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Frequently Convergent Properties of Solutions for a Discrete Dynamical System*

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Abstract

The classic concept of limit is not enough to accurately describe the property of convergent sequence, however the definition of frequent convergence of sequence, defined by the concept of frequent measure, can get the better details of divergent sequence than the classic concept of convergence. In this thesis, using the definition and properties of frequent measure and frequent convergence, we study the frequently convergent properties of difference equations $x_{n+k} = 1 - x_n^2$. We first present a fixed point theorem and then define a polynomial function, which are both closely related to the above diffrence equations. Through different monotonic properties of the above polynomial function on a different intervals, we detailed disscuss the solution of the above diffrence equation as $k = 2$, that is $x_{n+2} = 1 - x_n^2$, when initial values in diffrent intervals, and then we generalize the conclusion to the case k being any positive integer.

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1 Introduction

Discrete-time dynamic systems are always represented by difference equations. For discrete-time dynamic system, we use t to represent time, the system can be represented by the following equation:

$$x_{t+1} = f(x_t) \tag{1}$$

where f is a function [1]. If f is a linear function, then the danymic system is linear, if f is a nonlinear function, then the danymic system is nonlinear. This paper explores the frequently convergent properties of a class of nonlinear discrete dynamic system.

For difference equation (1), given an initial value x_0, we can use this difference equations to determine a sequence $X = \{x_n\}_{n=1}^{\infty}$, the sequence is called a solution of the differential equation (1) . Since the solutions of difference equations are sequences, we can transfer some convergent properties of sequences to the solutions of difference equations. The classic concept of limit has been insufficient to accurately describe the convergence and divergence of the aequence, so Chuanjun Tian [2] first introduced the concept of frequent measure of sequence, and thus defined the definitions of frequent convergence and frequent oscillation of sequences $[3-6]$. Now the concept of frequent measure has become a basic tool of studying discrete danymic systems. In recent years, there are much attention about frequent oscillation of solutions of difference equations $[6-20]$, but little concern on frequent convergence of solutions of difference equations $[5, 21-23]$.

In 2006, Chuanjun Tian and Suisheng Zheng [5] first discussed the frequent convergence of solutions of the following difference equations with the initial value x_0 in $[0, 1]$:

$$x_{n+1} = 1 - {x_n}^2 \tag{2}$$

For the above nonlinear differential equation, the selection of initial values does not guarantee that the solution of difference equation belongs to the same range. In view of the initial value can be taken over entire real axis, Hui Li and Yuanhong Tao [23] discussed the frequently convergent properties of diffrence equation (2) as the initial value took on different intervals of real axis. This paper intends to disscuss the frequently convergent properties of the following diffrence equation:

$$x_{n+k} = 1 - {x_n}^2 \tag{3}$$

where k is an arbitary positive integer.

2 Preliminary Notes

Let Z be the set of integers, for any $k, l \in Z$, denoting $Z[k, \infty) = \{i \in Z | i \geq k\}$, $Z[k, l] = \{i \in Z | k \leq i \leq l\}$, $Z(-\infty, l] = \{i \in Z | i \leq l\}$. If $\Omega \subseteq Z$, then $|\Omega|$

means the numbers of elements of set Ω. Denoting $\Omega^{(n)} = \Omega \cap Z(-\infty, n]$. Let $X = \{X_n\}$ be a real sequence, c be any real number, then the set $\{n \in Z[k, \infty) | X_n > c\}$ will be denoted by $(v > c)$, the notations $(v \geq c), (v < c)$ and $(v \leq c)$ will be defined similarly.

Definition 2.1 [2] *Let Ω be a subset of Z^+, if the limit $\limsup_{n\to\infty} \frac{|\Omega^n|}{n}$ exists, then we call it upper frequent measure of the set Ω, denoting by $\mu^*(\Omega)$; if the limit $\lim_{n\to\infty} \inf \frac{|\Omega^n|}{n}$ exists, then we call it lower frequent measure of Ω, denoting by $\mu_*(\Omega)$. Specially, if $\mu^*(\Omega) = \mu_*(\Omega)$, then we call it the frequent measure of the set Ω, denoting by $\mu(\Omega)$, we also say that Ω is measurable. If Ω can not be measured, we say that Ω is unmeasurable.*

The following are some properties of frequency measurement:

Proposition 2.2 [2] *If $\Omega \subseteq Z^+$, $\mu_*(\Omega)$ and $\mu^*(\Omega)$ both exist, then*

$$0 \leq \mu_*(\Omega) \leq \mu^*(\Omega) \leq 1.$$

If Ω is a finite set, then $\mu(\Omega) = 0$, $\mu(Z^+) = 1$. Especially $\mu(\phi) = 0$.

