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A Study on Difference Equations with Asymptotic Stability

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Abstract

In this paper, we study stochastic difference equation with asymptotic stability condition to the equilibrium. This equilibrium can be taken to be zero, without loss of generality. In this difference equation, linearization of the equation close to the equilibrium does not determine the asymptotic behaviour, because the terms which depend on the state of the system are o(x) as $x \to 0$. The non-exponential convergence of solutions of stochastic difference and functional difference equations with bounded delay has been studied.

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1 Introduction

The present work comes from the fact that we attempt to determine the exact nonlinear rate of decay of a stochastic difference equation when the terms depending on the state are o(x) as $x \to 0$ [2]. The non-autonomous features responsible for the polynomial decay of solutions; very rapid decay of the external perturbation is permitted. In doing, we prove some new results for the asymptotic decay of perturbed nonlinear deterministic difference equations. These results are quoted separately, as they may be of independent interest. We attempt to connect our results, as the difference equation studied here can be viewed as a discretisation of the stochastic difference equation [7]. In doing so, we show that the Euler-Maruyama method is dynamically consistent with the stochastic difference equation, in that, for a sufficiently small step size, solutions of the difference equation converge almost surely under conditions which imply the almost sure convergence of the stochastic difference equation.

2 III. Stochastic Difference Equations of decay property

Let A be a non-random number, h > 0 be fixed, and let $\{B_n\}$, $n \in \mathbb{N}$ be a sequence of independent random variables with

$$\mathbb{E}B_n = 0, \quad \mathbb{E}[B_n^2] = 1, \; n \in \mathbb{N} \tag{1}$$

We consider the equation

$$x_{n+1} = x_n \left(1 - hf\,(x_n) + \sqrt{hg}\,(x_n)\,B_{n+1}\right), \quad n = 0, 1, \ldots .x_0 \;\in\; \mathbb{R}^1 \tag{2}$$

We suppose that $f \in\; C(\mathbb{R};;\mathbb{R})$ is a non-random function, and $(\sigma_n)\,n \in \mathbb{N}$ is a non-random sequence. In order that zero be an equilibrium of the untroubled equation, we ask that

$$Let f\,(0) = 0. \tag{3}$$

We now impose some sufficient conditions on f and σ to ensure that all solutions of (2) tend to zero almost surely. We do not claim that these conditions are optimal ones for this purpose; rather we wish to show that solutions of (2) can be almost surely asymptotically stable [9]. In other results, we take the asymptotic stability as a fact, and then determine the rate at which solutions tend to the limit, given information on the asymptotic bahaviour of σ_n as $n \to \infty$ and $f\,(x)\, as\; x \to 0$.
Turning to the asymptotic stability result, we suppose that zero is a stable solution of the unperturbed equation, in the sense that

$$xf\,(x) > 0 \;\; x \neq 0. \tag{4}$$

3 Stochastic Difference Equations of decay property

If we define the stochastic process $\{T_n\}\;\, n \in \mathbb{N}$ by

$$T_n = \sqrt{h\sigma_n B_{n+1}}, \quad n \;\in\; \mathbb{N}, \tag{5}$$

Then the realization $X\,(\omega) = (X_n\,(\omega))_{n\,\in\,\mathbb{N}}$ of the solution of equation (2) can be identified with the solution, with $x_n = X_n(\omega)$ and $t_n = T_n\,(\omega)$. Due to the results of the last section, if we can show that $t_n = T_n\,(\omega)$ exhibits the decay property for almost all $\omega \in \Omega$, we can apply the deterministic results of the last section to each path associated with the outcome ω. Therefore if we can show

$$\underset{n \to \infty}{\lim sub}\; \frac{\log |T_n|}{\log n} < -\frac{\beta}{1-\beta} = \frac{p}{q}\;, \quad where\; p + q = 1 \tag{6}$$

All the results in the last section apply to the stochastic difference equation (2)

In order to prove (6), we request that Φ, the distribution function of each of the $B_{n,}$ has tends which decay faster than any polynomial function, namely

$$\lim_{n \to \infty} (1 - \Phi(x))\, x^{\gamma} = 0,$$

$\lim_{n \to \infty} \Phi(x)\, x^{\gamma} = 0$ For every $\gamma = 0$.

This condition is true for bounded random variables, two-sided exponential random variables (i.e., a random variable with density ϕ given by $\phi(x) = \frac{1}{2}\alpha e^{-\alpha|x|}$for all $x \in \mathbb{R}$ and some $\alpha > 0$), and log normally distributed random variables [10, 11]. It is also true for normally distributed random variables; therefore, it applies in the case when we view (2) as a strong Euler-Maruyama approximation of a stochastic difference equation with uniform mesh size h > 0.

