Mathematica Aeterna, Vol. 5, 2015, no. 4, 649 - 656

Ricci Curvature Of Einstein Kropina Metric

[1]Chethana B.C., [1*]Narasimha murthy S.K. and [2]Latha Kumari G.N.

[1]Department of P.G. Studies and Research in Mathematics,
Kuvempu University,
Shankaraghatta - 577451,
Shimoga, Karnataka, INDIA.
E-mail ID: chethanaanil@gmail.com, *nmurthysk@hotmail.com

[2]Department of Mathematics
GFGC, Tarikere, Chikkamagalore (D),
Karnataka, INDIA.
E-mail ID: nslathams@yahoo.com

Abstract

The purpose of this paper is to establish curvature properties of projectively related two Einstein Kropina metrics $L = \alpha^2/\beta$ and $\bar{L} = \bar{\alpha}^2/\bar{\beta}$. Further we proved that α and $\bar{\alpha}$ are Einstein and also showed that the Einstein Kropina metric with Killing 1-form have non-positive Ricci-curvature.

Mathematics Subject Classification: 53C60, 53B40.

Keywords: Finsler metric, Kropina metric, Scalar curvature, Ricci-curvature.

1 Introduction

The projective changes between two Finsler spaces have been studied by many geometers like Z. Shen, H. Shimada, I. Y. Lee, H. S. Park etc. Two regular metrics on an n-space are said to be *pointwise projectively related* if they have the same geodesics as point sets i.e., two Finsler metrics on a manifold M^n are projectively equivalent if they have the same geodesics as point sets. Two regular metric spaces are said to be *projectively related*, if there is a diffeomorphism between them such that the pull-back metric is pointwise projectively related to another one. The notion of an (α, β)-metric was introduced by M.Matsumoto and has been studied by many authors.

1∗ Corresponding Author.

By definition, every 2-dimensional Riemann metric is an Einstein metric, but generally not of Ricci constant. In higher dimensions, it is well known that every Einstein Riemann metric is of Ricci constant and, in particular, every 3-dimensional Einstein Riemann metric is of constant sectional curvature.

Since, every Riemann surface is Einstein. However, Finsler surfaces are typically not Einstein, with counterexamples provided by Numata metrics. Thus Einstein metrics in Finsler geometry are more complicated than those in Riemanns. Recently, some progress has been made on Finsler Einstein metrics of (α, β) type. An (α, β)-metrics form a special and important class of Finsler metrics which can be expressed in the form $L = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = \beta(y) = b_i(x)y^i$ is a 1-form on M^n and $\phi(s)$ is a C^∞ positive function on some open interval. In particular, when $\phi(s) = \frac{1}{s}$, the Finsler metric $L = \alpha^2/\beta$ is called a Kropina metric. Kropina metrics were first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by V. K. Kropina [8]. Such a metric is of physical interest in the sense that it describes the general dynamical system represented by a Lagrangian function, although it has the singularity. However, Randers metrics are regular Finsler metrics but Kropina metrics are Finsler metrics with sigularity. Kropina metrics seem to be among the simplest nontrivial Finsler metrics with many interesting applications in physics.

Kropina metrics have been received more attention recently. There are so many authors studied on Kropina metrics [1]. D.Bao, Colleen Robles were worked Randers spaces on constant flag curvature and also studied Ricci and flag curvatures in Finsler geometry [2][3], S. Bacso et al were studied curvature properties of (α, β)-metrics [4], X. Chen, and Z. Shen studied projectively flat Finsler metrics with almost isotropic S-curvature and also worked on Randers metrics with special curvature properties [5][6]. Guojun Yang studied on m-Kropina metrics of scalar flag curvature [7]. S. K. Narasimhamurthy et al worked on projectively related Kropina metric [9]. H. G. Nagaraj worked only on Kropina metrics [10]. and also S. K. Narasimhamurthy et al worked projectively flat Kropina metric with K=0 [11]. B. Rezaei, A. Razavi and N. Sadeghzadeh studied on Einstein (α, β)-metrics [12]. Z. Shen, On projectively related Einstein metrics [13]. H. Yasuda and H. Shimada studied on Randers spaces of scalar curvature [14]. Further R. Yoshikawa and K. Okubo worked Kropina Spaces of constant curvature [15] and also X. Zhang and Y. B. Shen worked on Einstien Kropina metrics [16].