Proposition 2.3 [2] *If Ω and Γ are the subsets of Z^+, $\Omega \subseteq \Gamma$, then $\mu^*(\Omega) \leq \mu^*(\Gamma)$ and $\mu_*(\Omega) \leq \mu_*(\Gamma)$.*

Proposition 2.4 [2] *If Ω and Γ are two subsets of Z^+, then we have*

$$\mu_*(\Omega) + \mu^*(\Gamma) - \mu^*(\Omega \cap \Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cap \Gamma)$$

$$\mu_*(\Omega) + \mu_*(\Gamma) - \mu^*(\Omega \cap \Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cap \Gamma)$$

Besides, if Ω and Γ are mutually disjoint, then

$$\mu_*(\Omega) + \mu_*(\Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma).$$

Proposition 2.5 [2] *For any set $\Omega \subseteq Z^+$, we have $\mu_*(\Omega) + \mu^*(Z^+ \setminus \Omega) = 1$.*

Proposition 2.6 [2] *If Ω and Γ are two subsets of Z^+, and $\Omega \subseteq \Gamma$, then we have*

$$\mu^*(\Gamma) - \mu^*(\Omega) \leq \mu^*(\Gamma \setminus \Omega) \leq \mu^*(\Gamma) - \mu_*(\Omega),$$

$$\mu_*(\Gamma) - \mu^*(\Omega) \leq \mu_*(\Gamma \setminus \Omega) \leq \mu_*(\Gamma) - \mu_*(\Omega).$$

Proposition 2.7 [2] *If Ω and Γ are two subsets of Z^+, and $\mu^*(\Omega) + \mu_*(\Gamma) \geq 1$, then the set $\Omega \cap \Gamma$ must be an infinite set.*

Definition 2.8 [3] *Let $X = \{x_n\}_{n=k}^{\infty}$ be a real sequence and c any real number. If for any given number $\varepsilon > 0$, there is a constant $\omega \in [0,1)$ such that $\mu^*(|X-c| \geq \varepsilon) \leq \omega$ (or $(\mu_*(|X-c| \geq \varepsilon) \leq \omega)$), then c is called a frequent limit of upper (respectively lower) degree ω of the sequence X, and X is said to be frequently convergent to c of upper (respectively lower) degree ω.*

If there exists a constant ε_0 such that $\mu\{|X-c| \geq \varepsilon\} = \omega$ for any number $\varepsilon \in (0, \varepsilon_0)$ then the sequence X is said to be frequently convergent to c of degree ω and c is said to be a frequent limit of degree ω of X. In particular, if $\omega = 0$, we say that X frequently converges to c, and c is the frequent limit of X, denoting by $f\lim_{n\to\infty} x_n = c$.

The following are properties of frequent limit, where $X = \{x_n\}$, $Y = \{y_n\}$, $Z = \{z_n\}$ are all real sequences.

Proposition 2.9 [3] *If $f\lim_{n\to\infty} x_n = f\lim_{n\to\infty} y_n = a$, if $\mu(X \leq Z \leq Y) = 1$, then $f\lim_{n\to\infty} z_n = a$.*

Proposition 2.10 [3] *If $f\lim_{n\to\infty} x_n = a$ and $f\lim_{n\to\infty} y_n = b \neq 0$, then $f\lim_{n\to\infty}(x_n \pm y_n) = a \pm b$ and $f\lim_{n\to\infty}(x_n y_n) = ab$.*

Proposition 2.11 [3] *If $f\lim_{n\to\infty} x_n = a$ and $f\lim_{n\to\infty} y_n = b \neq 0$, then the sequence $\{x_n/y_n\}$ is frequent convergence, and $f\lim_{n\to\infty}(x_n/y_n) = a/b$.*

Proposition 2.12 [3] *If $f\lim_{n\to\infty} x_n = a$ and function g(t) is continuous near point a, then $f\lim_{n\to\infty} g(x_n) = g(a)$.*

Definition 2.13 [3] *Let $X = \{x_n\}_{n=k}^{\infty}$ be a real sequence and $I \subseteq R$. If there exists a constant $\omega \in [0,1]$ such that $\mu^*(X \notin I) \leq \omega$ (or equivalently, $\mu_*(X \in I) \geq 1-\omega$), then X is said to be frequently inside I of upper degree ω. If $\mu_*(X \notin I) \leq \omega$ (or equivalently, $\mu^*(X \in I) \geq 1-\omega$), then X is said to be frequently inside I of lower degree ω.*

In particular, if $\mu^*(X \notin I) = 0$, then X is said to be frequently inside I.

