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On moments of the Cantor and related distributions

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Abstract

We provide several simple recursive formulae for the moment sequence of infinite Bernoulli convolution. We relate moments of one infinite Bernoulli convolution with others having different but related parameters. We give examples relating Euler numbers to the moments of infinite Bernoulli convolutions. One of the examples provides moment interpretation of Pell numbers as well as new identities satisfied by Pell and Lucas numbers.

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1 Introduction

The aim of this note is to add a few simple observations to the analysis of the distribution of the so called fatigue symmetric walk (term appearing in [12]). These observations are based on the reformulation of known results scattered through the literature. We however pay more attention to the moment sequences and less to the properties of distributions that produce these moment sequences. It seems that the main novelty of the paper lies in the probabilistic interpretation of Pell and Lucas numbers and easy proofs of some identities satisfied by these numbers. However in order to place these results in the proper context we recall the definition and basic properties of infinite Bernoulli convolutions. In deriving properties of these convolutions we recall some known, important results.

The paper is organized as follows. After recalling the definition and basic facts about the fatigue random walks we concentrate on the moment sequences of infinite Bernoulli convolutions. We formulate a corollary of the results of the paper expressed in terms of a moment sequence. This corollary formulated in terms of number sequences provides identities of Pell and Lucas numbers of even order (Remark 8).

2 Infinite Bernoulli Convolutions

Let $\{X_n\}_{n\geq 1}$ be the sequence of i.i.d. random variables such that $P(X_1 = 1) = P(X_1 = -1) = 1/2$. Further let $\{c_n\}_{n\geq 1}$ be a sequence of reals such that $\sum_{n\geq 1} c_n^2 < \infty$. We define random variable:

$$S = \sum_{n\geq 1} c_n X_n.$$

By Kolmogorov Three-Series theorem S exists and moreover it is square integrable. We have $ES^2 = \sum_{n\geq 1} c_n^2$. Obviously $ES = 0$. Let $\varphi(t)$ denote characteristic function of S. By the standard argument we have $\varphi(t) = E\exp(it\sum_{s\geq 1} c_n X_n) = E\prod_{n\geq 1}\exp(itc_n X_n) = \prod_{n\geq 1}(\exp(itc_n)/2+\exp(-itc_n)/2) = \prod_{n\geq 1}\cos(tc_n)$.

We will concentrate on the special form of the sequence c_n, namely we will assume that $c_n = \lambda^{-n}$ for some $\lambda > 1$. The reason for this is simple. There are practically no results in the literature for other than $c_n = \lambda^{-n}$ sequences. The problem of describing distributions appearing in 'fatigue random walk' is very difficult although simply formulated.

It is known (see [13]) that for all λ, the distribution of $S = S(\lambda)$ is continuous that is $P_S(\{x\}) = 0$ for all $x \in \mathbb{R}$. Moreover it is also known (see [16], [17]) that if for almost $\lambda \in (1, 2]$ this distribution is absolutely continuous and for almost all $\lambda \in (1, \sqrt{2}]$ it has square integrable density. Garsia in [7, Theorem 1.8] showed examples of such λ leading to an absolutely continuous distribution. Namely such $\lambda \in (1, 2)$ are the roots of monic polynomials P with integer coefficients such that $|P(0)| = 2$ and $\lambda\prod_{|\alpha_i|>1}|\alpha_i| = 2$, where $\{\alpha_i\}$ are the remaining roots of P.

There are known (see [4], [5]) countable many instances of $\lambda \in (1, 2]$ where this distribution is singular. We will denote by φ_λ the characteristic function of $S(\lambda)$. Following [4] we know that the values λ such that $\varphi_\lambda(t)$ does not tend to zero as $t \to \infty$ are the so called Pisot or PV- numbers. Recall that those are sole roots of such monic irreducible polynomials P with integer coefficients having the property that all other roots have absolute values less than 1. Obviously we must then have $P(0) = 1$. For such $\lambda's$ the related distribution is singular (by the Riemann–Lebesgue Lemma). Examples of such numbers are the so called 'golden ratio' $\left(1+\sqrt{5}\right)/2$ or the so called 'silver ratio' $1+\sqrt{2}$. Moreover following [14] one knows that PV numbers are the only numbers $\lambda \in (1, 2]$ for which φ_λ does not tend to zero. Of course singularity of the distribution of $S(\lambda)$ can occur for λ not being a PV number.

