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Extension of Measure without Recourse to Outer Measure

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Abstract

Let μ be a measure on a cofinal monotonically dense subring $\mathfrak{R}$ of a Boolean δ-ring $\mathfrak{D}$. Denote by $\mathfrak{R}^{\searrow}$ and $\mathfrak{R}^{\nearrow}$ the classes of those $A \in \mathfrak{D}$ which are the greatest lower (respectively: least upper) bound of some decreasing (respectively: increasing) sequence in $\mathfrak{R}$. First we extend μ to these classes by monotonic continuity and then introduce the functions $\mu_*(A) = \sup\limits_{B \in \mathfrak{R}^{\searrow},\ B \le A} \mu(B)$ and $\mu^*(A) = \inf\limits_{B \in \mathfrak{R}^{\nearrow},\ B \ge A} \mu(B)$ on $\mathfrak{D}$. Denote $\mathfrak{A} = \{A \in \mathfrak{D} : \mu_*(A) = \mu^*(A)\}$. For $A \in \mathfrak{A}$ we set $\mu(A) = \mu_*(A)$, or, equivalently, $\mu(A) = \mu^*(A)$. It is shown that $\mathfrak{A} = \mathfrak{D}$ and thus extended function μ is a measure on $\mathfrak{D}$.

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1 Introduction

The result of this paper is not a theorem but a new proof of the classical Lebesgue's theorem on extension of measure. It relies on the synthesis of two approaches to the extending which apart of each other do not yield the final solution of the problem.

The first approach comes back to ancient Greeks. It defines the measure of a set (say the area of a figure) as the common value the least upper bound of the measures of inscribed elementary sets (say polygons) and the greatest lower bound of the measures of described elementary sets. In the modern terminology, this is the Jordan measure. But its domain of definition is not closed with respect to the monotonic bounded passage to the limit.

The second approach is due to Borel. Each Borel set in $\mathbb{R}^d$ arises from elementary ones (say finite unions of parallelepipeds) by virtue of at most countably many monotonic passages to the limit. So if A is the limit of a monotonic sequence (A_n) of Borel sets with already determined measures, and the measure of A has not been determined yet, then we put $\mu(A) = \lim \mu(A_n)$ and continue this process, encompassing more and more sets. Thus we obtain the transfinite sequence of extensions of function μ. The union of their domains of definition coincides with the Borel σ-algebra — but "…the justification of Borel's scheme (i. e., the proof of countable additivity of thus constructed function on the σ-algebra) has been given only within the frames of Lebesgue's approach (though later on it was shown that the straightforward justification with the aid of transfinite induction is also possible)" [2, vol. 1, p.498].

Lebesgue's idea to use the induced outer measure as the tool (see [4] for the authentic version and [2, vol. 1] for the modern exposition) issued in the complete solution of the problem and pushed the preceding approaches aside. But at the same time it divorced with the geometric nature of the problem, since Lebesgue's approach is rather set-theoretical. The question arises, whether there exists a scheme of extension based on the duly modified antique approach. In this article, we construct such a scheme. More definitely, we adapt for our goal the proposed in [8] scheme of extending abstract integral. Just note that Daniell's scheme for integrals (see items 16–18, 61, 62 in [5] or section 7.8 in [2]) has nothing common with Lebesgue's scheme for measures. A remarkable fact is that the respective alternative schemes proposed here and in [8] differ one from another only with technical details (and, inevitably, notation and terminology).

To state the result we define some notions figuring in its assertion. All of them are well known but named differently in different works.

We will say that a subset X_0 of an ordered set X is *cofinal* (in X) if for any $x \in X$ there exist $\underline{x} \in X_0$ and $\overline{x} \in X_0$ such that $\underline{x} \le x \le \overline{x}$. If an ordered set contains the greatest lower (least upper) bound x of a decreasing (increasing) sequence (x_n) of its members, then they write $x_n \searrow x$ (respectively: $x_n \nearrow x$). In both cases, we write $x = \lim x_n$ and say that this sequence *converges to* x . Note that we define convergence only of monotonic sequences. A function f on an ordered set X will be called *monotonically continuous at a point* x if $f(x_n) \to f(x)$ for every converging to x monotonic sequence $(x_n) \in X^{\mathbb{N}}$.

A subset of an ordered set will be called *monotonically closed* if it contains the greatest lower bound of every bounded decreasing sequence and the least upper bound of every bounded increasing sequence of its members (the boundedness demand in this definition is essential). A subset X_0 of a monotonically closed ordered set X will be called *monotonically dense* (in X) if there are no containing X_0 monotonically closed subsets of X except X itself.

One can easily show that, for any elements A and B of a distributive lattice L having the least element $\mathbf{0}$, there exists at most one element $C \in L$ such that $C \vee B = A \vee B$ and $C \wedge B = \mathbf{0}$. In the lattice theory, this element, if it exists, is usually denoted by $A \ominus B$. Thus, by definition,

$$(A \ominus B) \vee B = A \vee B, \quad (A \ominus B) \wedge B = \mathbf{0}. \tag{1}$$

A distributive lattice is called *Boolean lattice* if it has the least element (this will be the tacit assumption throughout below) and $A \ominus B$ exists for any two its elements A and B. A fundamental theorem of Stone establishes the one-to-one correspondence between Boolean lattices and Boolean rings. Neither this theorem nor even the algebraic definition of Boolean ring is used below, but the theorem entitles us to use the terms 'Boolean lattice' and 'Boolean ring' as the synonyms. It is Stone's theorem that justifies the term 'ring of sets' in measure theory (though the definition prompts 'lattice of sets').

We will say that a mapping ν of a Boolean ring into an additive semigroup is *Boolean-additive* if $\nu(A_1 \vee A_2) = \nu(A_1) + \nu(A_2)$ for any two disjoint A_1 and A_2. In measure theory, such mappings are called simply additive, but this terminology is unacceptable in the context where we consider the genuine (not Boolean) additivity entering, for example, the axiomatic definition of integral. An $\mathbb{R}_+$-valued Boolean-additive monotonically continuous (at all points) function on a Boolean ring is called *measure*. As is seen from this definition, we consider only finite measures.

A monotonically closed lattice will be called *δ-lattice*. This is, due to the boundedness condition in the definition of monotonic closedness, an analog of δ-ring (not σ-ring) of sets. Obviously, a δ-lattice contains both exact bounds of any (not certainly monotonic) bounded sequence of its members.