3 Main Results

In this section, we will discuss the frequently convergence of solutions of difference equation (3). We first establish an fixed point theorem closely related to the diffrence equation (3), and which will be used in the sequel.

Provided the following two difference equations:

$$x_{n+k} = 1 - (1 - {x_n}^2)^2 \tag{4}$$

$$x_{n+k} = 1 - [1 - (1 - {x_n}^2)^2]^2 \tag{5}$$

Theorem 3.1 *The fixed points of difference equation (3) must also be fixed points of difference equations (4) and (5), but the fixed points of difference equation (4) are not always fixed points of difference equations (5).*

Proof. We first solve the fixed points of difference equation (3) : suppose that $x = 1 - x^2$, namely $x^2 + x - 1 = 0$, then two fixed points of (3) are $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$.

Then we find the fixed points of difference equation (4): set that $x = 1-(1-x^2)^2$, namely $x^4-2x^2+x = 0$, obviously $x^4-2x^2+x = x(x-1)(x^2+x-1)$, so four fixed points of (4) are $\frac{-1-\sqrt{5}}{2}, 0, \frac{-1+\sqrt{5}}{2}, 1$.

Therefore, the fixed points of difference equation (3) must also be fixed points of difference equations (4).

We next to seek the fixed points of difference equation (5): set that $x = 1 - [(1 - x^2)^2]^2$, namely $x^8 - 4x^6 + 4x^4 - 1 = 0$, obviously

$$x^8 - 4x^6 + 4x^4 - 1 = (x^2 + x - 1)(1 + x^2 + x^3 - 2x^4 - x^5 + x^6)$$

So using numerical methods, we can get six fixed points of (5):

$$\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, 0.0871062\pm0.655455i, -1.00914\pm0.324759i, 1.42203\pm0.114188i.$$

where $i = \sqrt{-1}$.

Thus the fixed points of difference equations (5) are two real numbers and four complex numbers, while the fixed points of difference equations (4) are four real numbers, so the fixed points of difference equation (3) must also be fixed points of difference equations (5), and the fixed points of difference equation (4) are not always fixed points of difference equations (5). Hence the theorem holds. #

Now we begin to discuss the diffrence equation (2) as $n = 2$, that is,

$$x_{n+2} = 1 - {x_n}^2 \qquad (6)$$

Obiviously, given two initial-values x_0, x_1, we can use equation (6) to deduce sequence $X = \{x_n\}_{n=0}^{\infty}$, which is the solution of the difference equation (6). Obviously, if the initial-values $x_0, x_1 = \frac{-1\pm\sqrt{5}}{2}$, then we can deduce that $x_n = \frac{1\pm\sqrt{5}}{2}$, $n = 0, 1, 2\cdots$, which means the solution of the difference equation (6) is constant-valued. If the initial-value x_0, x_1 doesn't equal to $\frac{-1\pm\sqrt{5}}{2}$, then we have the following theorem:

Theorem 3.2 *Let x_0, x_1 be the initial-values of the difference equation (6), $X = \{x_n\}_{n=0}^{\infty}$ be the solution , then we have the following results:*

1) If $x_0, x_1 \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, then $X = \{x_n\}_{n=0}^{\infty}$ belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$;

2) If $x_0, x_1 \in (\frac{-1-\sqrt{5}}{2}, -1] \cup [1, \frac{1+\sqrt{5}}{2})$, *then* $X = \{x_n\}_{n=0}^{\infty}$ *belongs to* $(\frac{-1-\sqrt{5}}{2}, 1]$*;*

3)If $x_0, x_1 \in [-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, *then* $X = \{x_n\}_{n=0}^{\infty}$ *has two frequent limits 0 and 1 of the same degree* 0.5.