For $\lambda > 2$ it is known that the distribution of S is singular [10].

To simplify notation we will write $\operatorname{supp} X$, where X is a random variable meaning $\operatorname{supp} P_X$, where P_X denotes a distribution of X. Similarly $X * Y$ denotes a random variable whose distribution is a convolution of distributions of X and Y.

We have a simple Lemma.

Lemma 1 *i)* $\operatorname{supp}(S(\lambda)) \subset \left[-\frac{1}{\lambda-1}, \frac{1}{\lambda-1}\right]$.

In particular:

ia) if $\lambda = 2$ *then* $S \sim U([-1,1])$ *and*

ib) if $\lambda = 3$ *then* $\operatorname{supp}(S + 1/2)$ *is equal to the Cantor set.*

In general if λ *is a positive integer then* $\operatorname{supp}(S + 1/(\lambda-1))$ *consist of all numbers of the form* $\sum_{j\geq 1} r_j \lambda^{-j}$ *where* $r_j \in \{0,2\}$. *Moreover the distribution of* $(S(\lambda) + 1/(\lambda-1))$ *is 'uniform' on this set.*

ii) $\forall k \geq 1$:

$$\varphi_\lambda(\lambda^k t) = \varphi_\lambda(t) \prod_{j=0}^{k-1} \cos\left(\lambda^j t\right). \tag{1}$$

iii) $\forall k \geq 1 : S(\lambda) \sim \sum_{i=1}^{k} \lambda^{i-1} S_i(\lambda^k)$, *where* $S_i(\tau)$ $(i = 1, \ldots, k)$ *are i.i.d. random variables each having distribution* $S(\tau)$. *Consequently* $\varphi_\lambda(t) = \prod_{j=1}^{k} \varphi_{\lambda^k}\left(\lambda^{j-1} t\right)$.

iv) Let us denote $m_n(\lambda) = ES(\lambda)^n$. *Then* $\forall n \geq 1 : m_{2n-1}(\lambda) = 0$ *and*

$$m_{2n}(\lambda) = \frac{1}{\lambda^{2kn} - 1} \sum_{j=0}^{n-1} \binom{2n}{2j} m_{2j}(\lambda) W_{2(n-j)}^{(k)}(\lambda),$$

with $m_0 = 1$, *where* $W_n^{(1)} = 1$, $W_n^{(k)}(\lambda) = \frac{d^n}{dt^n} (\prod_{j=1}^{k} \cosh(\lambda^{j-1} t))\Big|_{t=0}$ $= \frac{1}{2^{k-1}} \sum_{i_1=0,\ldots,i_{k-1}=0}^{1} (1 + \sum_{j=1}^{k} (2i_j - 1)\lambda^j)^{2n}$.

In particular we have:

$$m_{2n}(\lambda) = \frac{1}{\lambda^{2n} - 1} \sum_{j=0}^{n-1} m_{2j}(\lambda) \binom{2n}{2j}, \tag{2}$$

$$m_{2n}(\lambda) = \frac{1}{\lambda^{4n} - 1} \sum_{j=0}^{n-1} \binom{2n}{2j} m_{2j}(\lambda) \sum_{l=0}^{2(n-j)} \binom{2(n-j)}{2l} \lambda^{2l}. \tag{3}$$

v) $\forall k \geq 1 : m_{2k}(\lambda) = \frac{-1}{\lambda^{2k}-1} \sum_{j=0}^{k-1} \binom{2k}{2j} \lambda^{2j} E_{2(k-j)} m_{2j}(\lambda)$, *where* E_k *denotes* $k-$*th Euler number.*

