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Application of differential transformation method to nonlinear q-symmetric damped systems

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Abstract

In this paper, we first induce the recursive formula of q-symmetric polynomials and prove the product rule of two q-symmetric polynomials at 0.Then differential transformation method can be applied to find approximate solution of nonlinear q-symmetric difference equation. Last but not least, the approximate solutions are also contrasted with the numerical solutions to verify their accuracy.

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1 Introduction

Differential transformation method based on the Taylor series expansion is a semi-numerical-analytic method for solving nonlinear ordinary differential problems. It was first proposed by Zhou [1] in 1986 for the solution of linear and nonlinear initial value problems that appear in electrical circuits. Besides, it has been used to obtain numerical and analytical solutions of various equations [2-10]. Recently, there are many studies on the applications of it to the damped systems in literature. For instance, Jang and Chen [11] employed it to investigate the response of a strongly nonlinear damped system. Kuo and Lo [12] applied it to analyze the response of a damped system with high nonlinearity. Moreover, Hsuan-Ku Liu [13] extended the use of differential transformation method to the case of strongly nonlinear damped q-difference equations.

With the development of q-analysis, q-symmetric analysis motivates the interests of many scientists. Actually, Artur M.C. Brito da Cruz [14] has introduced a wealth of knowledge about q-symmetric variational calculus, which laid a good foundation to the continue study. On the other hand, q-symmetric quantum calculus does play an important role in various areas, such as conformal quantum mechanics [15] and so on. As noticed in [15], the q-symmetric derivative let the q-exponential function have unique properties. We believe that the q-symmetric derivative has, in general, better properties than the h-derivative $\frac{f(t+h)-f(t)}{h}$ and the q-derivative $\frac{f(qt)-f(t)}{(q-1)t}$. However, to the best of our knowledge, there are no papers concerned with the theory of approximate solutions for nonlinear q-symmetric difference equations.

This dissertation aims to extend the differential transformation method to the nonlinear damped q-symmetric difference equation, which defined on

$$\overline{q^{\mathbb{N}}} = \{q^n | n \in \mathbb{N}\} \bigcup \{0\}$$

for $q \in [0, 1]$. To put it precisely, the strongly nonlinear damped q-symmetric difference equation is described as

$$\widetilde{d}_q^2 x + (2\gamma + \epsilon\gamma_1 x)\widetilde{d}_q x + \Omega^2 x + x^2 = 0. \tag{1}$$

with $x(0) = a$, $\widetilde{d}_q x(0) = b$. Where, γ and γ_1 represent linear damping parameters, ϵ is nonlinear parameter, Ω is the frequency of underdamped motion.

For convenience, we assume that $\sum_{n=i}^{j} A_n = 0$ and $\prod_{n=i}^{j} A_n = 1$ for this.

2 q-symmetric polynomials

It is known that Hilger [16] proposed the calculus of time scales to make a connection between discrete and continuous analysis. However, as to the same problem, different derivatives result in different effects on literature. This section will study q-symmetric polynomials on the time scale $\overline{q^{\mathbb{N}}}$ for $q \in [0, 1]$ on which q-symmetric derivative is defined.

Let $q \in [0, 1]$ and let I be an interval (bounded or unbounded) of $\mathbb{R}$ containing 0. We will denote by I_q the set

$$I_q := qI := \{qx : x \in I\}.$$

Note that

$$I_q \subset I.$$

Definition 2.1 ([19]) *Let f be a real function defined on $\overline{q^{\mathbb{N}}}$. The q-symmetric difference operator of f is defined by*

$$\widetilde{d}_q[f](t) = \frac{f(qt) - f(q^{-1}t)}{(q - q^{-1})t},$$

if $t \in \overline{q^{\mathbb{N}}} \setminus \{0\}$, *and* $\widetilde{d}_q[f](0) := f'(0)$, *provided* f *is differentiable at* 0. *We usually call* $\widetilde{d}_q[f]$ *the q-symmetric derivative of* f.

