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Local Regularity for Minimizers of Obstacle Problems of Some Integral Functionals

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Abstract

A local regularity result is obtained for minimizers $u \in \mathcal{K}_\psi = \Big\{u \in W^{1,p}_{loc}(\Omega) : u \geq \psi\}$, $1 < p < \infty$, of integral functionals of the type

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x,u,Du)dx,$$

where the Carathéodory function $f(x,u,Du) = f_0(x,u,Du)+f_1(x,u,Du)$, $f_0(x,s,z)$ grows like $|z|^p$ with $1 < p < \infty$, and $f_1(x,s,z)$ satisfies some controllable growth condition.

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1 Introduction and Statement of Result.

Let $\Omega \subset \mathrm{R}^n$, $n \geq 2$, be a bounded domain. We consider integral functionals of the type

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x,u,Du)dx, \tag{1.1}$$

where the Carathéodory function $f(x,s,z)$ satisfies the following assumptions:

(i) $f(x,s,z):\Omega\times\mathrm{R}\times\mathrm{R}^n\to\mathrm{R}$ can be written as

$$f(x,s,z)=f_0(x,s,z)+f_1(x,s,z);$$

(ii) $f_0(x,s,z)$ satisfies the growth condition

$$L^{-1}|z|^p\le f_0(x,s,z)\le L|z|^p+\varphi_1,$$

where $L>1, 1<p<n$ and $\varphi_0\in L^r_{loc}(\Omega)$ with $1<r<\frac{n}{p}$;

(iii) there exist $0\le m<p$ and $0\le h(x)\in L_{loc}^{\frac{pr}{p-m}}(\Omega)$ such that

$$|f_1(x,s,z)|\le h(x)|z|^m.$$

In the present paper we shall consider minimizers $u\in\mathcal{K}_\psi=\Big\{u\in W^{1,p}_{loc}(\Omega):u\ge\psi\}$ for (1.1), that is,

$$\mathcal{F}(u,\mathrm{supp}(u-v))\le\mathcal{F}(v,\mathrm{supp}(u-v)) \tag{1.2}$$

for every $v\in K_\psi$. The main result is the following theorem.

Theorem 1.1 *Assume that the integral function (1.1) satisfies conditions (i), (ii) and (iii). Let $\psi\in W^{1,pr}_{loc}(\Omega)$. If $u\in\mathcal{K}_\psi$ satisfies (1.2), then it belongs to $L^{(pr)^*}_{loc}(\Omega)$.*

We refer the reader to [1-6] for some results related to local regularity property.

2 Preliminary Lemmas.

For $x_0\in\Omega$ and $t\in\mathrm{R}$, we denote by $B_t=B_t(x_0)$ the ball of radius t centred x_0. For $k>0$, let

$$A_k=\{x\in\Omega:|u(x)|>k\}\ \text{ and }\ A_{k,t}=A_k\cap B_t. \tag{2.1}$$

Moreover, if $m<n$, m^* is the real number satisfying $m^*=\frac{nm}{n-m}$.

In order to prove Theorem 1.1, we need the following two preliminary lemmas.

Lemma 2.1 *Let $u\in W^{1,p}_{loc}(\Omega),\varphi_0\in L^r_{loc}(\Omega)$, where $1<p<n$ and r satisfies*

$$1<r<\frac{n}{p}. \tag{2.2}$$

Assuming that the following integral estimate holds

$$\int_{A_{k,\tau}} |Du|^p dx \le c_0 \left[\int_{A_{k,t}} \varphi_0 dx + (t-\tau)^{-\alpha} \int_{A_{k,t}} |u|^p dx \right], \qquad (2.3)$$

for every $k \in N$ *and* $R_0 \le \tau < t \le R_1$, *where* c_0 *is a positive constant that depends only on* n, p, r, R_0, R_1 *and* $|\Omega|$, *and* α *is a real positive constant. Then* $u \in L^{(pr)^*}_{loc}(\Omega)$.

The proof can be found in [1, Theorem 2.1].

Lemma 2.2 *Let* $f(\tau)$ *be a non-negative bounded function defined for* $0 \le R_0 \le t \le R_1$. *Suppose that for* $R_0 \le \tau < t \le R_1$ *we have*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \theta f(t), \qquad (2.4)$$

where A, B, α, θ *are non-negative constants, and* $\theta < 1$. *Then there exist a constant* c, *depending only on* α *and* θ *such that for every* $\rho, R, R_0 \le \rho < R \le R_1$ *we have*

$$f(\rho) \le c[A(R-\rho)^{-\alpha} + B]. \qquad (2.5)$$

The proof can be found in [7, p.161, Lemma 3.1].

3 Proof of Theorem 1.1.

In the sequel the letter c will stands for a genetic constant which may vary from line to line. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily fixed. Let

$$T_\psi = \max\{T_k(u), \psi\},$$

where $T_k(u)$ is the usual truncation of u at level $k > 0$, that is

$$T_k(u) = \max\{-k, \min\{k, u\}\}.$$

Choose $v = u - \eta(u - T_\psi)$ in (1.2), where η is a cut-off function such that

$$\eta \in C_0^\infty(B_t), 0 \le \eta \le 1, \eta = 1 \text{ in } B_\tau \text{ and } |D\eta| \le 2(t-\tau)^{-1}.$$