Proof. i) First of all notice that $\frac{1}{\lambda-1} = \sum_{n\geq 1} 1/\lambda^n$, hence $S + \frac{1}{\lambda-1} = \sum_{n\geq 1} \frac{1}{\lambda^n}(X_n + 1)$. Now since $P(X_n + 1 = 0) = P(X_n + 1 = 2) = 1/2$ we see that $\operatorname{supp}(S+\frac{1}{\lambda-1})\subset[0,\frac{2}{\lambda-1}]$. Notice also that if $\lambda=2$ then $S+1=2\sum_{n\geq1}\frac{1}{2^n}Y_n$, where $P(Y_n=0)=P(Y_n=1)=1/2$. In other words $(S+1)/2$ is a number chosen from $[0,1]$ with 'equal chances' that is $(S+1)/2$ has uniform distribution on $[0,1]$.

When $\lambda=3$ we see that $S+1/2$ is a number that can be written with the help of '0' and '2' in ternary expansion. In other words $S+1/2$ is number drawn from the Cantor set with equal chances. For λ an integer we argue in the similar way.

ii) We have $\varphi_S(\lambda^k t)=\prod_{n\geq1}\cos(\lambda^k t\frac{1}{\lambda^n})=\varphi_S(t)\prod_{j=0}^{k-1}\cos\left(\lambda^j t\right)$.

iii) Fix integer k. Notice that we have:

$$S(\lambda)=\sum_{n\geq1}\lambda^{-n}X_n=$$

$$\sum_{j\geq1}\lambda^{-kj}X_{kj}+\sum_{j\geq1}\lambda^{-kj+1}X_{kj-1}+\ldots+\sum_{j\geq1}\lambda^{-kj+k-1}X_{kj-k+1}=$$

$$\sum_{m=1}^{k}\lambda^{m-1}\sum_{j\geq1}\left(\lambda^k\right)^{-j}X_{kj-m+1}.$$

Now since by assumption all X_i are i.i.d. we deduce that $S_i(\lambda^k)$ are i.i.d. random variables with distribution defined by $\varphi_{\lambda^k}(t)$. Hence we have $\varphi_\lambda(t)=\prod_{j=1}^{k}\varphi_{\lambda^k}(\lambda^{j-1}t)$.

iv) First of all we notice that $\varphi_S(t)$ is an even function hence all derivatives of odd order at zero are equal to 0. Secondly let $\psi_\lambda(t)$ denote the moment generating function of $S(\lambda)$. It is easy to notice that $\psi_\lambda(s)=\varphi_{S(\lambda)}(-is)$. Let us denote $m_{2n}(\lambda)=\psi_\lambda^{(2n)}(0)$. Basing on the elementary formula

$$\cosh(\alpha)\cosh(\beta)=\frac{1}{2}(\cosh(\alpha+\beta)+\cosh(\alpha-\beta)),$$

we can easily obtain by induction the following identity:

$$\prod_{j=0}^{k-1}\cosh(\lambda^j t)=\frac{1}{2^{k-1}}\sum_{i_1=0,\ldots,i_{k-1}=0}^{1}\cosh(t(1+\sum_{j=1}^{k}(2i_j-1)\lambda^j)).$$

Since $\left.(\cosh\alpha t)^{(2n)}\right|_{t=0}=\alpha^{2n}$, we have

$$\left.\left(\prod_{j=0}^{k-1}\cosh(\lambda^j t)\right)^{(2n)}\right|_{t=0}=$$

$$\frac{1}{2^{k-1}}\sum_{i_1=0,\ldots,i_{k-1}=0}^{1}\left.\left(\cosh\left(t\left(1+\sum_{j=1}^{k}(2i_j-1)\lambda^j\right)\right)\right)^{(2n)}\right|_{t=0}=W_{2n}^{(k)}(\lambda).$$

Now using Leibnitz formula for differentiation applied to (1) we get

$$f^{(2n)}(\lambda^k t)\lambda^{2kn} = \sum_{j=0}^{2n} \binom{2n}{j} (\prod_{i=0}^{k-1} \cosh \lambda^i t)^{(j)} f^{(2n-j)}(t) .$$