Definition 2.2 ([17]) *On a time scale* $\mathbb{T}$, *the generalized polynomials* $h_k(\cdot, t_0) : \mathbb{T} \to \mathbb{R}$ *are defined recursively as follows:*

$$h_0(t,s) = 1, \qquad h_{k+1} = \int_s^t h_k(\tau, s)\widetilde{d}_q\tau,$$

where

$$k = 0, 1, 2, \cdots .$$

Definition 2.3 ([15]) *Assume that* $f : \overline{q^{\mathbb{N}}} \to \mathbb{R}$ *is a function. The function* F *which is pre-differentiable in* $\overline{q^{\mathbb{N}}}$ *such that*

$$\widetilde{d}_q F(t) = f(t), \forall t \in \overline{q^{\mathbb{N}}}$$

is called a pre-antiderivative of f.

We define the indefinite integral of the function f *by*

$$\int f(t)\widetilde{d}_q t = F(t) + C,$$

where C *is an arbitrary constant. Moreover, the definite integral is defined as*

$$\int_a^b f(t)\widetilde{d}_q t = \int_0^b f(t)\widetilde{d}_q t - \int_0^a f(t)\widetilde{d}_q t,$$

Let $a, b \in I$ *and* $a < b$. *For* $f : \overline{q^{\mathbb{N}}} \longrightarrow \mathbb{R}$ *and for* $q \in]0, 1[$ *the q-symmetric integral of* f *from* a *to* b *is given by*

$$\int_a^b f(t)\widetilde{d}_q t = F(b) - F(a), \forall a, b \in \overline{q^{\mathbb{N}}}.$$

Lemma 2.4 *If* t^n *has q-symmetric derivative on* $\overline{q^{\mathbb{N}}}$, *then*

$$\int t^n \widetilde{d}_q t = \frac{t^{n+1}}{A_n} + C,$$

Where

$$A_n = \sum_{j=0}^{n} q^{n-j} q^{-j}, n = 0, 1, 2, \cdots .$$

and C *is an arbitrary constant.*

Proof. *Let* $F(t)=\frac{t^{n+1}}{A_n}+C$, *it is obviously that* $F(t)$ *is q-symmetric differentiable. Then, we obtain*

$$\begin{aligned}\widetilde{d}_qF(t) &= \frac{\frac{(qt)^{n+1}}{A_n}+C-\frac{(q^{-1}t)^{n+1}}{A_n}-C}{qt-q^{-1}t}\\ &= \frac{(qt)^{n+1}-(q^{-1}t)^{n+1}}{A_n(qt-q^{-1}t)}\\ &= t^n.\end{aligned}$$

□

For convenience, we assume $\sum_{n=i}^{j}A_n=0$ and $\prod_{n=i}^{j}A_n=1$ for $i>j$.

Theorem 2.5 *On the time scale* $\overline{q^{\mathbb{N}}}$, *Based on the above knowledge, after computing the recurrence relation, the q-symmetric polynomials are described as*

$$h_k(t,s)=\frac{t^k-s^k}{\prod_{n=0}^{k-1}A_n}-\frac{s(t^{k-1}-s^{k-1})}{\prod_{n=0}^{k-2}A_n}-\sum_{i=2}^{k-1}B_{i-1}\frac{s^i(t^{k-i}-s^{k-i})}{\prod_{n=0}^{k-i-1}A_n},\ \ k=0,1,2,\cdots, \tag{2}$$

where

$$A_n=\sum_{j=0}^{n}q^{n-j}q^{-j}, n=0,1,2,\cdots,$$

$$B_1=\frac{1}{q+q^{-1}}-1,$$

$$B_{i-1}=\frac{1}{\prod_{n=0}^{i-1}A_n}-\frac{1}{\prod_{n=0}^{i-2}A_n}-\frac{B_1}{\prod_{n=0}^{i-3}A_n}-\cdots-\frac{B_{i-3}}{\prod_{n=0}^{1}A_n}-B_{i-2}.\ \ i=3,4,5,\cdots.$$

on $\overline{q^{\mathbb{N}}}$.