Theorem 1.1. *Let $\mathfrak{R}$ be a cofinal monotonically dense subring of a Boolean δ-ring $\mathfrak{D}$. Then any measure on $\mathfrak{R}$ uniquely extends to a measure on $\mathfrak{D}$.*

2 Preliminaries

This section contains ancillary results on lattices. Most of them are known (sometimes in less general form), but they are scattered in the literature (see, e. g., [1, 3, 5, 6, 7]), so we adduce both the assertions and the proofs, the latter being pretty standard. The reader inclined to consider rather rings of sets than general Boolean rings may skip this section.

Lemma 2.1. *Let $x_1, x_2 \dots, x, y_1, y_2 \dots y$ be elements of an ordered set X containing the greatest lower bound of each its finite subset. Suppose that $x_n \searrow x$ and $y_n \searrow y$. Then $x_n \wedge y_n \searrow x \wedge y$.*

Proof. Obviously, the sequence $(x_n \wedge y_n)$ decreases and $x_n \wedge y_n \geq x \wedge y$ for all n. So it suffices to show the following: if z is such an element of X that $z \leq x_n \wedge y_n$ for all n, then $x \leq x \wedge y$.

For $n > m$ we have $x_m \wedge y_n \geq x_n \wedge y_n$ (since (x_n) decreases) and therefore $x_m \wedge y_n \geq z$. Consequently, $z \leq \inf_m \inf_n (x_m \wedge y_n \wedge z)$. It remains to recall that, for any $u, v_1, v_2 \ldots \in X$, $\inf_k (u \wedge v_k) = u \wedge \inf_k v_k$ provided $\inf_k v_k$ exists. □

Likewise proved is

Lemma 2.2. *Let $x_1, x_2 \ldots, x, y_1, y_2 \ldots y$ be elements of an ordered set X containing the least upper bound of each its finite subset. Suppose that $x_n \nearrow x$ and $y_n \nearrow y$. Then $x_n \vee y_n \nearrow x \vee y$.*

The following statement is obvious (however, the proof is adduced in [8]).

Lemma 2.3. *Let (x_n) and (y_n) be convergent monotonic sequences in an ordered set. Suppose that $x_n \leq y_n$ for all n. Then $\lim x_n \leq \lim y_n$.*

For an arbitrary subset L of an ordered set X we denote by $L^{\nearrow}$ (respectively, $L^{\searrow}$) the set of the limits of all increasing (respectively, decreasing) convergent sequences in X. Otherwise speaking, $x \in L^{\nearrow}$ if and only if there exists an increasing sequence $(x_n) \in X^{\mathbb{N}}$ such that $x_n \nearrow x$. The following statement is immediate from Lemmas 2.1 and 2.2.

Lemma 2.4. *Let L be a subset of an ordered set and let L contain the exact upper (lower) bound of each its finite subset. Then so does $L^{\nearrow}$ (respectively, $L^{\searrow}$).*

Lemma 2.5. *Let A and B be elements of a distributive lattice and let $A \ominus B$ exist. Then $A \ominus B \leq A$.*

Proof. The first equality in formula (1) implies that $A \ominus B \leq A \vee B$; the second yields $(A \ominus B) \wedge A = (A \ominus B) \wedge (A \vee B)$. □

Lemma 2.6. *Let A and B be elements of a distributive lattice and let $A \ominus B$ exist. Then, for any element C of this lattice, $(A \ominus B) \wedge C = (A \wedge C) \ominus (B \wedge C)$.*

Proof. Denote $D = (A \ominus B) \wedge C$. By construction $D \leq A \ominus B$, which together with the second equality (1) implies that $B \wedge D = \mathbf{0}$ and all the more

$$B \wedge C \wedge D = \mathbf{0}. \tag{2}$$

Also by construction $D \leq C$. So

$$(B \wedge C) \vee D = (B \vee D) \wedge C. \tag{3}$$

Further, $D \vee B = ((A \ominus B) \vee B) \wedge (C \vee B) \overset{(1)}{=} (A \vee B) \wedge (C \vee B)$, whence $(B \vee D) \wedge C = (A \vee B) \wedge C$, which converts (3) to

$$D \vee (B \wedge C) = (A \wedge C) \vee (B \wedge C).$$

And this jointly with (2) implies that $D = (A \wedge C) \ominus (B \wedge C)$. □

Lemma 2.7. *Let A, A', B, B' be elements of a Boolean lattice and let $A' \leq A,\ B' \geq B$. Then $A' \ominus B' \leq A \ominus B$.*

Proof. Lemma 2.6 and the second equality (1) yield $(A \ominus B) \wedge (A' \ominus B) = (A \ominus B) \wedge A'$, whence by Lemma 2.6

$$(A \ominus B) \wedge (A' \ominus B) = (A \wedge A') \ominus (B \wedge A'). \tag{4}$$

Herein $(A \wedge A') = A' \wedge A'$ by assumption $A' \leq A$), so Lemma 2.6 asserts that $(A \wedge A') \ominus (B \wedge A') = (A' \ominus B) \wedge A'$. This together with (4) and Lemma 2.5 shows that $(A \ominus B) \wedge (A' \ominus B) = (A' \ominus B)\ A' \ominus B \leq A \ominus B$.

Further, $(A' \ominus B') \wedge (A' \ominus B) = ((A' \ominus B') \wedge A') \ominus ((A' \ominus B') \wedge B)$ by Lemma 2.6. Herein $(A' \ominus B') \wedge B \leq (A' \ominus B') \wedge B' = \mathbf{0}$. So $(A' \ominus B') \wedge (A' \ominus B) = (A' \ominus B') \wedge A' = A' \ominus B'$ (the last equality relies on Lemma 2.5). Thus $A' \ominus B' \leq A' \ominus B$. □

Lemma 2.8. *For any elements $E, F_1, \ldots, E_n$ of a Boolean lattice*

$$E \ominus (F_1 \wedge \ldots \wedge F_n) = \bigvee_{i=1}^{n} E \ominus F_i.$$

Proof. Denote $G_i = E \ominus F_1$, so that $G_i \vee F_i = E \vee F_i,\ G_i \wedge F_i = \mathbf{0}$. Then

$$(G_1 \vee G_2) \wedge (F_1 \wedge F_2) = \mathbf{0} \tag{5}$$

and $(G_1 \vee G_2) \vee (F_1 \wedge F_2) = (E \vee F_1 \vee G_2) \wedge (G_1 \vee E \vee F_2)$. Noting that $G_i \leq E$ by Lemma 2.5, we convert the last equality to $(G_1 \vee G_2) \vee (F_1 \wedge F_2) = (E \vee F_1) \wedge (\vee E \vee F_2) \equiv E \vee (F_1 \wedge F_2)$, which together with (5) proves the lemma for $n = 2$. To deduce hence the general statement by induction it suffices to write $F_1 \wedge \ldots \wedge F_{n+1} = F_1' \wedge \ldots \wedge F_n'$, where $F_n' = F_n \wedge F_{n+1}$ and $F_i' = F_i$ for $i < n$. □

An order-preserving mapping of one ordered set into another is otherwise called *isotonic*.

