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On basicity of double and unitary systems in generalized Lebesgue spaces

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Abstract

We consider a unitary system of functions in generalized Lebesgue space of functions $L_{p(\cdot)}(0,a)$ with variable summability exponent $p(\cdot)$. We define a double system of functions in $L_{p(\cdot)}(-a,a)$, associated with a unitary system, and establish the relation between the basicity of these systems in the spaces $L_{p(\cdot)}$.

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1 Introduction

In this paper we study the basicity of unitary system of functions

$$\vartheta_n^{\pm}(t) \equiv a(t)\,\omega_n^{+}(t) \pm b(t)\,\omega_n^{-}(t), \quad n \in N, \tag{1}$$

and the associated double system of functions

$$\{A(t)\,W_n(t)\,; A(-t)\,W_n(-t)\}_{n\in N}, \tag{2}$$

in generalized Lebesgue spaces $L_{p(\cdot)}(0,a)$ and $L_{p(\cdot)}(-a,a)$, respectively. The systems of the form (1) are generalizations of the perturbed systems of cosines

$$\{\cos(nt + \alpha(t))\}_{n\in N}, \tag{3}$$

where $\alpha : [0,\pi] \to R$ is some function. The systems of type (3) arise when solving a lot of partial equations by the Fourier method. For this information see the papers [1-4]. By this reason, a number of papers (see for example [1-17] have been devoted to studying basis properties (completeness, minimality, basicity). It should be noted that the basis properties of the systems of the form (3) are being studied as long ago as the beginning of the last century in the papers of Paley, Wiener [18], Levinson [19] and others. As it was established in the papers [17;18;20], the basis properties of the system of the form (3) in $L_p(0,\pi)$ are closely connected with similar properties of double system of exponents

$$\left\{A(t)\,e^{int}; B(t)\,e^{-int}\right\},$$

in $L_p(-a,a)$, whose coefficients $A(\cdot)$ and $B(\cdot)$, are determined by the data of the system (3).

In the present paper this approach is extended to the systems of the form (1) and (2) in Lebesgue space $L_{p(\cdot)}$ with variable summability exponent $p(\cdot)$. The relations between basicity of these systems in $L_{p(\cdot)}$ are established. For more detailed information with respect to spaces $L_{p(\cdot)}$ you can see the papers [21;22] and the monograph [23].

2 Necessary information

We need some information on theory of bases and from theory of Lebesgue spaces with variable summability exponent.

Let X be some Banach space.

The system $\{x_n\}_{n\in N}\subset X$ is said to be complete in X, if $\overline{L\left[\{x_n\}_{n\in N}\right]}\equiv X$, where $L[M]$ is a linear hull of M, while $\bar{M}$ denotes the closure of M in X.

The following statement is known well.

Statement 2.1. *The system $\{x_n\}_{n\in N}$ is complete in X $\Leftrightarrow$from $x^*(x_n)=0,\forall n\in N$, it follows that $x^*=0$.*

The systems $\{x_n\}_{n\in N}\subset X$ and $\{x_n^*\}_{n\in N}\subset X^*$ are called biorthogonal if

$$x_n^*(x_m)=\begin{cases}\neq 0\ , & n=m\,,\\ 0\ , & n\neq m\ ,\quad \forall n,m\in N\,.\end{cases}$$

In the case

$$x_n^*(x_m)=\delta_{nm},\forall n,m\in N,$$

they are called biorthonormalized.

The system $\{x_n\}_{n\in N}\subset X$ is said to be minimal in X if $x_r\notin\overline{L\left[\{x_n\}_{n\neq r}\right]}$, for $\forall r\in N$.

The following statement is valid.

Statement 2.2. *The system $\{x_n\}_{n\in N}\subset X$ is minimal in X $\Leftrightarrow$there exists a system biorthogonal to it.*

We call the Banach space a B-space, the Banach space of sequences from the scalars over the field K a K-space.

Give some information on Lebesgue spaces with variable summability exponent.

Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. We denote the class of all measurable in $[-\pi, \pi]$ (with respect to Lebesgue measure) functions by $\mathscr{L}_0$. Accept the denotation

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

With respect to ordinary linear operations of addition of functions and multiplication by a number for

$$p^+ = \sup \underset{[-\pi,\pi]}{vrai}\, p(t) < +\infty,$$

$\mathscr{L}$ turns into a linear space. With respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ is a Banach and we denote it by $L_{p(\cdot)}$. Assume

$$\begin{aligned} WL_0 \stackrel{def}{\equiv} \{p : \exists C > 0, \quad \forall t_1, t_2 \in [0, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\right\}. \end{aligned}$$

Everywhere $q(\cdot)$ denotes the function $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$ conjugated to: $p(\cdot)$. We accept $p^- = \inf \underset{[-\pi,\pi]}{vrai}\, p(t)$. It holds Holder's generalized inequality

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

The following property that we will use follows immediately from the definition.

Property . *If* $|f(t)| \le |g(t)|$ *a.e. on* $(-\pi, \pi)$*, then* $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.

For detailed information about space $L_{p(\cdot)}$ see the papers [21;22] and the monograph [23].

3 Basic results

Let us consider the unitary system of functions

$$\vartheta_n^{\pm}(t) \equiv a(t)\,\omega_n^{+}(t) \pm b(t)\,\omega_n^{-}(t)\,, \quad n \in N,$$

given on the interval $[0, a]$. Let $p(-x) = p(x)$, $\forall x \in [-a, a]$, Assume

$$A(t) \equiv \begin{cases} a(t)\,, & t \in [0,a]\,, \\ b(-t)\,, & t \in [-a,0)\,, \end{cases} \qquad W_n(t) \equiv \begin{cases} \omega_n^{+}(t)\,, & t \in [0,a]\,, \\ \omega_n^{-}(t)\,, & t \in [-a,0)\,. \end{cases}$$

Let us consider the double system

$$V_{n;m} \equiv (A(t)\,W_n(t)\,; A(-t)\,W_m(-t))\,, \quad n, m \in N.$$

The following theorem is valid.

Theorem 3.1. *Let $1 \le p^- \le p^+ < +\infty$ and $p(-x) = p(x)$, $\forall x \in [-a, a]$. The double system $\{V_{n;n}\}_{n \in N}$ forms a basis for $L_{p(\cdot)}(-a, a)$ if and only if each of unitary systems $\{\vartheta_n^+\}_{n \in N}$ and $\{\vartheta_n^-\}_{n \in N}$ forms a basis for $L_{p(\cdot)}(0, a)$.*

Proof. Let $\{V_{n;n}\}_{n \in N}$ be complete and minimal in $L_{p(\cdot)}(-a, a)$ and $\{h_n^+; h_n^-\}_{n \in N} \subset L_{q(\cdot)}(-a, a)$ be a system biorthogonal to it, i.e.

$$\int_{-a}^{a} A(t)\,W_n(t)\,\overline{h_n^{\pm}(t)}\,dt = \frac{1}{2}\left[\delta_{nk} \pm \delta_{nk}\right],$$

$$\int_{-a}^{a} A(-t)\;W_n(-t)\,\overline{h_n^{\pm}(t)}\,dt = \frac{1}{2}\left[\delta_{nk} \mp \delta_{nk}\right], \quad \forall\; n, k \in N.$$

We have

$$\int_{-a}^{a} A(-t)\,W_n(-t)\;\overline{h_n^{\pm}(t)}\,dt = \int_{-a}^{a} A(t)\,W_n(t)\,\overline{h_n^{\pm}(-t)}\,dt = \frac{1}{2}\left[\delta_{nk} \mp \delta_{nk}\right].$$

From the uniqueness of the biorthogonal system to the complete system and from the previous relation we immediately get that $h_n^-(t) = h_n^+(-t)$. Take $\forall\, f \in L_{p(\cdot)}(-a, a)$ and consider

$$I_m^{(-a,a)} = \left\| \sum_{n=1}^{m} \left[\int_{-a}^{a} f(t)\;\overline{h_n^{+}(t)}\,dt\,A(x)\;W_n(x)\; + \right.\right.$$

$$\left.\left. + \int_{-a}^{a} f(t)\;\overline{h_n^{-}(t)}\,dt\,A(-x)\;W_n(-x) \right] - f(x) \right\|_{L_{p(\cdot)}(-a,a)}.$$