Proof. *Clearly, according to the definition* 2.2 *and lemma* 2.1, *we have*

$$\begin{aligned}h_1(t,s) &= t-s,\\ \widetilde{d}_qh_1(t,s) &= \frac{qt-s-q^{-1}t+s}{(q-q^{-1})t}=1=h_0(t,s),\\ h_2(t,s) &= \frac{t^2-s^2}{A_1}-\frac{s(t-s)}{A_0}=\frac{t^2-s^2}{q+q^{-1}}-s(t-s),\\ \widetilde{d}_qh_2(t,s) &= \frac{1}{(q-q^{-1})t}\left(\frac{(qt)^2-s^2}{q+q^{-1}}-s(qt-s)-\frac{(q^{-1}t)^2-s^2}{q+q^{-1}}+s(q^{-1}t-s)\right)=h_1(t,s).\end{aligned}$$

Next, we assume (2) holds when $k = m, m \in \mathbb{N}$, then

$$\begin{aligned}
\widetilde{d}_q h_{m+1}(t,s) &= \frac{1}{(q-q^{-1})t}\Big(h_{m+1}(qt,s) - h_{m+1}(q^{-1}t,s)\Big)\\
&= \frac{1}{(q-q^{-1})t}\Bigg(\frac{(qt)^{m+1}-s^{m+1}}{\prod_{n=0}^{m} A_n} - \frac{s((qt)^m - s^m)}{\prod_{n=0}^{m-1} A_n} - \sum_{i=2}^{m} B_{i-1}\frac{s^i((qt)^{m+1-i} - s^{m+1-i})}{\prod_{n=0}^{k-i-1} A_n}\\
&\quad -\frac{(q^{-1}t)^{m+1}-s^{m+1}}{\prod_{n=0}^{m} A_n} + \frac{s((q^{-1}t)^m - s^m)}{\prod_{n=0}^{m-1} A_n} + \sum_{i=2}^{m} B_{i-1}\frac{s^i((q^{-1}t)^{m+1-i} - s^{m+1-i})}{\prod_{n=0}^{m-i} A_n}\Bigg)\\
&= \frac{1}{(q-q^{-1})t}\Bigg(\frac{(q^{m+1}-(q^{-1})^{m+1})t^{m+1}}{\prod_{n=0}^{m} A_n} - \frac{s((qt)^m - (q^{-1}t)^m)}{\prod_{n=0}^{m-1} A_n}\\
&\quad -\sum_{i=2}^{m} B_{i-1}\frac{s^i((qt)^{m-i+1} - (q^{-1}t)^{m-i+1})}{\prod_{n=0}^{m-i} A_n}\Bigg)\\
&= \frac{t^m}{\prod_{n=0}^{m-1} A_n} - \frac{st^{m-1}}{\prod_{n=0}^{m-2} A_n} - \sum_{i=2}^{m} B_{i-1}\frac{s^i((qt)^{m-i+1} - (q^{-1}t)^{m-i+1})}{\prod_{n=0}^{m-i} A_n (q-q^{-1})t}\\
&= \frac{t^m}{\prod_{n=0}^{m-1} A_n} - \frac{st^{m-1}}{\prod_{n=0}^{m-2} A_n} - \sum_{i=2}^{m} B_{i-1}\frac{s^i t^{m-i}}{\prod_{n=0}^{m-i-1} A_n}\\
&= \frac{t^m}{\prod_{n=0}^{m-1} A_n} - \frac{st^{m-1}}{\prod_{n=0}^{m-2} A_n} - \sum_{i=2}^{m-1} B_{i-1}\frac{s^i t^{m-i}}{\prod_{n=0}^{m-i-1} A_n} - B_{m-1}s^m\\
&= \frac{t^m}{\prod_{n=0}^{m-1} A_n} - \frac{st^{m-1}}{\prod_{n=0}^{m-2} A_n} - \sum_{i=2}^{m-1} B_{i-1}\frac{s^i t^{m-i}}{\prod_{n=0}^{m-i-1} A_n}\\
&\quad -\frac{s^m}{\prod_{n=0}^{m-1} A_n} + \frac{s^m}{\prod_{n=0}^{m-2} A_n} + \frac{B_1 s^m}{\prod_{n=0}^{m-3} A_n} + \frac{B_2 s^m}{\prod_{n=0}^{m-4} A_n} + \cdots + s^m B_{m-2}\\
&= \frac{t^m - s^m}{\prod_{n=0}^{m-1} A_n} - \frac{s(t^{m-1}-s^{m-1})}{\prod_{n=0}^{m-2} A_n} - B_1\frac{s^2(t^{m-2}-s^{m-2})}{\prod_{n=0}^{m-3} A_n}\\
&\quad -B_2\frac{s^3(t^{m-3}-s^{m-3})}{\prod_{n=0}^{m-4} A_n} - \cdots - B_{m-2}s^{m-1}t + s^m B_{m-2}\\
&= \frac{t^m - s^m}{\prod_{n=0}^{m-1} A_n} - \frac{s(t^{m-1}-s^{m-1})}{\prod_{n=0}^{m-2} A_n} - \sum_{i=2}^{m-1} B_{i-1}\frac{s^i(t^{m-i}-s^{m-i})}{\prod_{n=0}^{m-i-1} A_n} = h_m(t,s). \qquad \square
\end{aligned}$$