In our earlier paper [9], we have studied projectively related Kropina metric and some curvature properties. In this paper, we extended to study the two Einstein Kropina metric with Killing 1-form have Ricci-curvature property.

2 Preliminaries

Definition 2.1 *A Finsler metric is a scalar field $L(x, y)$ which satisfies the following three conditions:*

1. *It is defined and differential for any point of $TM\backslash\{0\}$.*

2. *It is positively homogeneous of first degree in y^i, that is,* $L(x, \lambda y) = \lambda L(x, y)$, *for any positive number* λ.

3. *It is regular, that is,* $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, *constitute the regular matrix* g_{ij}, *where* $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x, y)$ is called Finsler space $F^n = (M^n, L)$

Example: Let M^n be a real n-dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable 1-form ω. Let H be a closed subset of $\phi(U) \times R^n$ consisting of all points (x^i, y^i) such that $\omega_i(x) = 0$.
Let $L(x^i, y^i) = \frac{g_{ij}(x)y^iy^j}{\omega_i(x)y^i}$ be the real valued function defined on the open set $U^* = \phi(U) \times R^n - \{H\}$. Let B denote the union of all open sets $\phi^{-1}(U^*)$. It is clear that L satisfies the homogeneity property on B and satisfies $Rank(\dot{\partial}_i\dot{\partial}_j L^2/2) = n$ on an open submanifold A of B. The pair $F^n = (M^n, L)$ is a Finsler space called *Kropina space*.

There is a class of Finsler metrics defined by a Riemannian metric and 1-form on a manifold, which is relatively simple with interesting curvature properties called (α, β)-metrics, which are defined generally as following.

Definition 2.2 *The Finsler space $F^n = (M^n, L)$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is differentiable 1-form.*

The space $R^n = (M^n, \alpha)$ is called the *associated Riemannian space* and the covariant vector field b_i is the *associated vector field.*
An (α, β)-metric is expressed in general form as,

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval (-b_0, b_0). The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by,

$$\| \beta_x \|_\alpha = sup_{y \in T_xM}\{\beta(x, y), \alpha(x, y)\} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In order to define L, β must satisfy the condition $\| \beta_x \|_\alpha < b_0$ for all $x \epsilon M^n$. The following is the simple example of the (α, β)-metric.
Example: A Finsler space having the fundamental function:

$$L(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}$$

is called a Kropina space.
The generalized (α, β)-metric:

$$L(x, y) = \frac{\alpha^{m+1}(x, y)}{\beta^m(x, y)}, \quad m \neq 0, -1,$$

is called a generalized m-Kropina metric and the Finsler space equipped with this metric is called a generalized m-Kropina space.
Generalizing the (α, β)-metric we obtain the different class of interesting (α, β)-metric. In this paper we consider one of them.

Definition 2.3 *The scalar curvature is also called the curvature scalar or Ricci scalar, is given by,*

$$R = g^{ij} R_{ij},$$

where, g^{ij} is the metric tensor and R_{ij} is the Ricci curvature tensor.

Definition 2.4 *The Ricci curvature tensor, also simply known as the Ricci tensor is defined by,*

$$R_{ik} = R^j_{ijk},$$

where, R^j_{ijk} is the Riemannian tensor.