4) If $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, $x_1 \in (\frac{-1-\sqrt{5}}{2}, -1] \cup [1, \frac{1+\sqrt{5}}{2})$, *then* $X = \{x_n\}_{n=0}^{\infty}$ *is frequently inside* $(-\infty, \frac{-1-\sqrt{5}}{2})$ *and* $(\frac{-1-\sqrt{5}}{2}, 1)$ *of the same degree* 0.5 *;*

5) If $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, $x_1 \in [-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, *then* $X = \{x_n\}_{n=0}^{\infty}$ *is frequently inside* $(-\infty, \frac{-1-\sqrt{5}}{2})$ *of degree* 0.5 *and has two frequent limits 0 and 1 of the same degree* 0.25*;*

6) If $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1] \cup [1, \frac{1+\sqrt{5}}{2})$, $x_1 \in [-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, *then* $X = \{x_n\}_{n=0}^{\infty}$ *belongs to* $(\frac{-1-\sqrt{5}}{2}, 1)$ *and has two frequent limits 0 and 1 of the same degree* 0.5.

Proof. Let $G(t) = t - [1 - (1 - t^2)^2]$. If $G(t) = 0$, then from Theorem 1 we can get four roots: $t_1 = \frac{-1-\sqrt{5}}{2}$, $t_2 = 0$, $t_3 = \frac{-1+\sqrt{5}}{2}$, $t_4 = 1$. Obviously, $t_1 < t_2 = 0 < t_3 < t_4$.

By elementary analysis, it is easy to see that $G(t) > 0$ for $t \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup [1, +\infty)$ and $G(t) < 0$ for $t \in (\frac{-1-\sqrt{5}}{2}, 0] \cup (\frac{-1+\sqrt{5}}{2}, 1]$, that is,

$$\begin{cases} t > 1-(1-t^2)^2, & t \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup [1, +\infty) \\ t < 1-(1-t^2)^2, & t \in (\frac{-1-\sqrt{5}}{2}, 0] \cup (\frac{-1+\sqrt{5}}{2}, 1] \end{cases}$$

In order to fully descibe the frequently convergent properties of the solotion of (6) as the initial values in different intervals, we can discuss the following five intervals which the two initial values belong to:

$$(-\infty, \frac{-1-\sqrt{5}}{2}); \quad (\frac{-1-\sqrt{5}}{2}, 0]; \quad [0, \frac{-1+\sqrt{5}}{2}); \quad (\frac{-1+\sqrt{5}}{2}, 1]; \quad (1, +\infty).$$

But the function $y = x^2$ is an even function, so we can only consider the case of initial values in negative half of the real axis, then we can deduce the case of initial values in positive half of the real axis. Thus the aymmetric points t_1, t_2, t_3, t_4 of $t_1' = \frac{1+\sqrt{5}}{2}, t_2' = t_2 = 0, t_3' = \frac{1-\sqrt{5}}{2}, t_4' = -1$ can also be regard as the terminals of intervals in real axis. Since $t_1 < t_4' < t_3' < t_2 = 0 < t_3 < t_4 < t_1'$, the negative half of the real axis can be separated into the following intervals:

$$(-\infty, \frac{-1-\sqrt{5}}{2}); \quad (\frac{-1-\sqrt{5}}{2}, -1]; \quad [-1, \frac{1-\sqrt{5}}{2}); \quad (\frac{1-\sqrt{5}}{2}, 0].$$

Then we should analyze the following eight cases of initial values

$I : x_0, x_1 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $\quad II : x_0, x_1 \in (\frac{-1-\sqrt{5}}{2}, -1]$;

$III : x_0, x_1 \in [-1, \frac{1-\sqrt{5}}{2})$; $\quad IV : x_0, x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$;

$V: x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in (\frac{-1-\sqrt{5}}{2}, -1]$; $VI:\ x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$;

$VII:\ x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$; $VIII:\ x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$; $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$;

$VIV:\ x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$; $VV:\ x_0 \in [-1, \frac{1-\sqrt{5}}{2})$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$;

We next to discuss each case in details:

Case I: $x_0, x_1 \in (-\infty, \frac{-1-\sqrt{5}}{2})$.