Agarwal and Bohner [18] gave the Taylor formula for functions on a general time scale. On $\overline{q^{\mathbb{N}}}$ the Taylor formula is written as

Theorem 2.6 *Let $n \in \mathbb{N}$. Suppose f is n times continuously differentiable on $\overline{q^{\mathbb{N}}}$. Let $\alpha, t \in \overline{q^{\mathbb{N}}}$. Then the Taylor formula of f near $x = \alpha$ is given by*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha)\widetilde{d}_q^{\,k} f(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau))\widetilde{d}_q^{\,n} f(\tau)\widetilde{d}_q\tau.$$

To develop an approximate solution of (1), the product rule of two q-polynomials at 0 is derived.

Lemma 2.7 *Let $h_i(t,0)$ and $h_j(t,0)$ be two q-symmetric polynomials at zero. Then the product rule of two q-symmetric polynomials is*

$$h_i(t,0)h_j(t,0) = \frac{(q^{-2(i+1)};q^{-2})_j}{(q^{-2};q^{-2})_j}h_{i+j}(t,0),$$

where

$$(q^{-2(i+1)};q^{-2})_j = \prod_{n=0}^{j-1} q^{n+i}(1-q^{-2(i+1)}q^{-2n}).$$

Proof. By

$$h_{i+j}(t,0) = \frac{t^{i+j}}{\prod_{n=0}^{i+j-1} A_n} = \frac{t^i}{\prod_{n=0}^{i-1} A_n}\frac{t^j}{\prod_{n=0}^{j-1} A_n}\frac{\prod_{n=0}^{j-1} A_n}{\prod_{n=i}^{i+j-1} A_n},$$

then we have

$$\begin{aligned}
h_i(t,0)h_j(t,0) &= \frac{\prod_{n=i}^{i+j-1} A_n}{\prod_{n=0}^{j-1} A_n}h_{i+j}(t,0) \\
&= \frac{\prod_{n=i}^{i+j-1}\sum_{u=0}^{n} q^{n-u}q^{-u}}{\prod_{n=0}^{j-1}\sum_{u=0}^{n} q^{n-u}q^{-u}}h_{i+j}(t,0) \\
&= \prod_{n=0}^{j-1}\frac{\sum_{u=0}^{n+i} q^{n-u+i}q^{-u}}{\sum_{u=0}^{n} q^{n-u}q^{-u}}h_{i+j}(t,0) \\
&= \prod_{n=0}^{j-1}\frac{q^{n+i}-q^{-(n+i+2)}}{q^{n}-q^{-(n+2)}}h_{i+j}(t,0) \\
&= \frac{(q^{-2(i+1)};q^{-2})_j}{(q^{-2};q^{-2})_j}h_{i+j}(t,0).
\end{aligned}$$