Geometrically, the Ricci curvature is the mathematical object that controls the growth rate of the volume of metric balls in a manifold.
All over the paper, the range of Latin indices is $1, 2, ..., n$ and the Einstein summation convection is adopted, $b_{i/j}$ denotes the covariant derivative of b_i with respect to α. From the differential 1-form $\beta(x, y) = b_i(x)y^i$, we define

$$2r_{ij} = b_{i/j} + b_{j/i}, \quad 2s_{ij} = b_{i/j} - b_{j/i},$$
$$s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad s^i_0 = s^i_j y^j.$$

3 Einstein Kropina Metrics

Let L be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of L, which are defined by,

$$G^i = \frac{1}{4} g^{il} [L^2]_{x^k y^l} y^k - [L^2]_{x^l}.$$

For any $x \epsilon M$ and $y \epsilon TM \backslash \{0\}$, the Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \bigotimes dx^k$ is defined by,

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}.$$

Ricci curvature is the trace of the Riemannian curvature, which is defined by,

$$Ric = R^m_m.$$

For the Kropina metric $L = \alpha^2/\beta$, the Ricci curvature of L is given by,

$$Ric = {}^{\alpha}Ric + T,$$

where, ${}^{\alpha}Ric$ denotes the Ricci curvature of α, and

$$\begin{aligned} T &= -\frac{\alpha^2}{b^4\beta}s_0r - \frac{r}{b^4}r_{00} + \frac{\alpha^2}{b^2\beta}b^ks_{0|k} + \frac{1}{b^2}b^kr_{00|k} + \frac{n-2}{b^2}s_{0|0} + \frac{n-1}{b^2\alpha^2}\beta r_{00|0} \\ &+ \frac{1}{b^2}\left\{\frac{\alpha^2}{\beta}s_0 + r_{00}\right\}r^k_k - \frac{\alpha^2}{\beta}s^k_{0|k} - \frac{1}{b^2}r_{0|0} - \frac{2(2n-3)}{b^4}r_0s_0 - \frac{n-2}{b^4}s^2_o - \frac{4(n-1)}{b^4\alpha^2}\beta r_{00}r_0 \\ &+ \frac{2(n-1)}{b^4\alpha^2}\beta r_{00}s_0 + \frac{3(n-1)}{b^4\alpha^4}\beta^2r^2_{00} + \frac{2n}{b^2}s^k_0r_{0k} + \frac{1}{b^4}r^2_0 - \frac{\alpha^2}{b^2\beta}s^k_0r_k + \frac{n-1}{b^2\beta}\alpha^2s^k_0s_k \\ &- \frac{\alpha^4}{2b^2\beta^2}s^ks_k - \frac{\alpha^2}{b^2\beta}s^kr_{0k} - \frac{\alpha^4}{4\beta^2}s^j_ks^k_j. \end{aligned}$$

A Finsler metric L on a manifold M is called an *Einstein metric* if Ricci curvature of L satisfies

$$Ric = \sigma L^2, \tag{1}$$

where $\sigma = \sigma(x)$ is a scalar function on M. It is shown in [16] that if a Kropina metric $L = \alpha^2/\beta$ is Einstein, then

$$r_{00} = c\alpha^2, \tag{2}$$

where c is a constant. We prove the following result :

Theorem 3.1 *A Kropina metric $L = \alpha^2/\beta$ be an Einstein metric, $\alpha^2 \neq \lambda(x)\bar{\alpha}^2$. If $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ is pointwise projectively related to L, then α and $\bar{\alpha}$ are Einstein, α has non-positive scalar curvature and L have non-positive Ricci-curvature.*

Proof: Since $L = \alpha^2/\beta$ is an Einstein, then (1) and (2) holds. Thus we have [16],

$$\begin{aligned} 0 &= {}^{\alpha}Ricb^4\beta^2 + (n-2)\left[b^2s_{0|0} + b^2c_0\beta - 2c\beta s_0 - s^2_0 - c^2\beta^2\right]\beta^2 \\ &+ b^2\left[(n-3)cs_0 + (n-2)c^2\beta + b^kc_k\beta + b^ks_{0|k} - b^2s^k_{0|k} + (n-1)s^k_0s_k\right]\beta\alpha^2 \\ &- b^2\left[\frac{1}{2}s^ks_k + \frac{b^2}{4}s^i_js^j_i + \sigma b^2\right]\alpha^4. \end{aligned} \tag{3}$$