For $u \in \mathcal{K}_\psi$, from $\psi \in W^{1,pr}_{loc}(\Omega)$ and

$$v = u - \eta(u - T_\psi) = (1-\eta)u + \eta T_\psi \ge (1-\eta)\psi + \eta\psi = \psi,$$

we know that $v \in \mathcal{K}_\psi$. (1.2) implies

$$\begin{aligned} \int_{B_t} f(x,u,Du)dx &\le \int_{B_t} f(x,v,Dv)dx \\ &= \int_{A_{k,t}} f(x, u - \eta(u-T_\psi), Du - D(\eta(u - T_\psi)))dx \\ &\quad + \int_{B_t \cap \{|u| \le k\}} f(x,u,Du)dx, \end{aligned} \qquad (3.1)$$

from which we derive

$$\int_{A_{k,t}} f(x,u,Du)dx \le \int_{A_{k,t}} f(x, u-\eta(u-T_\psi), Du - D(\eta(u-T_\psi)))dx. \quad (3.2)$$

Using (i), (ii) in (3.2) we have

$$\begin{aligned}
& L^{-1}\int_{A_{k,t}} |Du|^p dx \\
\le\ & \int_{A_{k,t}} f(x, u-\eta(u-T_\psi), Du - D(\eta(u-T_\psi)))dx - \int_{A_{k.t}} f_1(x,u,Du)dx \\
\le\ & \int_{A_{k,t}\setminus A_{k,\tau}} f(x,(1-\eta)u+\eta T_\psi,(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi)dx \\
& + \int_{A_{k,\tau}} f(x,T_\psi,DT_\psi)dx - \int_{A_{k.t}} f_1(x,u,Du)dx \\
\le\ & \int_{A_{k,t}\setminus A_{k,\tau}} f_0(x,(1-\eta)u+\eta T_\psi,(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi)dx \\
& + \int_{A_{k,t}\setminus A_{k,\tau}} f_1(x,(1-\eta)u+\eta T_\psi,(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi)dx \\
& + \int_{A_{k,\tau}} f_0(x,T_\psi,DT_\psi)dx + \int_{A_{k,\tau}} f_1(x,T_\psi,DT_\psi)dx - \int_{A_{k,t}} f_1(x,u,Du)dx \\
=\ & I_1+I_2+I_3+I_4+I_5.
\end{aligned} \quad (3.3)$$

Using (ii), (iii) and Young's inequality, $|I_i|$, $i = 1,2,\cdots,5$, can be estimated as follows:

$$\begin{aligned}
|I_1| & \le L\int_{A_{k,t}\setminus A_{k,\tau}} |(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi|^p dx + \int_{A_{k,t}\setminus A_{k,\tau}} \varphi_1 dx \\
& \le c\int_{A_{k,t}\setminus A_{k,\tau}} (|Du|^p + (t-\tau)^{-p}|u|^p + |D\psi|^p)dx + \int_{A_{k,t}\setminus A_{k,\tau}} \varphi_1 dx,
\end{aligned}$$

$$\begin{aligned}
|I_2| & \le \int_{A_{k,t}\setminus A_{k,\tau}} h|(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi|^m dx \\
& \le \varepsilon\int_{A_{k,t}\setminus A_{k,\tau}} |(1-\eta)Du-(u-T_\psi)D\eta+\eta DT_\psi|^p dx + c(\varepsilon)\int_{A_{k,t}\setminus A_{k,\tau}} h^{\frac{p}{p-m}} dx \\
& \le c\varepsilon\int_{A_{k,t}\setminus A_{k,\tau}} (|Du|^p + (t-\tau)^{-p}|u|^p + |D\psi|^p)dx + c(\varepsilon)\int_{A_{k,t}\setminus A_{k,\tau}} h^{\frac{p}{p-m}} dx,
\end{aligned}$$

$$|I_3| \le L\int_{A_{k,\tau}} |D\psi|^p dx + \int_{A_{k,\tau}} \varphi_1 dx,$$

$$|I_4| \le \int_{A_{k,\tau}} h|D\psi|^m dx \le \varepsilon\int_{A_{k,\tau}} |D\psi|^p dx + c(\varepsilon)\int_{A_{k,\tau}} h^{\frac{p}{p-m}} dx,$$

$$|I_5| \le \int_{A_{k,t}} h|Du|^m dx \le \varepsilon\int_{A_{k,t}} |Du|^p dx + c(\varepsilon)\int_{A_{k,t}} h^{\frac{p}{p-m}} dx.$$

In the above estimates we have used the facts

$$|u-T_\psi| \le |u|,\ |DT_\psi| \le |D\psi| \quad \text{in } A_{k,t}.$$

Substituting the above estimates into (3.3), we have

$$\begin{aligned}\int_{A_{k,\tau}} |Du|^p dx \;\; &\le \;\; c\int_{A_{k,t}\setminus A_{k,\tau}} (|Du|^p + (t-\tau)^{-p}|u|^p)dx \\ &\quad + c\int_{A_{k,t}} (\varphi_1 + h^{\frac{p}{p-m}} + |D\psi|^p)dx.\end{aligned} \tag{3.4}$$

Adding to both sides c times the left-side and dividing by $1+c$ we get

$$\begin{aligned}\int_{A_{k,\tau}} |Du|^p dx \;\; &\le \;\; \theta\int_{A_{k,t}} |Du|^p dx + \frac{\theta}{(t-\tau)^p}\int_{A_{k,t}} |u|^p dx \\ &\quad + c\int_{A_{k,t}} (\varphi_1 + h^{\frac{p}{p-m}} + |D\psi|^p)dx,\end{aligned} \tag{3.5}$$

where $\theta = \frac{c}{1+c} < 1$. Lemma 2.2 yields that for any ρ and R with $R_0 \le \rho \le \tau < t \le R \le R_1$, we have

$$\int_{A_{k,\rho}} |Du|^p \le \frac{c}{(R-\rho)^p}\int_{A_{k,R}} |u|^p dx + c\int_{A_{k,R}} (\varphi_1 + h^{\frac{p}{p-m}} + |D\psi|^p)dx. \tag{3.6}$$

Theorem 1.1 follows from Lemma 2.1.

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