Setting $t = 0$ and using the fact that all derivatives of both f and $\cosh t$ of odd order at zero are zeros we get the desired formula.

v) We use the result of [18] that states that for each N, the inverse of the lower triangular matrix of degree $N \times N$ with (i,j) entry $\binom{2i}{2j}$ is the lower triangular matrix with $(i,j)th$ entry equal to $\binom{2i}{2j} E_{2(i-j)}$. ■

Remark 1 *Formula (2) is known in a slightly different form. It appeared in [8], [6] and [1].*

Remark 2 *Notice that polynomials* $\left\{ W_n^{(k)}(\lambda) \right\}_{k,n \geq 1}$ *satisfy the following recursive relationship for* $k > 1$ *:*

$$W_n^{(k)}(\lambda) = \sum_{j=0}^{n} \binom{2n}{2j} \lambda^{2j} W_j^{(k-1)}(\lambda) ,$$

with $W_n^{(1)}(\lambda) = 1$. *Hence its generating functions* $\Theta_k(t, \lambda)$ *satisfy the following relationship*

$$\Theta_k(t, \lambda) = \Theta_{k-1}(\lambda t, \lambda) \cosh t,$$

where we have have denoted: $\Theta_k(t, \lambda) = \sum_{n \geq 0} \frac{t^{2n}}{(2n)!} W_n^{(k)}(\lambda)$.

Remark 3 *Notice that the above mentioned lemma provides an example of two singular distributions whose convolution is a uniform distribution. Namely we have* $S(4) * 2S(4) = S(2)$. *Similarly we have* $S(2) = S(8) * 2S(8) * 4S(8)$ *or* $S(2) = S(2^k) * \ldots * 2^{k-1} S(2^k)$. *The first example was already noticed by Kersher and Wintner in [10, equation 22a].*

Remark 4 *We can deduce even more from these examples, namely, following the result of Kersher [9, p. 451], that characteristic functions* $\varphi_n(t)$ *of* $S(n)$ *(where n is an integer* > 2*) do not tend to zero as* $t \to \infty$. *Thus since we have* $\varphi_4(t)\varphi_4(2t) = \sin t/t$ *and* $\varphi_8(t)\varphi_8(2t)\varphi_8(4t) = \sin t/t$ *we deduce that if* $t_k \to \infty$ *is a sequence such that* $|\varphi_4(t_k)| > \varepsilon > 0$ *for suitable* ε *then* $\varphi_4(2t_k) \to 0$. *Similarly if* $t_k \to \infty$ *such that* $|\varphi_8(t_k)| > \varepsilon > 0$ *then* $\varphi_8(2t_k)\varphi_8(4t_k) \to 0$. *Similar observations can be made can be made in more general situation. It is known from the papers of Erdős [4], [5] the situation that* $|\varphi_\lambda(t_k)| > \varepsilon > 0$ *for some sequence* $t_k \to \infty$ *occurs when* λ *is a Pisot number (briefly PV-number). On the other hand as it is known roots of Pisot numbers are not*

Pisot, hence using the above mentioned result of Salem that $|\varphi_{\lambda^{1/k}}(t)| \to 0$ *as* $t \to \infty$, *where* λ *is some PV number and* $k > 1$ *any integer. But we have* $\varphi_{\lambda^{1/k}}(t_n) = \prod_{j=1}^{k} \varphi_\lambda\left(\lambda^{(j-1)/k} t_n\right) \to 0$, *where* $t_n \to \infty$ *is such a sequence that* $|\varphi_\lambda(t_n)| > \varepsilon > 0$.

