s-a)^\alpha}{\alpha}} (s-a)^{\alpha-1} f(s, y(s)) ds \quad (9)$$

Combing (8) with(9) we obtain

$$\begin{aligned} y_\varepsilon(t) - y(t) \le \frac{\varepsilon}{A}(e^{A\frac{(t-a)^\alpha}{\alpha}} - 1) + e^{A\frac{(t-a)^\alpha}{\alpha}} \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} (f(s, y_\varepsilon(s) \\ - f(s, y(s)) d_\alpha(s, a) \end{aligned} \quad (10)$$

On the other hand,

$$T_\alpha^a y_\varepsilon(t) - A y_\varepsilon(t) - f(t, y_\varepsilon(t)) \ge -\varepsilon$$

In the same manner, we have

$$\begin{aligned} y_\varepsilon(t) - y(t) \ge -\frac{\varepsilon}{A}(e^{A\frac{(t-a)^\alpha}{\alpha}} - 1) - e^{A\frac{(t-a)^\alpha}{\alpha}} \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} (f(s, y_\varepsilon(s) \\ - f(s, y(s)) d_\alpha(s, a) \end{aligned} \quad (11)$$

A combination of (10), (11) and the Lipschitz assumption on f, results yields

$$\begin{aligned} |y_\varepsilon(t) - y(t)| &\le \frac{\varepsilon}{A}(e^{A\frac{(t-a)^\alpha}{\alpha}} - 1) + e^{A\frac{(t-a)^\alpha}{\alpha}} \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} |(f(s, y_\varepsilon(s) \\ &\quad - f(s, y(s)) d_\alpha(s, a)| \\ &\le \frac{\varepsilon}{A}(e^{A\frac{(t-a)^\alpha}{\alpha}} - 1) + e^{A\frac{(t-a)^\alpha}{\alpha}} L \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} |(y_\varepsilon(s) \\ &\quad - y(s))| d_\alpha(s, a). \end{aligned}$$

$$\begin{aligned} e^{-A\frac{(t-a)^\alpha}{\alpha}} |y_\varepsilon(t) - y(t)| \le \frac{\varepsilon}{A}(1 - e^{-A\frac{(t-a)^\alpha}{\alpha}}) + L \int_a^t e^{-A\frac{(s-a)^\alpha}{\alpha}} |(y_\varepsilon(s) \\ - y(s))| d_\alpha(s, a). \end{aligned}$$

In view of the corollary 3.4 , we get

$$\begin{aligned}
e^{-A\frac{(t-a)^\alpha}{\alpha}}|y_\varepsilon(t)-y(t)| &\leq \frac{\varepsilon}{A}(1-e^{-A\frac{(t-a)^\alpha}{\alpha}}) + Le^{L\frac{(t-a)^\alpha}{\alpha}}\int_a^t \frac{\varepsilon}{A}(1-e^{-A\frac{(s-a)^\alpha}{\alpha}}) \\
&\quad \times e^{-L\frac{(s-a)^\alpha}{\alpha}} d_\alpha(s,a) \\
&\leq \frac{\varepsilon}{A}(1-e^{-A\frac{(t-a)^\alpha}{\alpha}}) + \frac{L\varepsilon}{A}e^{L\frac{(t-a)^\alpha}{\alpha}}\int_a^t (e^{-L\frac{(s-a)^\alpha}{\alpha}} \\
&\quad - e^{-(A+L)\frac{(s-a)^\alpha}{\alpha}})d_\alpha(s,a) \\
&\leq \frac{\varepsilon}{A}(1-e^{-A\frac{(t-a)^\alpha}{\alpha}}) + \frac{L\varepsilon}{A}e^{L\frac{(t-a)^\alpha}{\alpha}}(-\frac{1}{L}e^{-L\frac{(t-a)^\alpha}{\alpha}} \\
&\quad + \frac{1}{A+L}e^{-(A+L)\frac{(t-a)^\alpha}{\alpha}} + \frac{1}{L} - \frac{1}{A+L}) \\
&\leq \frac{\varepsilon}{A}e^{-A\frac{(t-a)^\alpha}{\alpha}}(\frac{L}{A+L}-1) + \frac{\varepsilon}{A}e^{L\frac{(t-a)^\alpha}{\alpha}}(1-\frac{L}{A+L}).
\end{aligned} \tag{12}$$

Multiplying both sides of (12) by $e^{A\frac{(t-a)^\alpha}{\alpha}}$, then

$$\begin{aligned}
|y_\varepsilon(t)-y(t)| &\leq \frac{\varepsilon}{A}(\frac{L}{A+L}-1)(1-e^{(L+A)\frac{(t-a)^\alpha}{\alpha}}) \\
&\leq \frac{\varepsilon}{A+L}(e^{(L+A)\frac{(t-a)^\alpha}{\alpha}}-1).
\end{aligned}$$

Hence $K(\varepsilon) = \frac{1}{A+L}(e^{(L+A)\frac{(t-a)^\alpha}{\alpha}}-1)$. □

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References

[1] Kai Diethelm, The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type, Springer-Verlag, 2010.

[2] S. G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: Theory and applications, Gordon and Breach, Yverdon, 1993.

[3] R. Khalil, M.Al Horani, A. Yousef, M.Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math, Vol. 264, pp. 65-70, 2014.

[4] D. H. Hyers, On the stability of the linear functional equation, Proc. National Acad. Sci. (USA), Vol. 27, pp. 222-224, 1941.

[5] I. A. Rus, Remarks on Ulam stability of the operatorial equations, Fixed Point Theory, Vol. 10, no. 2, pp. 305-320, 2009.

[6] I. A. Rus, Ulam stabilityof ordinary differential equations, Studia Universitatis Babes-Bolyai, Mathematical, Vol. 54, no. 4, pp. 125-133, 2009

[7] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations,Ann. Math, Vol. 20, pp. 292-296, 1919.

[8] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, Acta Math. Acad. Sci, Hungar, Vol.7, pp. 81-94, 1965.

[9] J. Shao and F. Meng, Gronwall-Bellman type inequalities and their applications to fractional differential equations, Abstr. Appl. Anal, Vol 2013(2013), 7 pages.

[10] S. Lin, Generalized Gronwall inequalities and their applications to fractional differential equations, J. Inequal. Appl, 2013, 2013:549.

[11] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and application to a fractional differential equation, J. Math. Anal. Appl, Vol. 328, pp. 1075-1081, 2007.

[12] P. Thiramanus, J.Tariboon, S.K.Ntouyas, Henry-Gronwall integral inequalities with "Maxima "and their applications to fractional differential equations, Abstr. Appl. Anal, Vol 2014 (2014), 10 pages.

[13] M.D. Ortigueira , J.A. Tenreiro Machado, What is a fractional derivative? J. Comput. Phys, Vol. 293, pp. 4-13, 2015.

[14] Thabet Abdeljawad , On conformable fractional calculus, J. Comput. Appl. Math, Vol.279, pp.57-66, 2015.

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