□

Corollary 2.8 *Let $h_i(t,0)$ and $h_j(t,0)$ be two q-symmetric polynomials at zero. Then*

$$h_i(t,0)h_j(t,0) = h_j(t,0)h_i(t,0).$$

3 The differential transformation technique on q-symmetric calculus

Based on earlier research [12,13], the basic definitions and operations of differential transformation method are extended. The definition of the Taylor series on a time scale can be found in [17].

Definition 3.1 *A real-valued function f defined on $\mathbb{T}$ is said to be q-symmetric analytic at t_0 if and only if there is a power series centered at t_0 that converges to f near t_0, i.e. there exists coefficients $\{a_k\}$ and points $c, d \in \mathbb{T}$. such that $c < t_0 < d$ and $f(t) = \sum_{k=0}^{n_1} a_k h_k(t, t_0)$ for $t \in (c, d) \bigcap \mathbb{T}$. If $x(t)$ is q-symmetric analytic in the time domain $\overline{q^{\mathbb{N}}}$, then $x(t)$ is continuously differentiable with respect to t.*

Definition 3.2 *Let the $\mathbb{K}$ domain be the set of nonnegative integers. The spectrum of $x(t)$ at t_i in the $\mathbb{K}$ domain is written as*

$$X(k) = [\frac{\widetilde{d}_q^k x(t)}{\widetilde{d}_q t^k}]_{t=t_i}, \forall k \in \mathbb{K}. \tag{3}$$

Definition 3.3 *If $x(t)$ can be expressed by the Taylor series on , then the differential transformation of $X(k)$ is represented as*

$$x(t) = \sum_{k=0}^{\infty} X(k) h_k(t, t_i), \tag{4}$$

where $h_k(t, t_i), k \in \mathbb{N}$ are q-symmetric polynomials with degree k and $X(t)$ is the spectrum of $x(t)$ at $t = t_i$.

Applying the differential transformation method in Definition 3.3, the solution of the nonlinear q-symmetric difference equation (1) can be represented as

$$x(t) = \sum_{k=0}^{\infty} X(k) h_k(t, 0), \tag{5}$$

where $X(k)$ is the spectrum of $x(t)$ at 0. The products of $x(t)\widetilde{d}_q(t)$ and $x(t)$ in (1) are expressed as follows:

$$x(t)\widetilde{d}_q x(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{k} X(k-i)X(i+1)h_{k-i}(t,0)h_i(t,0) = \sum_{k=0}^{\infty}[\sum_{i=0}^{k} X(k-i)X(i+1)H(k-i,i)]h_k(t,0), \tag{6}$$

$$x^2(t) = \sum_{k=0}^{\infty}[\sum_{i=0}^{k} X(k-i)X(i)H(k-i,i)]h_k(t,0), \tag{7}$$

where $H(i,j) = \frac{(q^{-2(i+1)};q^{-2})_j}{(q^{-2};q^{-2})_j}, H(i,0) = H(0,j) = 1, i,j = 0,1,2,\cdots$. Substituting (5),(6) and (7) into the nonlinear equation (1), it can be transformed into an algebraic equation as

$$\sum_{k=0}^{\infty}[X(k+2) + 2\gamma X(k+1) + \varepsilon\gamma_1 \sum_{i=0}^{k} X(k-i)X(i+1)H(k-i,i)$$
$$+\Omega^2 X(k) + \sum_{i=0}^{k} X(k-i)X(i)H(k-i,i)]h_k(t,0) = 0.$$