Lemma 2.9. *Let φ be an isotonic function on a Boolean lattice $\mathfrak{B}$ such that for all $B_1, B_2 \in \mathfrak{B}$*

$$\varphi(B_1 \vee B_2) \leq \varphi(B_1) + \varphi(B_2). \tag{6}$$

Then for all $n \in \mathbb{N},\ A_1, \ldots, A_n \in \mathfrak{B}$

$$\varphi(A_1 \vee \ldots \vee A_n) \leq \varphi(A_1) + \sum_{k=2}^{n} \varphi(A_k \ominus A_{k-1}). \tag{7}$$

Proof. Denote $E_k = A_1 \vee \ldots \vee A_k$. Then, for $k > 1$, $E_k = E_{k-1} \vee A_k \equiv E_{k-1} \vee (A_k \ominus E_{k-1})$, whence by condition (6) $\varphi(E_k) \leq \varphi(E_{k-1}) + \varphi(A_k \ominus E_{k-1})$. By Lemma 2.7 $A_k \ominus E_{k-1} \leq A_k \ominus A_{k-1}$, which together with isotonicity of φ implies that $\varphi(A_k \ominus E_{k-1}) \leq \varphi(A_k \ominus A_{k-1})$. Consequently, $\varphi(E_k) - \varphi(E_{k-1}) \leq \varphi(A_k \ominus A_{k-1})$. Summing this inequality by k from 2 to n and taking to account that $E_1 = A_1$, we arrive at (7). $\square$

Lemma 2.10. *Let ν be a Boolean-additive map of a Boolean ring into an additive semigroup. Then for every A and B from this ring*

$$\nu(A) + \nu(B \ominus A) = \nu(A \vee B), \quad \nu(A \wedge B) + \nu(B \ominus A) = \nu(B).$$

Proof. The first equality is immediate from (1) and Boolean additivity of ν. Replacing in it A with $A \wedge B$ and taking to account Lemma 2.6, we get

$$\nu(A \wedge B) + \nu((B \ominus A) \wedge B) = \nu(B).$$

It remains to note that $B \ominus A \leq B$ by Lemma 2.5. $\square$

Corollary 2.11. *Let ν be a Boolean-additive function on a Boolean ring. Then for every A and B from this ring*

$$\nu(A) + \nu(B) = \nu(A \vee B) + \nu(A \wedge B).$$

Corollary 2.12. *Let ν be an isotonic Boolean-additive function on a Boolean ring. Then for every C and D from this ring one has $\nu(C) - \nu(D) \leq \nu(C \ominus D)$.*

Proof. $\nu(C) - \nu(D) \leq \nu(C \vee D) - \nu(D) = \nu(C \ominus D)$. $\square$

Corollary 2.13. *Every nonnegative Boolean-additive function on a Boolean ring is isotonic.*

We will say that a lattice $\mathfrak{L}$ is *countably distributive* if it has the following two properties.

(i) If $\mathfrak{L}$ contains the exact upper bound of an increasing sequence (G_n) of its elements, then for any $F \in \mathfrak{L}$

$$F \wedge \sup_n G_n = \sup_n (F \wedge G_n).$$

(ii) If $\mathfrak{L}$ contains the exact lower bound of a decreasing sequence (H_n) of its elements, then for any $F \in \mathfrak{L}$

$$F \vee \inf_n H_n = \inf_n (F \vee H_n).$$

Lemma 2.14. *Let $A_1, A_2 \ldots, A, B_1, B_2 \ldots B$ be elements of a countably distributive lattice. Suppose that $A_n \searrow A$ and $B_n \searrow B$. Then $A_n \vee B_n \searrow A \vee B$.*

Proof. Obviously, $(A_n \vee B_n)$ decreases and for any $m \in \mathbb{N}$ $\inf_n(A_n \vee B_n) \leq \inf_n(A_m \vee B_n)$. By the assumptions of the lemma $\inf_n(A_m \vee B_n) = A_m \vee B$. So $\inf_n(A_n \vee B_n) \leq \inf_m(A_m \vee B)$. And the right-hand side equals $A \vee B$ by the assumptions of the lemma. Thus $\inf_n(A_n \vee B_n) \leq A \vee B$. The reverse inequality is evident. □

Likewise proved is

Lemma 2.15. *Let $A_1, A_2 \ldots, A, B_1, B_2 \ldots B$ be elements of a countably distributive lattice. Suppose that $A_n \nearrow A$ and $B_n \nearrow B$. Then $A_n \wedge B_n \searrow A \wedge B$.*

Corollary 2.16 (of Lemmas 2.4, 2.14 and 2.15)**.** *Let $\mathfrak{L}$ be a sublattice of a countably distributive lattice. Then $\mathfrak{L}^{\nearrow}$ and $\mathfrak{L}^{\searrow}$ are lattices, too.*

We will say that a subset of an ordered set X is *exactly bounded from above* (*from below*) if it has the least upper (respectively, greatest lower) bound in X.

Lemma 2.17. *Let $\{B_\xi, \xi \in \Xi\}$ be an exactly bounded from above subset of a Boolean lattice $\mathfrak{L}$. Then for any $A \in \mathfrak{L}$*

$$A \wedge \sup_{\xi\in\Xi} B_\xi = \sup_{\xi\in\Xi}(A \wedge B_\xi).$$

Proof. Denote $B = \sup B_\xi$. By this definition and Lemma 2.5

$$(B \ominus A) \vee B_\xi \leq B \tag{8}$$

for all ξ. Besides, $A \wedge B_\xi \leq A \wedge B$, so it suffices to show the following: if F is such an element of the lattice that $F \geq A \wedge B_\xi$ for all ξ, then $F \geq A \wedge B$.