Since $x_0^2 > \frac{3+\sqrt{5}}{2}$, we have $x_2 = 1 - x_0^2 < \frac{-1-\sqrt{5}}{2}$ and $x_2^2 > \frac{3+\sqrt{5}}{2}$, then $x_4 = 1 - x_1^2 < \frac{-1-\sqrt{5}}{2}$, thus we can easily deduce that $\{x_{2n}\}_{n=0}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2})$. In view of the inequality $t > 1-(1-t^2)^2$ on $t \in (-\infty, \frac{-1-\sqrt{5}}{2})$ and (6), we have

$$\frac{-1-\sqrt{5}}{2} > x_0 > x_2 > x_4 > \cdots > x_{2n} > \cdots > -\infty,$$

that is, $\{x_{2n}\}_{n=0}^{\infty}$ is a decreasing sequence which belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$. Similarly, $\{x_{2n+1}\}_{n=0}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2})$ and

$$\frac{-1-\sqrt{5}}{2} > x_1 > x_3 > x_5 > \cdots > x_{2n+1} > \cdots > -\infty,$$

that is, $\{x_{2n+1}\}_{n=0}^{\infty}$ is also a decreasing sequence which belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$, hence the solution X of the difference equation (6) belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$.

Case II: $x_0, x_1 \in (\frac{-1-\sqrt{5}}{2}, -1]$.

Since $1 \le x_0^2 < \frac{3+\sqrt{5}}{2}$, then we have $\frac{-1-\sqrt{5}}{2} < x_2 = 1 - x_0^2 \le 0$ and $0 \le x_2^2 < \frac{3+\sqrt{5}}{2}$, from $\frac{-1-\sqrt{5}}{2} < x_4 = 1 - x_1^2 \le 1$ and $0 \le x_4^2 < \frac{3+\sqrt{5}}{2}$, we have $\frac{-1-\sqrt{5}}{2} < x_6 = 1 - x_4^2 \le 1$, thus we can deduce that $\{x_{2n}\}_{n=0}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1]$. Similarly, $\{x_{2n+1}\}_{n=0}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1]$, hence the solution X of the difference equation (6) belongs to $(\frac{-1-\sqrt{5}}{2}, 1]$.

Case III: $x_0, x_1 \in [-1, \frac{1-\sqrt{5}}{2})$.

Since $\frac{3-\sqrt{5}}{2} < x_0^2 \le 1$, we have $0 \le x_2 = 1 - x_0^2 < \frac{-1+\sqrt{5}}{2}$ and $0 \le x_2^2 < \frac{3-\sqrt{5}}{2}$, then $\frac{-1+\sqrt{5}}{2} < x_4 = 1 - x_1^2 \le 1$ and $\frac{3-\sqrt{5}}{2} < x_4^2 \le 1$, then we have $0 \le x_6 = 1 - x_2^2 < \frac{-1+\sqrt{5}}{2}$, and $0 \le x_6^2 < \frac{3-\sqrt{5}}{2}$, then $\frac{-1+\sqrt{5}}{2} < x_8 = 1 - x_3^2 < 1$, thus we can deduce that

$$\{x_{4n+2}\}_{n=0}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2}); \quad \{x_{4n}\}_{n=1}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1].$$

Similarly we have

$$\{x_{4n+3}\}_{n=0}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2}); \quad \{x_{4n+1}\}_{n=1}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1].$$

In view of (6), we have

$$x_{n+4} = 1 - x_{n+2}^2 = 1 - (1 - x_n^2)^2, \ n = 0, 1, 2, \ldots. \tag{7}$$

It also follows from the inequality $t > 1 - (1 - t^2)^2$ on $t \in [0, \frac{-1+\sqrt{5}}{2})$ and $t < 1 - (1 - t^2)^2$ on $t \in (\frac{-1+\sqrt{5}}{2}, 1]$ that

$$\frac{-1+\sqrt{5}}{2} > x_2 > x_6 = 1-(1 - x_2^2)^2 > \ldots > x_{4n+2} = 1-(1 - x_{4n-2}^2)^2 > \ldots \geq 0;$$

$$\frac{-1+\sqrt{5}}{2} > x_3 > x_7 = 1-(1 - x_3^2)^2 > \ldots > x_{4n+3} = 1-(1 - x_{4n-1}^2)^2 > \ldots \geq 0;$$

$$\frac{-1+\sqrt{5}}{2} < x_4 < x_8 = 1 - (1 - x_4^2)^2 < \ldots < x_{4n} = 1 - (1 - x_{4n-4}^2)^2 < \ldots \leq 1;$$

$$\frac{-1+\sqrt{5}}{2} < x_5 < x_9 = 1-(1 - x_5^2)^2 < \ldots < x_{4n+1} = 1-(1 - x_{4n-3}^2)^2 < \ldots \leq 1.$$