which yields the following algebraic equation

$$X(k+2)+2\gamma X(k+1)+\varepsilon\gamma_1\sum_{i=0}^{k}X(k-i)X(i+1)H(k-i,i)+\Omega^2X(k)+\sum_{i=0}^{k}X(k-i)X(i)H(k-i,i)=0, \tag{8}$$

where $k=0,1,2,3,\cdots$. As $X(0)=a$ and $\widetilde{d}_qx(0)=b$, the initial estimate of the series solution can be given as $x(t)=a+bh_i(t,0)$, i.e. the first two terms in the algebraic equation are $X(0)=a$ and $X(1)=b$. Hence, the spectrum $X(k)$ of $x(t)$ at 0 satisfies the algebraic equation (8) with initial conditions $X(0)=a$ and $X(1)=b$, where the solution can be obtained iteratively from $X(0)=a$ and $X(1)=b$.

4 Numerical method

To display the accuracy of the differential transformation method, a numerical method for the q-symmetric difference equation is developed. Since 0 is a cluster point of $\overline{q^{\mathbb{N}}}$, the derivative of $x(t)$ at 0 is defined as $\widetilde{d}_qx(0)=\lim_{n\to\infty}\frac{x(q^n)-x(0)}{q^n}$, if $q<1$. Let n_0 be a nonnegative integer. To obtain an approximation for the q-symmetric derivative of $x(t)$ at $t=0$, we use $x(q^{n_0})=x(0)+q^{n_0}\widetilde{d}_qx(0)+\frac{q^{2n_0}}{2}\widetilde{d}_q^2x(0)+\cdots$. Rearrangement leads to

$$\begin{aligned}\widetilde{d}_qx(0) &\approx \frac{x(q^{n_0})-x(0)}{q^{n_0}}-\frac{q^{n_0}}{2}\widetilde{d}_q^2x(0)\\ &= \frac{x(q^{n_0})-x(0)}{q^{n_0}}+O(q^{n_0}).\end{aligned}$$

where the dominate term in the truncation error is $O(q^{n_0})$. As $\widetilde{d}_qx(0)=b$, that is to say $\frac{x(q^{n_0})-x(0)}{q^{n_0}}$, which yields $x(q^{n_0})=x(0)+q^{n_0}b=a+q^{n_0}b$. Set $t_0=0$ and $t_1=q^{n_0}$, Let $t_i=q^{n_0-(i-1)}$, where $i=2,\cdots,n_0+1$. For convenience, we let $t_i=0$, where $i=-1,-2,\cdots$. Then the interval $[0,1]$ is partitioned into n_0 subintervals. Now $x_i=x(t_i), i=0,1,2,\cdots,n_0+1$. The q-symmetric derivative of $x(t)$ at t_i can be calculated as

$$\widetilde{d}_qx_i=\frac{x_{i+1}-x_{i-1}}{t_{i+1}-t_{i-1}}=D_ix_{i+1}-D_ix_{i-1}, \tag{9}$$

and

$$\widetilde{d}_q^2x_i=\frac{\widetilde{d}_qx_{i+1}-\widetilde{d}_qx_{i-1}}{t_{i+1}-t_{i-1}}=A_ix_{i+2}-B_ix_i+C_ix_{i-2} \tag{10}$$

where

$$A_i=\frac{1}{(t_{i+2}-t_i)(t_{i+1}-t_{i-1})}, B_i=\frac{t_{i+2}-t_{i-2}}{(t_{i+2}-t_i)(t_{i+2}-t_i)(t_{i+1}-t_{i-1})},$$

$$C_i = \frac{1}{(t_i - t_{i-2})(t_{i+1} - t_{i-1})}, D_i = \frac{1}{t_{i+1} - t_{i-1}}.$$

Substituting (10) and (11) into (1) yields the following equation

$$A_i x_{i+2} + (\Omega^2 - B_i)x_i + 2\gamma D_i x_{i+1} + \varepsilon\gamma_1 D_i x_i x_{i+1} - D_i x_{i-1} + C_i x_{i-2} + x_i^2 = 0.$$

This implies that

$$x_{i+2} = -\frac{1}{A_i}[(\Omega^2 - B_i)x_i + 2\gamma D_i x_{i+1} + \varepsilon\gamma_1 D_i x_i x_{i+1} - D_i x_{i-1} + C_i x_{i-2} + x_i^2]. \quad (11)$$

5 Numerical results

The theoretical considerations introduced in the previous sections will be illustrated with some examples, where the approximate solutions are compared with numerical solutions.