Remark 5 *One knows that if* $\lambda = q/p$ *where* p *and* q *are relatively prime integers and* $p > 1$ *then* $\varphi_\lambda(t) = O((\log|t|)^{-\gamma})$ *where* $\gamma = \gamma(p,q) > 0$ *as* $t \to \infty$ *(see [9, equation (3)]). Besides we know that the distribution of* $S(\lambda)$ *is singular. Hence from our considerations it follows that if* $\lambda = (q/p)^{1/k}$ *for some integer* k, *then* $\varphi_\lambda(t) = O((\log|t|)^{-k\gamma})$. *Is it also singular?*

3 Moment sequences

To give a connection of certain moment sequences with some known integer sequences let us remark that some moment sequences satisfy the following identities:

Remark 6 *i)*

$$\begin{aligned} 9^n m_{2n}(3) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(3), \\ 81^n m_{2n}(3) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(3)(2^{4(n-j)-1} + 2^{2(n-j)-1}). \end{aligned}$$

ii)

$$\begin{aligned} 5^n m_{2n}\left(\sqrt{5}\right) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(\sqrt{5}), \\ 25^n m_{2n}\left(\sqrt{5}\right) &= \sum_{j=0}^{n} \binom{2n}{2j} m_{2j}(\sqrt{5}) 4^{n-j} L_{2(n-j)}/2, \end{aligned}$$

where L_n *denotes* $n-$*th Lucas number, defined below.*

Proof. i) The first assertion is a direct application of (2) while in proving the second one we use (3) and the fact that $\sum_{j=0}^{n}\binom{2n}{2j}9^j = 4^n(4^n+1)/2$ which is elementary to prove by generating function method and which is known (see [21] seq. no. A026244). ii) Again the first statement follows (2) while the second follows (3) and the fact that $\sum_{j=0}^{n}\binom{2n}{2j}5^j = 4^n T_n(3/2)$, where T_n denotes the Chebyshev polynomial of the first kind. (This identity is also elementary to prove by generating function method and which is known (see

[21] seq. no. A099140). Further we use the fact that $T_n(3/2) = L_{2n}/2$. ([21], seq. no. A005248). ■

We also have the following Lemma.

Lemma 2 $\forall n \geq 1, k \geq 2$:

$$m_{2n}(\lambda) = \sum_{i_1,\ldots,i_k=0}^{n} \frac{(2n)!}{(2i_1)!\ldots(2i_k)!} \lambda^{2(i_2+2i_3\ldots(k-1)i_k)} \prod_{j=1}^{k} m_{2i_j}(\lambda^k). \qquad (4)$$

In particular:

$$m_{2n}(\lambda) = \sum_{j=0}^{n} \binom{2n}{2j} \lambda^{2j} m_{2j}(\lambda^2) m_{2n-2j}(\lambda^2), \qquad (5)$$

$$m_{2n}(\lambda) = \sum_{\substack{i,j=0 \\ i+j\leq n}} \frac{(2n)!}{(2i)!(2j)!(2(n-i-j))!} \times \qquad (6)$$
$$\lambda^{2i}\lambda^{4j} m_{2j}(\lambda^3) m_{2i}(\lambda^3) m_{2n-2i-2j}(\lambda^3).$$

Proof. (4) follows directly Lemma 1, iii). ■

As a corollary we get the following four observations:

Corollary 3 *We have:*

i) $\forall n \geq 1 : 4^n = \sum_{j=0}^{n} \binom{2n+1}{2j+1}$ *and* $1 = \sum_{j=0}^{n} \binom{2n+1}{2j+1} 4^j E_{2(n-j)}$.

ii) $S\left(\sqrt{2}\right)$ *has density*

$$g(x) = \begin{cases} \sqrt{2}/4 & if \quad |x| \leq \sqrt{2}-1, \\ \sqrt{2}(\sqrt{2}+1-|x|)/8 & if \quad \sqrt{2}-1 \leq |x| \leq \sqrt{2}+1, \\ 0 & if \quad |x| > 1+\sqrt{2}. \end{cases}$$

iii) Let us denote $\delta_n = (\sqrt{2}+1)^n$, *then*

$$m_{2n}(\sqrt{2}) = \left(\delta_{2n+2} - \delta_{2n+2}^{-1}\right)/(4\sqrt{2}(n+1)(2n+1)).$$

Proof. Since for $\lambda = 2$, the random variable S is distributed as $U[-1,1]$ and its moments are equal to $ES^{2n} = \frac{1}{2n+1}$. Now we use Lemma 1 iii) and iv).