The systems $\{\vartheta_n^{\pm}\}_{n\in N}$ are minimal in $L_{p(\cdot)}(0,a)$ and the systems biorthogonal to them and $\left\{\nu_k^{\pm}(t)\right\} \subset L_{q(\cdot)}(0,a)$ to $\{V_{n;n}\}_{n\in N}$ systems are connected with the relations

$$\tilde{\nu}_k^{\pm}(t) \equiv \begin{cases} \nu_k^{\pm}(t), & t\in(0,a), \\ \pm\nu_k^{\pm}(-t), & t\in(-a,0), \end{cases}$$

$$h_k^{\pm}(t) \equiv \frac{1}{2}\left[\tilde{\nu}_k^{+}(t) \pm \tilde{\nu}_k^{-}(t)\right], \quad k\in N.$$

Taking into account these relations, we have

$$I_m^{(-a,a)} = \left\| \sum_{n=1}^{m} \left[\int_{-a}^{a} f(t)\, \overline{h_n^{+}(t)}\, dt\, A(x)\, W_n(x) + \right.\right.$$

$$\left.\left. + \int_{-a}^{a} f(-t)\, \overline{h_n^{+}(t)}\, dt\, A(-x)\, W_n(-x)\right] - f(x) \right\|_{L_{p(\cdot)}(-a,a)} =$$

$$= \left\| \sum_{n=1}^{m} \left[\frac{1}{2}\int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^{+}(t)}\, dt\, A(x)\, W_n(x) + \right.\right.$$

$$+\frac{1}{2}\int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^{-}(t)}\, dt\, A(x)\, W_n(x) +$$

$$+\frac{1}{2}\int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^{+}(t)}\, dt\, A(-x)\, W_n(-x) +$$

$$\left.\left. + \frac{1}{2}\int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^{-}(t)}\, dt\, A(-x)\, W_n(-x)\right] - f(x) \right\|_{L_{p(\cdot)}(-a,a)}.$$

Denote

$$I_1^n(x) = \int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^{+}(t)}\, dt\, A(x)\, W_n(x) =$$

$$= \int_{0}^{a} \left[f(t) + f(-t)\right] \overline{\nu_n^{+}(t)}\, dt\, A(x)\, W_n(x);$$

$$I_2^n(x) = \int_{-a}^{a} f(t)\, \overline{\tilde{\nu}_n^{-}(t)}\, dt\, A(x)\, W_n(x) =$$

$$= \int_{0}^{a} \left[f(t) - f(-t)\right] \overline{\nu_n^{-}(t)}\, dt\, A(x)\, W_n(x);$$

$$I_3^n(x) = \int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^{+}(t)}\, dt\, A(-x)\, W_n(-x) =$$

$$= \int_{0}^{a} \left[f(t) + f(-t)\right] \overline{\nu_n^{+}(t)}\, dt\, A(-x)\, W_n(-x);$$

$$I_4^n(x) = \int_{-a}^{a} f(-t)\, \overline{\tilde{\nu}_n^{-}(t)}\, dt\, A(-x)\, W_n(-x) =$$

$$= -\int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)}\, dt\, A(-x)\, W_n(-x)\,.$$

Summing up these relations, we obtain the following expressions.

Let $x \in (0, a)$.

$$I_1^n(x) + I_3^n(x) = \int_{-a}^a [f(t) + f(-t)] \overline{\nu_n^+(t)}\, dt\, \vartheta_n^+(x)\,;$$

$$I_2^n(x) + I_4^n(x) = \int_{-a}^a [f(t) - f(-t)] \overline{\nu_n^-(t)}\, dt\, \vartheta_n^-(x)\,.$$

Let $x \in (-a, 0)$. We have

$$I_1^n(x) = \int_{-a}^a [f(t) + f(-t)]\, \overline{\nu_n^+(t)}\, dt\, b(-x)\, \omega_n^-(-x)\,;$$

$$I_2^n(x) = \int_{-a}^a [f(t) + f(-t)]\, \overline{\nu_n^-(t)}\, dt\, b(-x)\, \omega_n^-(-x)\,;$$

$$I_3^n(x) = \int_0^a [f(t) + f(-t)]\, \overline{\nu_n^+(t)}\, dt\, a(-x)\, \omega_n^+(-x)\,;$$

$$I_4^n(x) = -\int_0^a [f(t) - f(-t)]\, \overline{\nu_n^-(t)}\, dt\, a(-x)\, \omega_n^+(-x)\,.$$

Similarly, summing up these expressions we get

$$I_1^n(x) + I_3^n(x) = \int_0^a [f(t) + f(-t)] \overline{\nu_n^+(t)}\, dt\, \vartheta_n^+(-x)\,;$$

$$I_2^n(x) + I_4^n(x) = -\int_0^a [f(t) - f(-t)] \overline{\nu_n^-(t)}\, dt\, \vartheta_n^-(-x)\,.$$