If $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ is pointwise projectively related to $L = \alpha^2/\beta$ and $\alpha^2 \neq \lambda(x)\bar{\alpha}^2$, by Theorem 3.3 [9] $s_{ij} = \bar{s}_{ij} = 0$ and $\bar{\alpha}$ is pointwise projectively related to α. Thus (3) became,

$$\begin{aligned} 0 &= {}^{\alpha}Ricb^4\beta^2 + (n-2)b^2c_0\beta^3 - (n-2)c^2\beta^4 + (n-2)b^2c^2\beta^2\alpha^2 \\ &+ b^kc_kb^2\beta^2\alpha^2 - \sigma b^4\alpha^4. \end{aligned} \tag{4}$$

Replace y by $-y$, then we get

$$\begin{aligned} 0 &= {}^{\alpha}Ricb^4\beta^2 - (n-2)b^2c_0\beta^3 - (n-2)c^2\beta^4 + (n-2)b^2c^2\beta^2\alpha^2 \\ &+ b^kc_kb^2\beta^2\alpha^2 - \sigma b^4\alpha^4. \end{aligned} \tag{5}$$

After simple seperation and arrangement of the terms of equations (4) and (5), finally we write

$$\begin{aligned} 0 &= {}^{\alpha}Ric b^4\beta^2 - (n-2)c^2\beta^4 - 2\sigma b^4\alpha^4 \\ 0 &= (n-2)b^2c^2\beta^2\alpha^2 + b^k c_k b^2\beta^2\alpha^2 + \sigma b^4\alpha^4 \end{aligned}$$

Thus we vhave,

$$\sigma = -\frac{\beta^2}{b^2\alpha^2}\left\{(n-2)c^2 - b^k c_k\right\}, \quad (6)$$

$${}^{\alpha}Ric = -\left\{\frac{2\alpha^2}{b^2}\left[(n-2)c^2 - b^k c_k\right] - (n-2)\frac{c^2\beta^2}{b^4}\right\}. \quad (7)$$

Eq. (7) implies that α is an Einstein metric with nonpositive scalar curvature. Then $\bar{\alpha}$ is pointwise projectively related to α is also Einstein [9]. Eq. (6) implies that L has nonpositive Ricci curvature.

Theorem 3.2 *Let $L = \alpha^2/\beta$ and $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ be two Einstein metrics, $\alpha^2 = \lambda(x)\bar{\alpha}^2$. Suppose that β- is a Killing 1-form and has constant length, equivalently $r_{ij} = 0, s_i = 0$. If $\bar{L}$ is pointwise projectively related to L, then they have non-positive Ricci-curvature of the form (14) and (15).*

Proof: Since $L = \alpha^2/\beta$ is Einstein, then (3) holds. Suppose that β is a Killing form of constant length. i.e., β satisfies

$$r_{ij} = 0 \ \ and \ \ s_j = 0,$$

and (3) becames,

$$\begin{aligned} 0 &= {}^{\alpha}Ric b^4\beta^2 + (n-2)\left[b^2c_0\beta - c^2\beta^2\right]\beta^2 \\ &+ b^2\left[(n-2)c^2\beta + b^k c_k\beta\right]\beta\alpha^2 - b^4\sigma\alpha^4. \end{aligned} \quad (8)$$