By the choice of F for any $\xi \in \Xi$ and $E \in \mathfrak{L}$ one has $E \vee F \geq E \vee (A \wedge B_\xi)$. Hence, writing

$$(B \ominus A) \vee (A \wedge B_\xi) = ((B \ominus A) \vee A) \wedge ((B \ominus A) \vee B_\xi) \overset{(1)}{=} \\ (B \vee A) \wedge ((B \ominus A) \vee B_\xi) \overset{(8)}{=} (B \ominus A) \vee B_\xi \geq B_\xi,$$

we get $(B \ominus A) \vee F \geq B_\xi$. Consequently, $(B \ominus A) \vee F \geq B$ and therefore $A \wedge ((B \ominus A) \vee F) \geq A \wedge B$. Herein $A \wedge ((B \ominus A) \vee F) = (A \wedge (B \ominus A)) \vee (A \wedge F) \overset{(1)}{=} \mathbf{0} \vee (A \wedge F)$. Thus $A \wedge F \geq A \wedge B$ and all the more $F \geq A \wedge B$. □

Lemma 2.18. *Let $\{D_\xi, \xi \in \Xi\}$ be an exactly bounded from below subset of a Boolean lattice $\mathfrak{L}$. Then for any $C \in \mathfrak{L}$*

$$C \vee \inf_{\xi\in\Xi} D_\xi = \inf_{\xi\in\Xi}(C \vee D_\xi).$$

Proof. Since for every ξ $C \vee D_\xi \geq C \vee D$, where $D = \inf D_\xi$, it suffices to show the following: if G is such an element of the lattice that $G \leq C \vee D_\xi$ for all ξ, then $G \leq C \vee D$.

Fix $\xi_0 \in \Xi$ and denote $B = C \vee D_{\xi_0}$. By the choice of G

$$C \vee G \leq B. \tag{9}$$

Again by the choice of G for any $\xi \in \Xi$ and $E \in \mathfrak{L}$ one has $E \wedge G \leq E \wedge (C \vee D_\xi)$. Hence, writing

$$(B \ominus C) \wedge (C \vee D_\xi) = ((B \ominus C) \wedge C) \vee ((B \ominus C) \wedge D_\xi) \overset{(1)}{=} (B \ominus C) \wedge D_\xi,$$

we get $(B \ominus C) \wedge G \leq D_\xi$. Consequently, $(B \ominus C) \wedge G \leq D$ and therefore $C \vee ((B \ominus C) \wedge G) \leq C \vee D$. Herein $C \vee ((B \ominus C) \wedge G) = (C \vee (B \ominus C)) \wedge (C \vee G) \overset{(1)}{=} (C \vee B) \wedge (C \vee G) \overset{(9)}{=} C \vee G$. Thus $C \vee G \leq C \vee D$ and all the more $G \leq C \vee D$. ☐

Corollary 2.19 (of Lemmas 2.17 and 2.18)**.** *Boolean lattice is countably distributive.*

Corollary 2.20 (of Corollaries 2.16 and 2.19)**.** $\mathfrak{R}^{\searrow}$ *and* $\mathfrak{R}^{\nearrow}$ *are lattices.*

Lemma 2.21. *Let $A_1, A_2 \ldots, A, B_1, B_2 \ldots B$ be elements of a Boolean lattice. Suppose that $A_n \searrow A$ and $B_n \nearrow B$. Then $A_n \ominus B_n \searrow A \ominus B$.*

Proof. Denote $E_n = A_n \ominus B_n$. By Lemma 2.7 the sequence (E_n) decreases, since (A_n) decreases and (B_n) increases. By the same lemma $E_n \geq A \ominus B$. So it suffices to show the following: if F is such an element of the lattice that $E_n \geq F$ for all n, then $(A \ominus B) \wedge F = F$.

The last equality is tantamount, since $(A \ominus B) \wedge F = (A \wedge F) \ominus (B \wedge F)$ by Lemma 2.6, to the pair of equalities $F \wedge B = \mathbf{0}$, $F \vee (B \wedge F) = A \wedge F \vee B \wedge F$, or the same, $B \wedge F = \mathbf{0}, F = A \wedge F$.

By Corollary 2.19 and the assumptions of the lemma $F \wedge B = \sup_n (F \wedge B_n)$. But $F \wedge B_n \leq E_n \wedge B_n$, and the right-hand side equals $\mathbf{0}$ by the construction of E_n.

By Lemma 2.5 $E_n \leq A_n$ and all the more $F \leq A_n$, whence $F \leq \inf_n A_n \equiv A$. ☐

Lemma 2.22. *Let $A_1, A_2 \ldots, A$ be elements of a Boolean lattice, and let $A_n \searrow A$. Then, for any element G of this lattice, $G \ominus A_n \nearrow G \ominus A$.*

Proof. Denote $B_n = G \ominus A_n$. By Lemma 2.7 the sequence (B_n) increases and $B_n \leq G \ominus A$. So it suffices to show the following: if F is such an element of the lattice that $F \geq B_n$ for all n, then $F \geq G \ominus A$.

By the choice of F and by the definition of B_n $A_n \vee F \geq A_n \vee B_n = A_n \vee G$. By the assumptions of the lemma and by Corollary 2.19 $A_n \vee F \searrow A \vee F$, $A_n \vee G \searrow A \vee G$, which together with the established above inequality $A_n \vee F \geq A_n \vee G$ implies, by lemma 2.1, that $A \vee F \geq A \vee G$. Hence we get with account of Lemma 2.5 $(A \vee F) \wedge (G \ominus A) \geq G \ominus A$. Herein $(A \vee F) \wedge (G \ominus A) = A \wedge (G \ominus A) \vee F \wedge (G \ominus A) \overset{(1)}{=} F \wedge (G \ominus A)$. Thus $F \wedge (G \ominus A) \geq G \ominus A$ and all the more $F \geq G \ominus A$. $\square$

3 Preparatory Extension

Lemma 3.1. *Let μ be a measure on $\mathfrak{R}$, and (C_n), (D_n) be monotonic convergent sequences in $\mathfrak{R}$ such that* $\lim C_n \leq \lim D_n$. *Then* $\lim \mu(C_n) \leq \lim \mu(D_n)$. *In particular, if* $\lim C_n = \lim D_n$, *then* $\lim \mu(C_n) = \lim \mu(D_n)$.