In what follows we use the obvious relation

$$I_m^{(-a,a)} = \left\| \frac{1}{2} \sum_{n=1}^m \sum_{k=1}^4 I_k^n(x) - f(x) \right\|_{L_{p(\cdot)}(-a,a)} \leq$$

$$\leq \|S_m(x)\|_{L_{p(\cdot)(0,a)}} + \|S_m(x)\|_{L_{p(\cdot)(-a,0)}}\,,$$

where

$$S_m = S_m^+ + S_m^- :$$

$$S_m^+(x) \equiv \sum_{n=1}^m \frac{I_1^n(x) + I_3(x)}{2} - \frac{f(x) + f(-x)}{2}, \tag{4}$$

$$S_m^-(x) \equiv \sum_{n=1}^m \frac{I_2^n(x) + I_4^n(x)}{2} - \frac{f(x) - f(-x)}{2}. \tag{5}$$

Consequently,

$$\|S_m(x)\|_{L_{p(\cdot)(0,a)}} \le \left\|S_m^+(x)\right\|_{L_{p(\cdot)(0,a)}} + \left\|S_m^-(x)\right\|_{L_{p(\cdot)(0,a)}};$$

$$\|S_m(x)\|_{L_{p(\cdot)(-a,0)}} \le \left\|S_m^+(x)\right\|_{L_{p(\cdot)(-a,0)}} + \left\|S_m^-(x)\right\|_{L_{p(\cdot)(-a,0)}} =$$

$$= \left\|S_m^+(-x)\right\|_{L_{p(\cdot)(0,a)}} + \left\|S_m^-(-x)\right\|_{L_{p(\cdot)(0,a)}} =$$

$$\left\|\sum_{n=1}^{m}\int_0^a \frac{f(t)+f(-t)}{2}\overline{\nu_n^+(t)}\,dt\,\vartheta_n^+(x) - \frac{f(x)+f(-x)}{2}\right\|_{L_{p(\cdot)}(0,a)} +$$

$$\left\|\sum_{n=1}^{m} -\int_0^a \frac{f(t)-f(-t)}{2}\overline{\nu_n^-(t)}\,dt\,\vartheta_n^-(x) - \frac{f(-x)-f(x)}{2}\right\|_{L_{p(\cdot)(0,a)}}.$$

Now let's as suppose that the systems $\{\vartheta_n^{\pm}\}_{n\in N}$ form a basis in $L_{p(\cdot)}(0,a)$. Then from the previous relations it directly follows that it holds
$\|S_m(x)\|_{L_{p(\cdot)}} \mapsto 0$, as $m\to\infty$,
and as a result

$$I_m^{(-a,a)} \to 0, \text{ as } m\to\infty.$$

Thus, the biorthogonal series of an arbitrary function $f(t)$ from $L_{p(\cdot)}(-a,a)$ in the system $\{V_{n;n}\}_{n\in N}$ converges to it in $L_{p(\cdot)}(-a,a)$ and as result the system $\{V_{n;n}\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(-a,a)$.

Now, let's assume that the system $\{V_{n;n}\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(-a,a)$. Take $\forall\ g \in L_{p(\cdot)}(0,a)$. Continue this function evenly (oddly) to the whole interval $[-a,a]$ and denote it by $f^+(x)$ $(f^-(x))$. Consider the expression

$$S_m(x) = S_m^+(x) + S_m^-(x)\,,\ \ \forall\ m\in N,$$

where $S_m^{\pm}(x)$ are determined by relations (4), (5).

At first consider the even case. In this case it is easy to notice that

$$S_m^-(x) \equiv 0,\ \ \forall\ m\in N.$$

We have

$$I_m^{(-a,a)} = \left\|S_m^+(x)\right\|_{L_{p(\cdot)(-a,a)}}.$$

In what follows we will use the following elementary reasoning. Let $\psi(\cdot)$ be an even function on $(-a,a)$. Then from the expression

$$\int_{-a}^{a}\left|\frac{\psi(t)}{\lambda}\right|^{p(t)}dt = 2\int_0^a\left|\frac{\psi(t)}{\lambda}\right|^{p(t)}dt = \int_0^a\left|\frac{2^{\frac{1}{p(t)}}\psi(t)}{\lambda}\right|^{p(t)}dt,$$

it directly follows that it holds

$$\left\|\psi\left(t\right)\right\|_{L_{p(\cdot)}(-a,a)}=\left\|2^{\frac{1}{p(t)}}\psi\left(t\right)\right\|_{L_{p(\cdot)}(0,a)}.$$