That is to say, the sequences $\{x_{4n+2}\}_{n=0}^{\infty}$ and $\{x_{4n+3}\}_{n=0}^{\infty}$ are two monotonically decreasing sub-sequence in $[0, \frac{-1+\sqrt{5}}{2})$, and the sequences $\{x_{4n}\}_{n=1}^{\infty}$ and $\{x_{4n+1}\}_{n=1}^{\infty}$ are two monotonically increasing sub-sequence in $(\frac{-1+\sqrt{5}}{2}, 1]$.

If we let $y_n = x_{4n}$ for $n = 1, 2, \ldots,$ and $z_n = x_{4n+2}$ for $n = 0, 1, 2, \ldots,$, then $\{y_n\}_{n=1}^{\infty}$ is a monotonically decreasing and bounded sequence in $[0, \frac{-1+\sqrt{5}}{2})$ and $\{z_n\}_{n=0}^{\infty}$ is a monotonically increasing and boubded sequence in $(\frac{-1+\sqrt{5}}{2}, 1]$, so

$$\lim_{n\to\infty} y_n = y_* \in [0, \frac{-1+\sqrt{5}}{2}); \quad \lim_{n\to\infty} z_n = z_* \in (\frac{-1+\sqrt{5}}{2}, 1].$$

We assert that $y_* = 0$ and $z_* = 1$. To see this, note that we can write (7) in the form

$$y_n = H(y_{n-1}), n = 1, 2, \ldots; \quad z_n = H(z_{n-1}), n = 0, 1, 2, \ldots;$$

where $H(u) = 1 - (1 - u^2)^2$. It is easy to see that $G(t) = t - H(t) = t - [1 - (1 - t^2)^2]$ and $y_* = 1 - (1 - y_*^2)^2; \quad z_* = 1 - (1 - z_*^2)^2$, i.e., $G(y_*) = G(z_*) = 0$. Note that the polynomial $G(t) = 0$ has only one root 0 in $[0, \frac{-1+\sqrt{5}}{2})$ and only one root 1 in $(\frac{-1+\sqrt{5}}{2}, 1]$,so $y_* = 0, \quad z_* = 1$, That is:

$$\lim_{n\to\infty} x_{4n} = 0; \quad \lim_{n\to\infty} x_{4n+2} = 1;$$

By similar arguments, we have

$$\lim_{n\to\infty} x_{4n+3} = 0; \quad \lim_{n\to\infty} x_{4n+1} = 1.$$

Thus from Definition 2.8, for any given number $\varepsilon > 0$ we have

$$\mu^*(|X - 0| \geq \varepsilon) = 0.5; \quad \mu^*(|X - 1| \geq \varepsilon) = 0.5.$$

thus the solution X of the difference equation (6) has two frequent limits 0 and 1 of the same degree 0.5.

Case IV : $x_0, x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$.

Since $0 \leq x_0^2 < \frac{3-\sqrt{5}}{2}$, we have $\frac{-1+\sqrt{5}}{2} < x_2 = 1 - x_0^2 \leq 1$ and $\frac{3-\sqrt{5}}{2} < x_2^2 \leq 1$, then $0 \leq x_4 = 1 - x_2^2 < \frac{-1+\sqrt{5}}{2}$ and $0 \leq x_4^2 < \frac{3-\sqrt{5}}{2}$, then $\frac{-1+\sqrt{5}}{2} < x_6 = 1 - x_4^2 \leq 1$ and $\frac{3-\sqrt{5}}{2} < x_6^2 \leq 1$, then we have $0 \leq x_8 = 1 - x_6^2 < \frac{-1+\sqrt{5}}{2}$, thus we can deduce that $\{x_{2n+1}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1)$ and

$$\{x_{4n}\}_{n=1}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2}); \quad \{x_{4n+2}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1].$$

Similarly we have

$$\{x_{4n+1}\}_{n=1}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2}); \quad \{x_{4n+3}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1].$$

By similar argument with Case III, we can say that the solution X of the difference equation (6) has two frequent limits 0 and 1 of the same degree 0.5.

Case V : $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in (\frac{-1-\sqrt{5}}{2}, -1]$.