The time scale $q^{\mathbb{Z}_+}$ is given as $\{0.9^n | n \in \mathbb{Z}_+\} \bigcup \{0\} = 1, 0.9, 0.81, 0.729, \cdots, 0$, where 0 is the cluster point of $q^{\mathbb{Z}_+}$. The maximum error and the average error are defined as

maximum error $= \max\{|\overline{x}_n(t) - \hat{x}(t)| | t \in q^{\mathbb{Z}_+}, t \geq 0.9^{100}\}$,

and

average error $= \frac{sum\{|\overline{x}_n(t) - \hat{x}(t)| | t \in q^{\mathbb{Z}_+}, t \geq 0.9^{100}\}}{100}$,

respectively, where $\overline{x}_n(t)$ is the approximate solution with n iterations and $\hat{x}(t)$ is the numerical solution obtained by (12).

The under-damped cases are considered with (i) $2\gamma = 0$, $\gamma_1 = 0.1$, $\varepsilon = 1$ and $\Omega = 1$; (ii) $2\gamma = 0.1$, $\gamma_1 = 0.1$, $\varepsilon = 1$ and $\Omega = 1$; (iii) $2\gamma = 2.5$, $\gamma_1 = 0.1$, $\varepsilon = 1$ and $\Omega = 1$. In order to satisfy the initial conditions, $x(0) = 1$ and $\widetilde{d}_q x(0) = 0.5$, the initial approximation can be given $x_0 = 1 + 0.5t$. By the recurrence relation(7), the first 4 components of $x_n(t)$ are obtained. In the same manner, the rest of components of the iteration formula were obtained using the symbolic toolbox in Mathematics package. For the numerical computations, the interval $[0, 1]$ is partitioned into 101 subintervals, i.e. $n_0 = 100$. For case (i)

$$\begin{aligned}
x_1 &= 1 + 0.5h_1(t, 0), \\
x_2 &= 1 + 0.5h_1(t, 0) - 2.05h_2(t, 0), \\
x_3 &= 1 + 0.5h_1(t, 0) - 2.05h_2(t, 0) - 1.32h_3(t, 0), \\
x_4 &= 1 + 0.5h_1(t, 0) - 2.05h_2(t, 0) - 1.32h_3(t, 0) + 5.988h_4(t, 0).
\end{aligned}$$

and so on.

For case (ii)

$$\begin{aligned} x_1 &= 1+0.5h_1(t,0),\\ x_2 &= 1+0.5h_1(t,0)-2.10h_2(t,0),\\ x_3 &= 1+0.5h_1(t,0)-2.10h_2(t,0)-1.11h_3(t,0),\\ x_4 &= 1+0.5h_1(t,0)-2.10h_2(t,0)-1.11h_3(t,0)+6.099h_4(t,0). \end{aligned}$$

and so on.

For case (iii)

$$\begin{aligned} x_1 &= 1+0.5h_1(t,0),\\ x_2 &= 1+0.5h_1(t,0)-4.30h_2(t,0),\\ x_3 &= 1+0.5h_1(t,0)-4.30h_2(t,0)+9.43h_3(t,0),\\ x_4 &= 1+0.5h_1(t,0)-4.30h_2(t,0)+9.43h_3(t,0)-17.587h_4(t,0). \end{aligned}$$

and so on.