Taking into account the obvious inequality

$$2^{\frac{1}{p^+}}\le 2^{\frac{1}{p(t)}}\le 2^{\frac{1}{p^-}},$$

we get

$$2^{-\frac{1}{p^-}}\left\|\psi\left(t\right)\right\|_{L_{p(\cdot)}(-a,a)}=\left\|\psi\left(t\right)\right\|_{L_{p(\cdot)}(0,a)}\le 2^{-\frac{1}{p^+}}\left\|\psi\left(t\right)\right\|_{L_{p(\cdot)}(-a,a)}. \tag{6}$$

As $S_m^+(\cdot)$ is an even function on $(-a,a)$, then in conformity to it from (6) we get

$$\left\|S_m^+\left(x\right)\right\|_{L_{p(\cdot)}(0,a)}\le 2^{-\frac{1}{p^+}}\left\|S_m^+\left(x\right)\right\|_{L_{p(\cdot)}(0,a)}. \tag{7}$$

From the basicity of the system $\{V_{n;n}\}_{n\in N}$ in $L_{p(\cdot)}\left(-a,a\right)$ it follows that

$$I_m^{(-a,a)}\to 0, m\to\infty.$$

As a result, from relation (7) it follows that

$$\left\|S_m^+\left(x\right)\right\|_{L_{p(\cdot)}}\mapsto 0,\ m\to\infty,$$

i.e. the biorthogonal series of an arbitrary function $g\left(t\right)$ from $L_{p(\cdot)}\left(0,a\right)$ converges to it in $L_{p(\cdot)}\left(0,a\right)$, as

$$\frac{f^+\left(x\right)+f^+\left(-x\right)}{2}=g\left(x\right).$$

This proves the basicity of the system $\{\vartheta_n^{\pm}\}_{n\in N}$ in $L_{p(\cdot)}\left(0,a\right)$. The odd case is considered in the same way and along with this we prove the basicity of the system $\{\vartheta_n^{-}\}_{n\in N}$ in $L_{p(\cdot)}\left(-a,a\right)$.

The theorem is proved.

The following theorem is proved similarly.

Theorem 3.2. *The double system $1\bigcup\{V_{n;n}\}_{n\in N}$ forms a basis for $L_{p(\cdot)}\left(-a,a\right)$ if and only if each of the systems $\{\vartheta_n^{-}\}_{n\in N}$ and $1\bigcup\{\vartheta_n^{+}\}_{n\in N}$ forms a basis for $L_{p(\cdot)}\left(0,a\right)$.*

In particular, taking as $\vartheta_n^{\pm}$ the following functions

$$\omega_n^+\left(t\right)\equiv e^{i(\lambda_n t+\mu_n)};\ \ \omega_n^-\left(t\right)\equiv e^{-i(\lambda_n t+\mu_n)}$$

on $[0,\pi]$, from these theorems we get the following statements.

Corollary 3.3. *Let $\{\lambda_n; \mu_n\}_{n\in N}$ be some sequences of complex numbers. The system of exponents $\left\{e^{\pm i(\lambda_n t+\mu_n signt)}\right\}_{n\in N}$ is complete, minimal, forms a basis for $L_{p(\cdot)}(-\pi,\pi)$, if and only if each of the systems of cosines $\{\cos(\lambda_n t+\mu_n)\}_{n\in N}$ and sines $\{\sin(\lambda_n t+\mu_n)\}_{n\in N}$ is complete, minimal and forms a basis for $L_{p(\cdot)}(0,\pi)$, respectively.*

Corollary 3.4. *The system of exponents $1\bigcup\left\{e^{\pm i(\lambda_n t+\mu_n signt)}\right\}_{n\in N}$ is complete, minimal, forms a basis for $L_{p(\cdot)}(-\pi,\pi)$ if and only if each of the system of cosines $1\bigcup\{\cos(\lambda_n t+\mu_n)\}_{n\in N}$ and sines $\{\sin(\lambda_n t+\mu_n)\}_{n\in N}$ is complete, minimal and forms a basis for $L_{p(\cdot)}(0,\pi)$, respectively.*

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