Proof. Let at first $C_n \searrow A, D_n \nearrow B \geq A$. Then by Lemma 2.21 $C_n \ominus D_n \searrow A \ominus B$ $(= \mathbf{0})$ and therefore $\mu(C_n \ominus D_n) \to 0$, whence by Corollaries 2.13 and 2.12 $\overline{\lim}(\mu(C_n) - \mu(D_n)) \leq 0$. Recall that, for arbitrary sequences (a_n) and (b_n) of real numbers, $\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$. Put $a_n = \mu(D_n)$, $b_n = \mu(C_n) - \mu(D_n)$.

Let now $C_n \nearrow A, D_n \nearrow B \geq A$. Then by what was proved $\mu(C_k) \leq \lim \mu(D_n)$ for all k. It remains to let $k \to \infty$.

Let further $C_n \searrow A, D_n \searrow B \geq A$. Denote $G = C_1 \vee D_1, E_n = G \ominus D_n, F_n = G \ominus C_n$. Then by Lemma 2.22 $E_n \nearrow G \ominus B, F_n \nearrow G \ominus A \geq G \ominus B$, so according to the previous paragraph, $\lim \mu(E_n) \leq \lim \mu(F_n)$. It remains to note that $\mu(E_n) = \mu(G) - \mu(D_n)$ and $\mu(F_n) = \mu(G) - \mu(C_n)$ since $D_n \leq G, C_n \leq G$ and $G \in \mathfrak{R}$.

Finally, the case $C_n \nearrow A, D_n \searrow B \geq A$ is trivial. $\square$

Let us denote $\widetilde{\mathfrak{R}} = \mathfrak{R}^{\nearrow} \cup \mathfrak{R}^{\searrow}$ and extend μ to $\widetilde{\mathfrak{R}}$, setting, for any $A \in \widetilde{\mathfrak{R}}$, $\mu(A) = \lim \mu(C_n)$, where (C_n) is an arbitrary converging to A monotonic sequence in $\mathfrak{R}$ (Lemma 3.1 asserts that $\mu(A)$ does not depend on the choice of the sequence (C_n) with these properties). From now on till Proposition 5.2, inclusive, μ signifies the just constructed (preparatory) extension of the likewise denoted measure defined initially on $\mathfrak{R}$.

Corollary 3.2 (of Lemma 3.1 Corollary 2.13and)**.** *μ is isotonic on $\widetilde{\mathfrak{R}}$.*

Lemma 3.3. *For any A and B both belonging either $\mathfrak{R}^{\searrow}$ or $\mathfrak{R}^{\nearrow}$ one has*

$$\mu(A) + \mu(B) = \mu(A \vee B) + \mu(A \wedge B).$$

Proof. Let there exist sequences (C_n) and (D_n) such that $C_n, D_n \in \mathfrak{R}$, $C_n \searrow A$, $D_n \searrow B$. Then by Lemmas 2.14 and 2.1,

$$C_n \vee D_n \searrow A \vee B, \quad C_n \wedge D_n \searrow A \wedge B.$$

These four relations imply that

$$\mu(C_n) \to \mu(A),\ \mu(D_n) \to \mu(B),\ \mu(C_n \vee D_n) \to \mu(A \vee B),\ \mu(C_n \wedge D_n) \to \mu(A \wedge B).$$

It remains to note that $\mu(C_n) + \mu(D_n) = \mu(C_n \vee D_n) + \mu(C_n \wedge D_n)$ by the choice of the sequences and by Corollary 2.11.

For $A, B \in \mathfrak{R}^{\nearrow}$, we argue likewise, referring to Lemma 2.15 instead of Lemma 2.14. □

Lemma 3.4. *Let (A_n) be an increasing sequence in $\mathfrak{R}^{\nearrow}$ (decreasing sequence in $\mathfrak{R}^{\searrow}$) converging to some A. Then $A \in \mathfrak{R}^{\nearrow}$ (respectively, $A \in \mathfrak{R}^{\searrow}$) and $\mu(A) = \lim \mu(A_n)$.*

Proof. Let $A_n \in \mathfrak{R}^{\nearrow}$,

$$A_n \nearrow A. \tag{10}$$

By the choice of the A_n's, for each n there exists an increasing sequence $(C_{nk}, k \in \mathbb{N}) \in \mathfrak{R}^{\mathbb{N}}$ such that

$$C_{nk} \nearrow A_n \quad \text{as} \quad k \to \infty. \tag{11}$$

Denote $D_k = C_{1k} \vee \ldots \vee C_{kk}$ ($\in \mathfrak{R}$ since $\mathfrak{R}$ is a lattice). By construction $D_k \le C_{1,k+1} \vee \ldots \vee C_{k,k+1}$ and all the more $D_k \le D_{k+1}$. Herein $C_{ik} \le A_i$ as $i \le k$ and all the more (since (A_k) increases) $C_{ik} \le A_k$ as $i \le k$. Consequently, $D_k \le A_k$, whence in view of (10) $D_k \le A$. Thus the sequence (D_k) increases and is bounded from above, which together with monotonic closedness of $\mathfrak{D}$ implies existence of an element $D \in \mathfrak{D}$ such that $D_k \nearrow D$. By construction $D_k \ge C_{nk}$ as $k \ge n$, which together with the preceding relation and (11) yields, by Lemma 2.3, $D \ge A_n$. Hence we get with account of (10) $D \ge A$. On the other hand, the relations $D_k \le A_k$, $D_k \nearrow D$ and assumption (10) imply, by the same lemma, that $D \le A$. Thus $D = A$ and therefore $D_k \nearrow A$ (so that $A \in \mathfrak{R}^{\nearrow}$), whence by the definition of μ as a function on $\mathfrak{R}^{\nearrow}$ we have $\mu(D_k) \nearrow \mu(A)$. Herein $\mu(D_k) \le \mu(A_k) \le \mu(A)$, because $D_k \le A_k \le A$ and μ is isotonic by Corollary 2.13. We have proved the lemma for the first of two cases. For the second case, the rationale is similar. □

4 The Tools for the Final Extension

Denote, for $A \in \mathfrak{D}$,

$$\mu_*(A) = \sup_{B \in \mathfrak{R}^{\searrow},\ B \le A} \mu(B), \quad \mu^*(A) = \inf_{B \in \mathfrak{R}^{\nearrow},\ B \ge A} \mu(B), \tag{12}$$

The next two statements are immediate from these equalities and Corollary 3.2.