By the same reason, for $\bar{L}$ we get,

$$\begin{aligned} 0 &= {}^{\alpha}Ric\bar{b}^4\bar{\beta}^2 + (n-2)\left[\bar{b}^2\bar{c}_0\bar{\beta} - \bar{c}^2\bar{\beta}^2\right]\bar{\beta}^2 \\ &+ \bar{b}^2\left[(n-2)\bar{c}^2 + \bar{b}^k\bar{c}_k\right]\bar{\beta}^2\bar{\alpha}^2 - \bar{b}^4\bar{\sigma}\bar{\alpha}^4. \end{aligned} \quad (9)$$

If $\bar{L}$ is pointwise projectively related to L and $\alpha^2 = \lambda\alpha^2$, by theorem 3.3 [9], we have $s_{ij} = \lambda\bar{s}_{ij}$ and $G^i_\alpha = \bar{G}^i_\alpha$. Then (9) becames,

$$\begin{aligned} 0 &= {}^{\alpha}Ric\bar{b}^4\bar{\beta}^2 + (n-2)\left[\bar{b}^2\bar{c}_0\bar{\beta} - \bar{c}^2\bar{\beta}^2\right]\bar{\beta}^2 \\ &+ \bar{b}^2\left[(n-2)\bar{c}^2 + \bar{b}^k\bar{c}_k\right]\bar{\beta}^2\frac{\bar{\alpha}^2}{\lambda} - \bar{b}^4\bar{\sigma}\frac{\bar{\alpha}^4}{\lambda^2}. \end{aligned} \quad (10)$$

Divide (10) by $\bar{b}^4\bar{\beta}^2$, $b^i = \bar{b}^i$ [9] and since β is a Killing form so we have,

$$0 = {}^{\alpha}Ric + (n-2)\frac{\bar{c}_0\beta}{b^2} - (n-2)\frac{\bar{c}^2\beta^2}{b^4} + (n-2)\frac{\bar{c}^2\alpha}{\lambda b^2} + b^k c_k\frac{\alpha^2}{\lambda b^2} - \bar{\sigma}\frac{\alpha^4}{\lambda^2\beta^2}. \quad (11)$$

Divide (8) by $b^4\beta^2$ and (11), from these two equations we have,

$$(n-2)\frac{c_0\beta}{b^2}-(n-2)\frac{c^2\beta^2}{b^4}+(n-2)\frac{c^2\alpha^2}{b^2}+b^kc_k\frac{\alpha^2}{b^2}-\sigma\frac{\alpha^4}{\beta^2}$$
$$=(n-2)\frac{\bar{c}_0\beta}{b^2}-(n-2)\frac{\bar{c}^2\beta^2}{b^4}+(n-2)\frac{\bar{c}^2\alpha^2}{\lambda b^2}+b^k\bar{c}_k\frac{\alpha^2}{\lambda b^2}-\bar{\sigma}\frac{\alpha^4}{\lambda^2\beta^2}. \qquad (12)$$

Replace y by $-y$, we have the following,

$$-(n-2)\frac{c_0\beta}{b^2}-(n-2)\frac{c^2\beta^2}{b^4}+(n-2)\frac{c^2\alpha^2}{b^2}+b^kc_k\frac{\alpha^2}{b^2}-\sigma\frac{\alpha^4}{\beta^2}$$
$$=-(n-2)\frac{\bar{c}_0\beta}{b^2}-(n-2)\frac{\bar{c}^2\beta^2}{b^4}+(n-2)\frac{\bar{c}^2\alpha^2}{\lambda b^2}+b^k\bar{c}_k\frac{\alpha^2}{\lambda b^2}-\bar{\sigma}\frac{\alpha^4}{\lambda^2\beta^2} \qquad (13)$$

From the above two equations we get the Eq. (14) and (15),

$$\sigma=-\frac{1}{b^2L^2}\left\{(n-2)\frac{c^2\beta^2}{b^2}-[(n-2)c^2+b^kc_k]\alpha^2\right\}, \qquad (14)$$

and

$$\bar{\sigma}=-\frac{\lambda^2}{b^2L^2}\left\{(n-2)\frac{\bar{c}^2\beta^2}{b^2}-[(n-2)\bar{c}^2+b^k\bar{c}_k]\frac{\alpha^2}{\lambda}\right\}. \qquad (15)$$

It means that L and $\bar{L}$ have non-positive Ricci curvatures.