The condition that the large fluctuations of B_n grow more slowly than any polynomially growing function.

IV. Numerical Methods and Stochastic Difference Equations

We show how the asymptotic stability of the solution of the stochastic difference equation (2) compares with that of the solution of the stochastic difference equation

$$dX(t) = -f(X(t))\, dt + \sigma(t)\, dB(t) \tag{7}$$

Here $(B_n)\, n \geq 0$ is a sequence of standardised normally distributed random variables (which satisfy (2.1)), and B is a scalar standard Brownian motion [15].

To explain the relationship between equations (2) and (7), let us recall the Euler-Maruyama numerical method that computes approximations $X_n(h) \approx X(nh)$ by

$$X_{n+1}(h) = X_n(h) - hf(X_n(h)) + \sigma(nh) \triangle B_{n+1}, \tag{8}$$

Where h > 0 is the constant step size and $\triangle B_{n+1} = B((n+1)h) - B(nh)$. We see that when

$$\sigma_n = \sigma(nh),$$

$$A_{n+1} = \frac{B((n+1)h) - B(nh)}{\sqrt{h}},$$

(8) coincides with (2). Moreover, we notice that the condition is satisfied for the common distribution function of the B_n, which is that of a standardized

normal random variable B were then generated according to $B = \sqrt{-2logU_1}\cos(2\pi U_2)$, where $?a_1$ and U_2 are uniform random numbers in [0,1].
In the simulations below, the function f and noise intensity (σ_n) always take the same form. We have chosen

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{1}{2}, & x < -1, \\ sgn(x)\,|x|^{3/2}, & x \in [-1, 1], \\ \frac{3}{2}x - \frac{1}{2}, & x > 0, \end{cases}$$

and taking $\sigma_n > 0$ where

$$\sigma_n = \frac{5}{(1 - nh)^{7/2}},\ n \geq 0.$$

In each case, we have taken $(B_n)_{n\geq 1}$ to be a sequence of independent normal random variables with zero mean and variance one. In the nomenclature of the problem, we see that $\beta = 3/2,\ \ a = 1,$ and we may define K > 0 according to

$$K = K_1 \vee K_2,$$

Where $K_1 = \begin{matrix} sup \\ x > 0 \end{matrix}\ \frac{f(x)}{x}$,

$$K_2 = \begin{matrix} sup \\ x < 0 \end{matrix}\ \frac{f(x)}{x}.$$

Therefore K $= 3/2$. By dint of this choice of (σ_n), the stochastic difference equation can be written as

$$dU(t) = -\beta U(t)\,dt + dF(t)\,, \tag{9}$$

Where $-\beta U(t)$ represents the systematic part due to the resistance of the medium and dF(t) represents the random component. It is assumed that these two parts are independent and that F(t) is a wiener process with drift $\mu = 0$ and variance parameter σ^2. The Markov process $\{U(t)\,,\ t \geq 0\}$ is such that in a small interval of time the change in U(t) is also small [15]. Since F(t) is a Wiener process, we have from (9)

$$\begin{matrix} lim \\ \triangle t \to 0 \end{matrix}\ \frac{E\{U(t + \triangle t) - U(t) \mid U(t) = u\}}{\triangle t}$$

$$= -\beta u + \begin{matrix} lim \\ \triangle t \to 0 \end{matrix}\ \frac{E\{\triangle F(t)\}}{\triangle t}$$

$$= -\beta u,$$

and

$$\lim_{\triangle t \to 0} \frac{var\{U(t+\triangle t) - U(t) \mid U(t) = u\}}{\triangle t}$$

$$= \lim_{\triangle t \to 0} \frac{(\triangle t)^2}{\triangle t} + \lim_{\triangle t \to 0} \frac{var\{\triangle F(t)\}}{\triangle t}$$

$$= \sigma^2$$

In other words, the limits exist. So the process $\{U(t),\ t \geq 0\}$ is a diffusion process and its transition p.d.f $p(\mu_0,\ u, t)$satisfies the forward Kolmogorov equation.

Conclusion: Asymptotic Stability and the rate of decay of solutions of the difference equation

$$x_{n+1} = x_n \left(1 - hf(x_n) + \sqrt{hg}(x_n) B_{n+1}\right), \quad n = 0, 1, \ldots \quad x_0 \ \in \ \mathbb{R}^1$$

Where B_{n+1} are independent random variables. The functions f and g are nonlinear and are assumed to be bounded. The small parameter h > 0 usually arises as the step size in numerical schemes. The conclusion may be viewed as stochastically disturbed version of a deterministic autonomous difference equation, where the random deviation of a system is state dependent. In general, it does not have linear leading order spatial dependence close to the equilibrium. Similarly to deterministic difference equations, analyzing asymptotic behavior of stochastic difference equations is often harder.

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