Since $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$, from the analysis of Case I, we can get $\{x_{2n}\}_{n=0}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2})$. Due to $x_1 \in (\frac{-1-\sqrt{5}}{2}, -1]$, from the analysis of Case II, we can get $\{x_{2n+1}\}_{n=1}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1]$. From Definition 2.13 we have

$$\mu(X \notin (-\infty, \frac{-1-\sqrt{5}}{2})) = 0.5, \quad \mu(X \notin (\frac{-1-\sqrt{5}}{2}, 1]) = 0.5.$$

hence the solution X of the difference equation (6) is frequently inside $(\frac{-1-\sqrt{5}}{2}, 1]$ and frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$ of the same degree 0.5.

Case VI : $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$;

Since $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$ and $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$, from the analysis of Case I and Case III, , we can get

$$\frac{-1-\sqrt{5}}{2} > x_0 > x_2 > x_4 > \cdots > x_{2n} > \cdots > -\infty,$$

$$\frac{-1+\sqrt{5}}{2} > x_3 > x_7 = 1-(1 - x_3^2)^2 > \ldots > x_{4n+3} = 1-(1 - x_{4n-1}^2)^2 > \ldots \geq 0;$$

$$\frac{-1+\sqrt{5}}{2} < x_5 < x_9 = 1-(1 - x_5^2)^2 < \ldots < x_{4n+1} = 1-(1 - x_{4n-3}^2)^2 < \ldots \leq 1.$$

From Definition 2.13 we have

$$\mu(X \notin (-\infty, \frac{-1-\sqrt{5}}{2})) = 0.5$$

similar to the argument of Case III, for any given number $\varepsilon > 0$ we have

$$\mu^*(|X-0| \geq \varepsilon) = 0.75; \quad \mu^*(|X-1| \geq \varepsilon) = 0.75.$$

hence the solution X of the difference equation (6) is frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$ of degree 0.5 and has two frequent limits 0 and 1 of the same degree 0.25.

Case VII : $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$.

Since $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$ and $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$, from the analysis of Case I and Case IV, we can get

$$\{x_{2n}\}_{n=0}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2});$$

$$\frac{-1+\sqrt{5}}{2} < x_3 < x_7 = 1-(1-x_3^2)^2 < \ldots < x_{4n+3} = 1-(1-x_{4n-1}^2)^2 < \ldots \leq 1;$$

$$\frac{-1+\sqrt{5}}{2} > x_5 > x_9 = 1-(1-x_5^2)^2 > \ldots > x_{4n+1} = 1-(1-x_{4n-3}^2)^2 > \ldots \geq 0.$$

By the similar argument of Case VI, we conclude that the solution X of the difference equation (6) is frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$ of degree 0.5 and has two frequent limits 0, 1 of the same degree 0.25.

Case VIII : $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$; $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$;

Since $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$ and $x_1 \in [-1, \frac{1-\sqrt{5}}{2})$, from the analysis of Case II and Case III, we can get

$$\{x_{2n}\}_{n=1}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1];$$

$$\frac{-1+\sqrt{5}}{2} > x_3 \geq x_7 = 1-(1-x_3^2)^2 > \ldots > x_{4n+3} = 1-(1-x_{4n-1}^2)^2 > \ldots \geq 0;$$

$$\frac{-1+\sqrt{5}}{2} < x_5 < x_9 = 1-(1-x_5^2)^2 < \ldots < x_{4n+1} = 1-(1-x_{4n-3}^2)^2 < \ldots \leq 1.$$

Since $[0, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1-\sqrt{5}}{2}, 1] \subset (\frac{-1-\sqrt{5}}{2}, 1]$, we can conclude that the solution X belongs to $(\frac{-1-\sqrt{5}}{2}, 1]$. By the similar argument of Case VI, we conclude that the solution X of the difference equation (6) has two frequent limits 0 and 1 of the same degree 0.25.

Case VIV : $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$.

Since $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1]$ and $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$, from the analysis of Case II and Case IV, we can get

$$\{x_{2n}\}_{n=1}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1];$$

$$\frac{-1+\sqrt{5}}{2} < x_3 < x_7 = 1-(1-x_3^2)^2 < \ldots < x_{4n+3} = 1-(1-x_{4n-1}^2)^2 < \ldots \leq 1;$$

$$\frac{-1+\sqrt{5}}{2} > x_5 > x_9 = 1-(1-x_5^2)^2 > \ldots > x_{4n+1} = 1-(1-x_{4n-3}^2)^2 > \ldots \geq 0.$$

By the similar argument of Case VIII, we conclude that the solution X of the difference equation (6) belongs to $(\frac{-1-\sqrt{5}}{2}, 1]$ and has two frequent limits 0 and 1 of the same degree 0.25

Case VV : $x_0 \in [-1, \frac{1-\sqrt{5}}{2})$; $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$.