Lemma 4.1. *For any $A \in \mathfrak{D}$ $\quad \mu_*(A) \le \mu^*(A)$.*

Lemma 4.2. *The functions μ_* and μ^* are isotonic.*

Lemma 4.3. *For any disjoint $B_1, B_2 \in \mathfrak{D}$*

$$\mu_*(B_1) + \mu_*(B_2) \leq \mu_*(B_1 \vee B_2).$$

Proof. Corollary 2.20 implies that

$$\sup_{E \in \mathfrak{R}^{\searrow} : E \leq B_1 \vee B_2} \mu(E) \geq \sup_{\substack{A_1, A_2 \in \mathfrak{R}^{\searrow} : \\ A_1 \leq B_1, A_2 \leq B_2}} \mu(A_1 \vee A_2). \tag{13}$$

Any two elements $A_1 \leq B_1$ and $A_2 \leq B_2$ are disjoint, since so are B_1 and B_2. So by Lemma 3.3 $\mu(A_1 \vee A_2) = \mu(A_1) + \mu(A_1)$ (if A_1 and A_2 taken from $\mathfrak{R}^{\searrow}$), which converts (13) to

$$\sup_{E \in \mathfrak{R}^{\searrow} : E \leq B_1 \vee B_2} \mu(E) \geq \sup_{A_1 \in \mathfrak{R}^{\searrow} : A_1 \leq B_1} \mu(A_1) + \sup_{A_2 \in \mathfrak{R}^{\searrow} : A_2 \leq B_2} \mu(A_2).$$

It remains to make use of (12). □

Lemma 4.4. *For any $B_1, B_2 \in \mathfrak{D}$*

$$\mu^*(B_1 \vee B_2) \leq \mu^*(B_1) + \mu^*(B_2).$$

Proof. Corollary 2.20 implies that

$$\inf_{F \in \mathfrak{R}^{\nearrow} : F \geq B_1 \vee B_2} \mu(F) \leq \inf_{\substack{A_1, A_2 \in \mathfrak{R}^{\nearrow} : \\ A_1 \geq B_1, A_2 \geq B_2}} \mu(A_1 \vee A_2),$$

which together with Lemma 3.3 yields

$$\inf_{F \in \mathfrak{R}^{\nearrow} : F \geq B_1 \vee B_2} \mu(F) \leq \inf_{A_1 \in \mathfrak{R}^{\nearrow} : A_1 \geq B_1} \mu(A_1) + \inf_{A_2 \in \mathfrak{R}^{\nearrow} : A_2 \geq B_2} \mu(A_2).$$

It remains to make use of (12). □

Corollary 4.5. *Let B_1 and B_2 be disjoint elements of $\mathfrak{D}$ such that $\mu_*(B_i) = \mu^*(B_i)$, $i = 1, 2$. Then*

$$\mu_*(B_1) + \mu_*(B_2) = \mu_*(B_1 \vee B_2) = \mu^*(B_1 \vee B_2) = \mu^*(B_1) + \mu^*(B_2).$$

Corollary 4.6 (of Lemmas 4.2, 4.4 and 2.9)**.** *For all $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathfrak{D}$*

$$\mu^*(A_1 \vee \ldots \vee A_n) \leq \mu^*(A_1) + \sum_{k=2}^{n} \mu^*(A_k \ominus A_{k-1}).$$

Corollary 4.7 (of Lemmas 4.4 and 2.8)**.** *For all* $n \in \mathbb{N}$, $E, F_1, \ldots, F_n \in \mathfrak{D}$

$$\mu^*(E \ominus (F_1 \wedge \ldots \wedge F_n)) \leq \sum_{k=1}^{n} \mu^*(E \ominus F_k).$$

Lemma 4.8. *For any* $A \in \mathfrak{R}^{\searrow}$ $\mu_*(A) = \mu(A)$*; for any* $A \in \mathfrak{R}^{\nearrow}$ $\mu^*(A) = \mu(A)$*. In particular, for any* $A \in \mathfrak{R}$

$$\mu_*(A) = \mu(A) = \mu^*(A). \tag{14}$$

Proof. Let $A \in \mathfrak{R}^{\searrow}$. Then $\mu_*(A) \geq \mu(A)$ because of (12) and the trivial inequality $A \geq A$. On the other hand if $A, B \in \mathfrak{R}^{\searrow}$ and $A \geq B$, then $\mu(A) \geq \mu(B)$ by Corollary 3.2. Hence and from (12) we get $\mu(A) \geq \mu_*(A)$.

The proof of the second statement is similar. □

Lemma 4.9. *Equalities* (14) *hold for all* $A \in \widetilde{\mathfrak{R}}$.

Proof. Let $C_n \nearrow A$, where $C_n \in \mathfrak{R}$. Then by the construction of the preparatory extension $\mu(C_n) \to \mu(A)$. Besides, $\mu_*(A) \geq \mu_*(C_n) = \mu(C_n)$, where the inequality follows from Lemma 4.2 and the equality – from Lemma 4.8. Thus $\mu_*(A) \geq \mu(A)$. Hence, noting that $\mu(A) = \mu^*(A)$ by Lemma 4.8, we get $\mu_*(A) \geq \mu^*(A)$, which together with Lemma 4.1 completes the proof for the case $A \in \mathfrak{R}^{\nearrow}$. The case $A \in \mathfrak{R}^{\searrow}$ is treated similarly. □

5 Final Extension

Denote $\mathfrak{A} = \{A \in \mathfrak{D} : \mu_*(A) = \mu^*(A)\}$. Lemma 4.9 asserts that $\widetilde{\mathfrak{R}} \subset \mathfrak{A}$.

Lemma 5.1. *Let* $A, B \in \mathfrak{A}$, $A \leq B$*. Then* $\mu_*(B \ominus A) = \mu_*(B) - \mu_*(A)$, $\mu^*(B \ominus A) = \mu^*(B) - \mu^*(A)$ *and therefore* $B \ominus A \in \mathfrak{A}$.

Proof. Applying Corollary 4.5 to $B_1 = A$, $B_2 = B \ominus A$, we get $\mu_*(B) = \mu_*(A) + \mu_*(B \ominus A) = \mu^*(A) + \mu^*(B \ominus A) = \mu^*(B)$. □

Proposition 5.2. *Let* (A_n) *be a monotonic convergent sequence in* $\mathfrak{A}$*. Then* $\lim A_n \in \mathfrak{A}$ *and*

$$\mu^*(\lim A_n) = \lim \mu^*(A_n). \tag{15}$$

By the choice of the sequence (A_n) and according to the first statement of the lemma equality (15) is tantamount to $\mu_*(\lim A_n) = \lim \mu_*(A_n)$.