ii) From the proof of Lemma 2 it follows that $S\left(\sqrt{2}\right) \sim S\left(2\right) + \sqrt{2}S\left(2\right)$. Now keeping in mind that $S\left(2\right) \sim U(-1,1)$ we deduce that $g(x) = \frac{\sqrt{2}}{8}\int_{-1}^{1} h(x-t)dt$, where $h\left(x\right) = \begin{cases} \frac{\sqrt{2}}{4} & if \quad |x| \leq \sqrt{2}, \\ 0 & if \quad otherwise. \end{cases}$. iii) By straightforward calculations we get $m_{2n}(\sqrt{2}) = 2\int_{0}^{\sqrt{2}+1} x^{2n} g(x)dx = \frac{\sqrt{2}}{2}\int_{0}^{\sqrt{2}-1} x^{2n}dx + \frac{\sqrt{2}}{4}\int_{\sqrt{2}-1}^{\sqrt{2}+1} x^{2n}(\sqrt{2}+1-x)dx$. ■

Remark 7 *Let us apply formulae: (5), (2), (3) and observe by direct calculation that* $2\sum_{j=0}^{n}\binom{2n}{2j}2^{j} = (1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}$. *We get the following identities:* $\forall n \geq 1:$

$$\begin{aligned} m_{2n}(\sqrt{2}) &= \sum_{j=0}^{n}\binom{2n}{2j}\frac{2^{j}}{(2n-2j+1)(2j+1)} \\ &= \frac{1}{2^{n}-1}\sum_{j=0}^{n-1}\binom{2n}{2j}m_{2j}(\sqrt{2}) \\ &= \frac{1}{4^{n}-1}\sum_{j=0}^{n-1}\binom{2n}{2j}m_{2j}(\sqrt{2})\tau_{2(n-j)}, \end{aligned}$$

where $\tau_{n} = (1+(-1)^{n})(\delta_{n}+\delta_{n}^{-1})/4$.

Now let us recall the definition of the so called Pell and Pell–Lucas numbers. Using sequence δ_n the Pell numbers $\{P_n\}$ and the Pell–Lucas numbers $\{Q_n\}$ are defined

$$P_{n} = (\delta_{n} + (-1)^{n+1}\delta_{n}^{-1})/(2\sqrt{2}), \tag{7}$$
$$Q_{n} = \delta_{n} + (-1)^{n}\,\delta_{n}^{-1}, \tag{8}$$

where δ_n is defined in 3,iii).

Using these definitions we can rephrase the assertions of Corollary 3 and Remark 7 adding to recently discovered ([15], [2]) new identities satisfied by the Pell and the Pell-Lucas numbers and of course a probabilistic interpretation of Pell numbers.

Remark 8 *i)* $m_{2n}(\sqrt{2}) = \frac{P_{2n+2}}{(2n+2)(2n+1)}$, $\tau_{2n} = Q_{2n}/2$.

ii) $\forall n \geq 1:$

$$P_{2n+2} = \sum_{j=0}^{n}\binom{2n+2}{2j+1}2^{j}, \tag{9}$$

$$Q_{2n} = 2\sum_{j=0}^{n}\binom{2n}{2j}2^{j}, \tag{10}$$

$$2^{n-1}P_{2n} = \sum_{j=0}^{n}\binom{2n}{2j}P_{2j}, \tag{11}$$

$$2^{2n-1}P_{2n} = \sum_{j=0}^{n}\binom{2n}{2j}P_{2j}Q_{2(n-j)}, \tag{12}$$

$$\sum_{j=0}^{n}\binom{2n}{2j}(1+\sqrt{2})^{2j} = 2^{n-1} + 2^{n-2}Q_{2n} + 2^{n-1}\sqrt{2}P_{2n}. \tag{13}$$

Proof. Only the last statement requires justification. First we find that $\sum_{j=0}^{n} \binom{2n}{2j} Q_{2j} = 2^n(1 + Q_{2n}/2)$ using (10). Then we use (7), (8) and (11). ■