Since $x_0 \in [-1, \frac{1-\sqrt{5}}{2})$ and $x_1 \in (\frac{1-\sqrt{5}}{2}, 0]$, from the analysis of Case III and Case IV, we can get

$$\frac{-1+\sqrt{5}}{2} > x_2 > x_6 = 1-(1-x_2^2)^2 > \ldots > x_{4n+2} = 1-(1-x_{4n-2}^2)^2 > \ldots \geq 0;$$

$$\frac{-1+\sqrt{5}}{2} < x_4 < x_8 = 1-(1-x_4^2)^2 < \ldots < x_{4n} = 1-(1-x_{4n-4}^2)^2 < \ldots \leq 1;$$

$$\frac{-1+\sqrt{5}}{2} < x_3 < x_7 = 1-(1-x_3^2)^2 < \ldots < x_{4n+3} = 1-(1-x_{4n-1}^2)^2 < \ldots \leq 1;$$

$$\frac{-1+\sqrt{5}}{2} > x_5 > x_9 = 1-(1-x_5^2)^2 > \ldots > x_{4n+1} = 1-(1-x_{4n-3}^2)^2 > \ldots \geq 0.$$

By the similar argument of Case III and IV, for any given number $\varepsilon > 0$ we have

$$\mu^*(|X-0| \geq \varepsilon) = 0.5; \quad \mu^*(|X-1| \geq \varepsilon) = 0.5.$$

we conclude that the solution X of the difference equation (6) has two frequent limits 0 and 1 of the same degree 0.5

Based on the above analysis and the symmetric intervals which initial values belong to, the theorem is proved. #

Actually we can use inductive method to get the corresponding theorem of (3) for arbitray positive integer k. Obviously, given k initial-values $x_0, x_1, \cdots, x_{k-1}$, we can use equation (3) to deduce sequence $X = \{x_n\}_{n=0}^{\infty}$, which is the solution of the difference equation (3). If the initial-values $x_0, x_1, \cdots, x_{k-1}$ do not equal to $\frac{-1\pm\sqrt{5}}{2}$, then we have the following theorem:

Theorem 3.3 *Let $x_0, x_1, \cdots, x_{k-1}$ be the initial-values of the difference equation (6), $X = \{x_n\}_{n=0}^{\infty}$ be the solution, then we have the following results:*

1) If $x_0, x_1, \cdots, x_{k-1} \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, then $X = \{x_n\}_{n=0}^{\infty}$ belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$;

2) If $x_0, x_1, \cdots, x_{k-1} \in (\frac{-1-\sqrt{5}}{2}, -1] \cup [1, \frac{1+\sqrt{5}}{2})$*, then* $X = \{x_n\}_{n=0}^{\infty}$ *belongs to* $(\frac{-1-\sqrt{5}}{2}, 1]$*;*

3) If $x_0, x_1, \cdots, x_{k-1} \in [-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$*, then* $X = \{x_n\}_{n=0}^{\infty}$ *has two frequent limits 0 and 1 of the same degree* 0.5.

4) If the number of initial values $x_0, x_1, \cdots, x_{k-1}$ *in* $(-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$ *is athe number of initial values* $x_0, x_1, \cdots, x_{k-1}$ *in* $(\frac{-1-\sqrt{5}}{2}, -1] \cup [1, \frac{1+\sqrt{5}}{2})$ *is b, and the number of initial values* $x_0, x_1, \cdots, x_{k-1}$ *in* $[-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$ *is* $k-a-b$*, then* $X = \{x_n\}_{n=0}^{\infty}$ *is frequently inside* $(-\infty, \frac{-1-\sqrt{5}}{2})$ *of degree* $\frac{a}{k}$*, frequently inside* $(\frac{-1-\sqrt{5}}{2}, 1)$ *of degree* $\frac{a}{k}$*, and there are two frequent limits* 0 *and* 1 *of same degree* $\frac{k-a-b}{2k}$.

Proof. Similar to the proof of Theorem 3.2.

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