4 Conclusion

The purpose of this paper is to investigate the ricci curvature for Einstein Kropina metrics. For an (α,β)-metrics, the form β is said to be Killing form if $r_{ij}=0$ (resp. $s_{ij}=0$). β is said to be a constant Killing form if it is a Killing form and has constant length with respect to α, equivalently $r_{ij}=0, s_i=0$.

There are several interesting curvatures in Finsler geometry, among them two important and interesting curvatures are Riemann curvature and the Ricci curvature. The Ricci curvature plays an important role in the geometry of Finsler manifolds and is defined as the trace of the Riemannian curvature on each tangent space. In this paper we have the Ricci curvature for Einstein Kropina metrics and also showed α and $\bar{\alpha}$ are Einstein.

References

[1] P.L. Antonelli, *Hand book of Finsler geometry.,* Kluwer Academic Publisher, London.

[2] D. Bao and Colleen. Robles, *On Randers spaces of constant flag curvature.,* Rep on Math Phys. 51 (2003), 9-42.

[3] D. Bao, C. Robles, *Ricci and Flag Curvatures in Finsler Geometry.,* A sampler of Finsler Geometry.

[4] S. B*á*cs*ó*, X. Cheng and Z. Shen, *Curvature Properties of* (α,β)*- metrics.,* Advanced Studies in Pure and Mathematics, Math. Soc. of Japan, 48 (2007), 73-110.

[5] X. Chen, and Z.Shen, *Projectively flat Finsler metrics with almost isotropic S-curvature.,* Acta Math. Scientia, Ser. B, Engl. Ed. No.2, 26(2006), 307-313.

[6] X. Chen and Z. Shen, *Randers metrics with special curvature properties.,* Osaka. J. Math.(1)40 (2003), 87-101.

[7] Guojun Yang, *On m-Kropina Metrics of Scalar Flag Curvature.,* arXiv:13024119v1 [math.DG], (2013), 1-19.

[8] V.K.Kropina, *On projective two-dimensional Finsler spaces with a special metric.,* Trudy Sem. Vektor. Tenzor. Anal., 11(1961), 277-292.

[9] S. K. Narasimhamurthy[1], B. C. Cheteana[1] and G. N. Latha Kumari[2], *On Projectively Related Kropina Metric.,* Brit. J. Of Math and Com. Sci. , 8(4) (2015) 309 - 317.

[10] H. G. Nagaraj, *On Kropina Metrics.,* Afr. Mat., Springer. , (2013).

[11] S. K. Narasimhamurthy, Latha Kumari G.N. and C. S. Bagewadi, *Projectively Flat Kropina Metric with K=0.,* Applied Mathematical Sciences, 5 (2011), 585 - 593.

[12] B. Rezaei, A. Razavi and N. Sadeghzadeh, *On Einstein* (α,β)*-metrics.,* Tranian J. of Scie. Tech., Tran. A. 31, No. A4, Printed in Islamic Republic of Iran, (2007).

[13] Z. Shen, *On Projectively Related Einstein Metrics.,* arXiv:math9910100v1 [math.MG], (1999) 1-23.

[14] H. Yasuda and H. Shimada, *On Randers spaces of scalar curvature.,* Rep. on Math. Phys. 11 (1977), 347-360.

[15] R. Yoshikawa and K. Okubo, *Kropina Spaces of constant Curvature.,* Tensor, n. S., 68(2007), 190-203.

[16] X. Zhang and Y. B. Shen, *On Einstien Kropina metrics.,* arXiv:1207.1944v1 [math.DG] (2012) 1-17.

Received: August, 2015