References

[1] Arnold, Barry C. The generalized Cantor distribution and its corresponding inverse distribution. *Statist. Probab. Lett.* **81** (2011), no. 8, 1098–1103. MR2803750 (2012f:60051)

[2] Benjamin, Arthur T.; Plott, Sean S.; Sellers, James A. Tiling proofs of recent sum identities involving Pell numbers. *Ann. Comb.* 12 (2008), no. **3**, 271–278. MR2447257 (2009g:05017)

[3] Calkin, Neil J.; Davis, Julia; Delcourt, Michelle; Engberg, Zebediah; Jacob, Jobby; James, Kevin. Discrete Bernoulli convolutions: an algorithmic approach toward bound improvement. *Proc. Amer. Math. Soc.* **139** (2011), no. 5, 1579–1584. MR2763747 (2012d:05049)

[4] Erdös, Paul. On a family of symmetric Bernoulli convolutions. *Amer. J. Math.* **61**, (1939). 974–976. MR0000311 (1,52a)

[5] Erdös, Paul. On the smoothness properties of a family of Bernoulli convolutions. *Amer. J. Math.* **62**, (1940). 180–186. MR0000858 (1,139e)

[6] Hosking, J. R. M. Moments of order statistics of the Cantor distribution. *Statist. Probab. Lett.* **19** (1994), no. 2, 161–165. MR1256706 (94k:62086)

[7] Garsia, Adriano M. Arithmetic properties of Bernoulli convolutions. *Trans. Amer. Math. Soc.* **102,** (1962), 409–432. MR0137961 (25 #1409)

[8] Lad, F. R.; Taylor, W. F. C. The moments of the Cantor distribution. *Statist. Probab. Lett.* **13** (1992), no. 4, 307–310. MR1160752

[9] Kershner, Richard. On Singular Fourier-Stieltjes Transforms. *Amer. J. Math.* **58** (1936), no. 2, 450–452. MR1507167

[10] Kershner, Richard; Wintner, Aurel. On Symmetric Bernoulli Convolutions. *Amer. J. Math.* **57** (1935), no. 3, 541–548. MR1507093

[11] Morrison, Kent E. Cosine products, Fourier transforms, and random sums. *Amer. Math. Monthly* **102** (1995), no. 8, 716–724. MR1357488 (96h:42001)

[12] Morrison, Kent E. Random Walks with decreasing steps, 1998, http://www.calpoly.edu/~kmorriso/Research/RandomWalks.pdf

[13] Jessen, Břrge; Wintner, Aurel. Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.* **38** (1935), no. 1, 48–88. MR1501802

[14] Salem, R. A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan. *Duke Math. J.* **11**, (1944). 103–108. MR0010149 (5,254a)

[15] S. F. Santana and J. L. Diaz–Barrero, Some properties of sums involving Pell numbers. *Missouri Journal of Mathematical Sciences* **18**(1), (2006), 33–40

[16] Solomyak, Boris. On the random series $\sum\pm\lambda\sp n$ (an Erdős problem). *Ann. of Math.* (2) **142** (1995), no. 3, 611–625. MR1356783 (97d:11125)

[17] Peres, Yuval; Solomyak, Boris. Absolute continuity of Bernoulli convolutions, a simple proof. *Math. Res. Lett.* **3** (1996), no. 2, 231–239. MR1386842 (97f:28006)

[18] Szabłowski, Paweł J. A few remarks on Euler and Bernoulli polynomials and their connections with binomial coefficients and modified Pascal matrices. *Math. Aeterna* 4 (2014), no. 1-2, 83–88. MR3176129 , *http://arxiv.org/abs/1307.0300*

[19] Wintner, Aurel. A Note on the Convergence of Infinite Convolutions. *Amer. J. Math.* **57** (1935), no. 4, 839. MR1507117

[20] Wintner, Aurel. On Convergent Poisson Convolutions. *Amer. J. Math.* **57** (1935), no. 4, 827–838. MR1507116

[21] The On-Line Encyclopedia of Integer Sequences® (OEIS®), http://oeis.org/.

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