Using the calculator, we get the corresponding values of the approximate solution $\overline{x}_n(t)$ at each point of $q^{\mathbb{Z}_+}$, while the values of the numerical solution $\hat{x}(t)$ can be easily seen from the Figures.1-3 generated by Mathematica 9.0. Besides, we calculate the maximum errors shown in the following table, which indicates the accuracy.

Table 1

The comparison table of each case.

Time domain	.109	.205	.313	.430	.531	.656	.729	.81	.9
case 1	.0007	.0023	.0085	.0125	.0150	.0156	.0032	.0138	.0308
case 2	.0007	.0023	.0083	.0118	.0148	.0142	.0029	.0156	.0312
case 3	.0008	.0011	.0101	.0115	.0142	.0287	.0103	.0147	.0186

6 Conclusions

In this study, a product rule of two generalized polynomials on $q^{\mathbb{Z}_+}$ is derived, which overcomes the difficulty of developing a theory of series solutions of q-symmetric difference equation. In future studies, the use of differential transformation method will be extended to other nonlinear q-symmetric difference equations.

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References

[1] J.K. Zhou, Differential Transform and its Applications for Electrical Circuits, Wuhan, Huarjung University Press, 1986.

[2] F. Ayaz, Solutions of the system of differential equations by differential transformation method, Applied Mathematics and Computation 147 (2004) 547-567.

[3] A. Kurnaz, G. Oturanc, The differential transform approximation for the system of ordinary differential equations, International Journal of Computer Mathematics 82 (2005) 709-719.

[4] I.H. Abdel-Halim Hassan, Application to differential transformation method for solving systems of differential equations, Applied Mathematical Modelling 32 (2008) 2552-2559.

[5] S.S. Chen, C.K. Chen, Application of the differential transformation method to the free vibrations of strongly non-linear oscillators, Nonlinear Analysis: Real World Applications 10 (2009) 881-888.

[6] M. Thongmoon, S. Pusjuso, The numerical solutions of differential transformation method and the Laplace transform method for a system of differential equations, Nonlinear Analysis: Hybrid Systems 4 (2010) 425-431.

[7] A. Al-rabtah, V.S. Ertrk, S. Momani, Solutions of a fractional oscillator by using differential transform method, Computers and Mathematics with Applications 59 (2010) 1356-1362.

[8] A.Arikoglu, I.Ozkol, Vibration analysis of composite sandwich beams with viscoelastic core by using differential transform method, Composite Structures 92 (2010) 3031-3039.

[9] A. Borhanifar, R. Abazari, Numerical study of nonlinear Schrödinger and coupled Schrödinger equations by differential transformation method, Optics Communications 283 (2010) 2026-2031.

[10] O.özkan, Numerical implementation of differential transformations method for integro-differential equations, International Journal of Computer Mathematics 87 (2010) 2786-2797.

[11] M.-J. Jang, C.-L. Chen, Analysis of the response of a strongly nonlinear damped system using differential transformation technique, Applied mathematics and Computation 88 (1997) 137-151.

[12] B.-L. Kuo, C.-Y. Lo, Application of the differential transformation method to the solution of a damped system with high nonlinearity, Nonlinear Analysis 70 (2009) 1732-1737.

[13] Hsuan-Ku Liu, Application of a differential transformation method to strongly nonlinear damped q-difference equations, Computers and Mathematics with Applications 61 (2011) 2555-2561.

[14] Artur M.C. Brito da Cruz, Computers and Mathematics with Applications 64 (2012) 2241-2250.

[15] A. Lavagno, G. Gervino, Quantum mechanics in q-deformed calculus, J.Phys.: Conf.Ser.174 (2009).

[16] S. Hilger, Analysis on measure chains a unified approach to continuous and discrete calculus, Results in Mathematics 18 (1990) 18-56.

[17] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[18] R. Agarwal, M. Bohner, Basic calculus on time scales and some of it applications, Results in Mathematics 35 (1999) 3-22.

[19] V.Kac, P. Cheung, Quantum Calculus, Springer, New York, 2002, Universitext.

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