Proof. 1°. Let $A_n \nearrow A$ and

$$\mu_*(A_n) = \mu^*(A_n). \tag{16}$$

Fix $\varepsilon > 0$ and take, for each $k \in \mathbb{N}$, an existing by the definition of μ^* element $B_k^\varepsilon \in \mathfrak{R}^{\nearrow}$ such that

$$B_k^\varepsilon \geq A_k \tag{17}$$

and

$$\mu(B_k^\varepsilon) < \mu^*(A_k) + 2^{-k}\varepsilon. \tag{18}$$

Cofinality of $\mathfrak{R}$ allows us to consider that the sequence (B_k^ε) is bounded from above (otherwise we will take an arbitrary $H \in \mathfrak{R}_{\geq A}$ and replace B_k^ε with $B_k^\varepsilon \wedge H$).

Denote $F_n^\varepsilon = B_1^\varepsilon \vee \ldots \vee B_n^\varepsilon$ ($\in \mathfrak{R}^{\nearrow}$ by construction and Lemma 2.4). By Lemma 4.9 and Corollary 4.5

$$\mu(F_n^\varepsilon) = \mu^*(F_n^\varepsilon) \leq \mu^*(B_1^\varepsilon) + \sum_{k=2}^{n} \mu^*(B_k^\varepsilon \ominus B_{k-1}^\varepsilon). \tag{19}$$

From (17) we have by Lemmas 2.7and 4.2

$$\mu^*(B_k^\varepsilon \ominus B_{k-1}^\varepsilon) \leq \mu^*(B_k^\varepsilon \ominus A_{k-1}). \tag{20}$$

By assumption $A_{k-1} \leq A_k$, whence because of (17) $A_{k-1} \leq B_k^\varepsilon$. Herein $B_k^\varepsilon \in \mathfrak{A}$ by Lemma 4.9; $A_{k-1} \in \mathfrak{A}$ in view of (16). From the last three relations we get by Lemma 5.1

$$\mu^*(B_k^\varepsilon \ominus A_{k-1}) = \mu^*(B_k^\varepsilon) - \mu^*(A_{k-1}). \tag{21}$$

By Lemma 4.9 $\mu^*(B_k^\varepsilon) = \mu(B_k^\varepsilon)$, which together with (19) – (21) implies that

$$\mu(F_n^\varepsilon) \leq \mu(B_1^\varepsilon) + \sum_{k=2}^{n} \left(\mu(B_k^\varepsilon) - \mu^*(A_{k-1})\right).$$

Herein $-\mu^*(A_{k-1}) < 2^{-(k-1)}\varepsilon - \mu(B_{k-1}^\varepsilon)$ because of (18). So $\mu(F_n^\varepsilon) < \mu(B_n^\varepsilon) + \varepsilon \sum\limits_{k=1}^{n-1} 2^{-k}$, which together with (18) yields

$$\mu(F_n^\varepsilon) < \mu^*(A_n) + \varepsilon. \tag{22}$$

By construction the sequence (F_n^ε) increases and is bounded from above. So there exists $F^\varepsilon \in \mathfrak{D}$ such that $F_n^\varepsilon \nearrow F^\varepsilon$. By Lemma 3.4 $F^\varepsilon \in \mathfrak{R}^{\nearrow}$ and

$$\mu(F^\varepsilon) = \lim \mu(F_n^\varepsilon). \tag{23}$$

Relations $A_n \leq B_n^\varepsilon \leq F_n^\varepsilon$ and $A_n \nearrow A$ imply by Lemma 3.1 that $A \leq F^\varepsilon$, hereon Lemma 4.2 asserts that $\mu^*(A) \leq \mu^*(F^\varepsilon)$. But $\mu^*(F^\varepsilon) = \mu(F^\varepsilon)$ by Lemma 4.9. Thus

$$\mu^*(A) \leq \mu(F^\varepsilon). \tag{24}$$

Take, for each $n \in \mathbb{N}$, an existing by the definition of μ_* element $G_n \in \mathfrak{R}^{\searrow}$ such that $G_n \le A_n$ and

$$\mu(G_n) > \mu_*(A_n) - 2^{-n}. \tag{25}$$

Since $G_n \le A_n \le F_n^\varepsilon$, Corollary 3.2 asserts that $\mu(G_n) \le \mu(F_n^\varepsilon)$, which together with (22) and (25) yields

$$\mu_*(A_n) - 2^{-n} < \mu(G_n) \le \mu(F_n^\varepsilon) < \mu^*(A_n) + \varepsilon.$$

Hence we get with account of (15) $0 \le \mu(F_n^\varepsilon) - \mu(G_n) < \varepsilon + 2^{-n}$. This jointly with (23) implies that $\mu(F^\varepsilon) \le \lim \mu(G_n) + \varepsilon$, which together with (24) yields

$$\mu^*(A) \le \lim \mu(G_n) + \varepsilon. \tag{26}$$

On the other hand, the inequalities $G_n \le A_n \le A$ imply by Lemma 4.2 that $\mu_*(G_n) \le \mu_*(A_n) \le \mu_*(A)$. Besides, by the choice of G_n and by Lemma 4.9 $\mu_*(G_n) = \mu(G_n)$. So $\lim \mu(G_n) \le \lim \mu_*(A_n) \le \mu_*(A)$. Comparing this with (26), we see that, for any $\varepsilon > 0$, $\mu^*(A) \le \lim \mu_*(A_n) \le \mu_*(A) + \varepsilon$. Hence and from (15) we get by Lemma 4.1 $\mu^*(A) = \lim \mu^*(A_n) = \lim \mu_*(A_n) = \mu_*(A)$. Thus we have proved the lemma for increasing sequences.

2°. Let now $A_n \in \mathfrak{A}$, $A_n \searrow A$ and (16) hold. Then:

$$A_1 \ominus A_n \in \mathfrak{A} \tag{27}$$

by Lemma 5.1; $A_1 \ominus A_n \nearrow A_1 \ominus A$ by Lemma 2.21. Hence we get by what was proved

$$\lim \mu^*(A_1 \ominus A_n) = \mu^*(A_1 \ominus A). \tag{28}$$

Writing

$$\begin{multline} \mu^*(A_1 \ominus A) + \mu^*(A) \ge \mu^*(A_1) = \mu_*(A_1) \ge \\ \mu_*(A_1 \ominus A_n) + \mu_*(A_n) = \mu^*(A_1 \ominus A_n) + \mu^*(A_n) \end{multline} \tag{29}$$

(the inequalities rely on Lemmas 4.4 and 4.3, the equalities — on (15) and (27)), we get with account of (28) $\mu^*(A) \ge \lim \mu^*(A_n)$. On the other hand $\mu^*(A) \le \mu^*(A_n)$ by Lemma 4.2. So $\mu^*(A) = \lim \mu^*(A_n)$. Now, it suffices, in view of (15) and (16), to show that, under the assumptions of this item,

$$\mu_*(A) = \lim \mu_*(A_n).$$

Fix $\varepsilon > 0$ and take, for each $k \in \mathbb{N}$, an existing by the definition of μ_* element $C_k^\varepsilon \in \mathfrak{R}^{\searrow}$ such that

$$C_k^\varepsilon \le A_k \tag{30}$$

and

$$\mu(C_k^\varepsilon) > \mu^*(A_k) - 2^{-k}\varepsilon. \tag{31}$$

Denote $D_n^\varepsilon = C_1^\varepsilon \wedge \ldots \wedge C_n^\varepsilon$. By Corollary 4.7

$$\mu^*(A_n \ominus D_n^\varepsilon) \leq \sum_{k=1}^{n} \mu^*(A_n \ominus C_k^\varepsilon). \tag{32}$$

By assumption $A_n \leq A_k$ as $n \geq k$, whence by Lemmas 2.7 and 4.2

$$\mu^*(A_n \ominus C_k^\varepsilon) \leq \mu^*(A_k \ominus C_k^\varepsilon) \quad \text{as} \quad n \geq k. \tag{33}$$

By Lemma 4.9 $C_k \in \mathfrak{A}$; by assumption $A_n \in \mathfrak{A}$, so Lemma 5.1 asserts that

$$\mu^*(A_k \ominus C_k^\varepsilon) = \mu_*(A_k) - \mu_*(C_k^\varepsilon). \tag{34}$$

By the choice of C_k^ε and by Lemma 4.9 $\mu_*(C_k^\varepsilon) = \mu(C_k^\varepsilon)$, which together with (31) – (34) yields $\mu^*(A_n \ominus D_n^\varepsilon) < \varepsilon$. By construction and Lemma 2.4 $D_n^\varepsilon \in \mathfrak{R}^{\searrow}$, whence by Lemma 4.9 $D_n^\varepsilon \in \mathfrak{A}$. Also by construction $D_n^\varepsilon \leq A_n$. By condition (15) $A_n \in \mathfrak{A}$. The last three relations imply by Lemma 5.1 that $\mu^*(A_n \ominus D_n^\varepsilon) = \mu^*(A_n) - \mu^*(D_n^\varepsilon)$. Herein $\mu^*(D_n^\varepsilon) = \mu(D_n^\varepsilon)$ by Lemma 4.9. Thus $\mu^*(A_n) \leq \mu(D_n^\varepsilon) + \varepsilon$, which in view of (15) can be written as

$$\mu_*(A_n) \leq \mu(D_n^\varepsilon) + \varepsilon. \tag{35}$$

The sequence (D_n^ε) decreases, by construction, so there exists $D^\varepsilon \in \mathfrak{D}$ such that $D_n^\varepsilon \searrow D^\varepsilon$. Lemma 3.4 asserts that $D^\varepsilon \in \mathfrak{R}^{\searrow}$ and $\mu(D^\varepsilon) = \lim \mu(D_n^\varepsilon)$, which together with (35) yields

$$\lim \mu_*(A_n) \leq \mu(D^\varepsilon) + \varepsilon. \tag{36}$$

Besides, $\mu(D^\varepsilon) = \mu_*(D^\varepsilon)$ (by Lemma 4.8) and $D^\varepsilon \leq A$ (by Lemma 2.3). From the last two relations we get by Lemma 4.2 $\mu(D^\varepsilon) \leq \mu_*(A)$, hereon formula (36) where ε is an arbitrary positive number yields $\lim \mu_*(A_n) \leq \mu_*(A)$. On the other hand, the assumed in this item relation $A_n \geq A$ implies by Lemma 4.2 that $\lim \mu_*(A_n) \geq \mu_*(A)$. □

Proof of Theorem 1.1. Every bounded monotonic sequence in $\mathfrak{A}$ converges in $\mathfrak{D}$, since $\mathfrak{D}$ is a δ-lattice. By Proposition 5.2 its limit belongs to $\mathfrak{A}$. So $\mathfrak{A}$ is monotonically closed.

For $A \in \mathfrak{A}$ we put $\mu(A) = \mu_*(A)$ ($= \mu^*(A)$ by the choice of A). Lemma 4.9 asserts that thus defined function μ is an extension of likewise denoted function defined hitherto only on $\widetilde{\mathfrak{R}}$ (and prior to the preparatory extension – only on $\mathfrak{R}$). By construction $\mathfrak{R} \subset \mathfrak{A} \subset \mathfrak{D}$, which together with monotonic closedness of $\mathfrak{A}$ and assumed monotonic denseness of $\mathfrak{R}$ in $\mathfrak{D}$ entails the equality

$\mathfrak{A} = \mathfrak{D}$. The extended function μ is isotonic by Lemma 4.2, Boolean additive by Corollary 4.5 and monotonically continuous by Proposition 5.2. So it is a measure.

If μ_1 and μ_2 are measures on $\mathfrak{D}$, then the set $\{A \in \mathfrak{D} : \mu_1(A) = \mu_2(A)\}$ is, obviously, monotonically closed. Consequently, any measure defined initially on a monotonically dense Boolean sublattice of $\mathfrak{D}$ admits at most one extension to $\mathfrak{D}$. $\square$

The above scheme of extension applies to integrals, as well. Namely, let I be an integral (= additive isotonic upper continuous at zero functional) on an additive sublattice F of an additive δ-lattice E. Suppose that F is cofinal and monotonically dense in E. First we extend I to $\mathrm{F}^{\searrow} \cup \mathrm{F}^{\nearrow}$ by monotonic continuity and then introduce the functionals $I_* x = \sup\limits_{u \in \mathrm{F}^{\searrow},\ u \leq x} Iu$ and $I^* x = \inf\limits_{v \in \mathrm{F}^{\nearrow}, v \geq x} Iv$ on E. Denote $\mathrm{L} = \{x \in \mathrm{E} : I_* x = I^* x\}$. For $x \in \mathrm{L}$ we put $Ix = I_* x$, or, equivalently, $Ix = I^* x$. It was shown in [8] that $\mathrm{L} = \mathrm{E}$ and thus extended I is an integral on E.

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