**Mathematica Aeterna, Vol. 5, 2015, no. 4, 571 - 581**

# On $h-$convex stochastic processes

**Daniel Barráez**
**Lysis González**
**Nelson Merentes**

Escuela de Matemática, Facultad de Ciencias
Universidad Central de Venezuela
Caracas, Venezuela.
daniel.barraez@gmail.com
lysis.gonzalez@gmail.com
nmerucv@gmail.com

**Ana M. Moros**

Escuela de Matemática, Facultad de Ciencias
Universidad de Los Andes
Mérida, Venezuela
mileula@gmail.com

**Abstract**

$h-$convex stochastic processes are introduced. Some results for $h-$convex functions, like Jensen and Hermite-Hadamard inequalities type, are extended to $h-$convex stochastic processes.

**Mathematics Subject Classification:** Primary 26D15; Secondary 26A51, 39B62, 46N10, 60G99.

**Keywords:** convex and $h$-convex stochastic processes, Jensen inequality, Hermite-Hadamard inequality, convex analysis.

## 1 Introduction

In 1980, the study of quadratic and convex stochastic processes were initiated by K. Nikodem in [6, 7]. Following this line of investigation, Skowroński described the properties of Jensen-convex and Wright-convex stochastic process in [10, 11]. More recently, D. Kotrys presented in [3, 4, 5] results on convex and strongly convex stochastic processes, among them, a Hermite-Hadamard type inequality for convex stochastic processes.

If $h : (0,1) \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, and $I \subseteq \mathbb{R}$ is an interval, a function $f : I \to \mathbb{R}$, is an $h-$convex function on $I$ if the inequality

$$f(\lambda t_1 + (1-\lambda)t_2) \leq h(\lambda)f(t_1) + h(1-\lambda)f(t_2)$$

holds for every $t_1, t_2 \in I$ and $\lambda \in [0,1]$. The $h-$convex functions appeared in 2007, when S. Varošanec (see [8, 9]) unified and generalize the classes of convex, s-convex, Godunova-Levin and $P-$functions, .

The aim of this paper is to introduce the notion of $h-$convex stochastic processes and present some properties obtained like generalizations of some properties of $h-$convex functions.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathcal{A}-$measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t,\cdot)$ is a random variable.

Fixed $h$ like above, we say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is an *$h-$convex stochastic process* if, for every $t_1, t_2 \in I$, $\lambda \in (0,1)$, the following inequality is satisfied

$$X(\lambda t_1 + (1-\lambda)t_2, \cdot) \leq h(\lambda)X(t_1,\cdot) + h(1-\lambda)X(t_2,\cdot) \quad (a.e)$$

Also, we say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is

1. *continuous in probability* in $I$, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t,\cdot) = X(t_0,\cdot),$$

where $P - \lim$ denotes the limit in probability.

2. *mean square continuous* in the interval $I$, if for all $t_0 \in I$

$$\lim_{t \to t_0} \mathbb{E}[(X(t,\cdot) - X(t_0,\cdot))^2] = 0,$$

where $\mathbb{E}[X(t,\cdot)]$ denotes the expectation value of the random variable $X(t,\cdot)$.

Note that mean-square continuity implies continuity in probability, but the converse is not true.

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $\mathbb{E}[X(t)^2] < \infty$ for all $t \in I$. Let $[a,b] \subseteq I$, $a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a,b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the *mean-square integral* of the process $X$ on $[a, b]$, if for a normal sequence of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} X(\Theta_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0$$

In such case, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds \quad (a.e)$$

For the existence of the mean-square integral is enought to assume the mean-square continuity of the stochastic process $X$. Basic properties of the mean-square integral can be read in [12].

Now, we shall present some examples of $h-$convex stochastic processes.

**Example 1.1.** *Every convex stochastic process is an $h-$convex stochastic process with $h$ equals to the identity function.*

**Example 1.2.** *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process. For every $k \leq 1$, consider the function*

$$\begin{array}{rcl} h_k : (0,1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^k \end{array}$$

*Note that $h_k(\lambda) \geq \lambda$ for all $\lambda \in (0, 1)$. Moreover, for every $t_1, t_2 \in I$ and $\lambda \in (0, 1)$, the following inequality is satisfied*

$$\begin{aligned} X(\lambda t_1 + (1-\lambda)t_2, \cdot) &\leq \lambda X(t_1, \cdot) + (1-\lambda)X(t_2, \cdot) \\ &\leq h_k(\lambda)X(t_1, \cdot) + h_k(1-\lambda)X(t_2, \cdot) \quad (a.e) \end{aligned}$$

*Then, $X$ is an $h_k-$convex stochastic process.*

**Example 1.3.** *Every $h-$convex function gives an example of an $h-$convex stochastic process.*
*Let $A : \Omega \to \mathbb{R}$ a random variable, $h : (0, 1) \to \mathbb{R}$ a non-negative function, $h \not\equiv 0$ and $f : I \to \mathbb{R}$ an $h-$convex function. The stochastic process*

$$\begin{array}{rcl} X : (0,1) \times \Omega & \longrightarrow & \mathbb{R} \\ (t, \omega) & \longmapsto & A(\omega)f(t) \end{array}$$

*is an $h-$convex stochastic process.*

# 2 Main Results

In this section we shall present some results concerning to the basic properties of $h-$convex stochastic processes and also, we prove some inequalities, among them, a Jensen-type, a conversion of Jensen-type and a Hermite-Hadamard-type inequality.

**Basic Properties of $h-$convex stochastic processes**

The following propositions show that the class of $h-$convex stochastic processes satisfies some monotony property and is closed under addition, product and positive scalar product.

**Proposition 2.1.** *If $h_1, h_2 : (0,1) \to \mathbb{R}$ are non negative functions with $h_2(\lambda) \le h_1(\lambda)$ for all $\lambda \in (0,1)$ and $X : I \times \Omega \to \mathbb{R}$ is a non- negative $h_2-$convex stochastic process, then $X$ is an $h_1-$convex stochastic process.*

*Proof.* Consider $t_1, t_2 \in I$, $\lambda \in (0,1)$ arbitrary. Then,

$$\begin{aligned} X(\lambda t_1 + (1-\lambda)t_2, \cdot) &\le h_2(\lambda)X(t_1,\cdot) + h_2(1-\lambda)X(t_2,\cdot) \\ &\le h_1(\lambda)X(t_1,\cdot) + h_1(1-\lambda)X(t_2,\cdot) \quad (a.e) \end{aligned}$$

□

**Proposition 2.2.** *Let $h : (0,1) \to \mathbb{R}$ be a non negative function. If $X, Y : I \times \Omega \to \mathbb{R}$ are $h-$convex stochastic processes, then $X+Y$ is an $h-$convex stochastic process. Also, if $\alpha > 0$ then $\alpha X$ is an $h-$convex stochastic process.*

*Proof.* Consider $t_1, t_2 \in I$, $\lambda \in (0,1)$ arbitrary.

$$\begin{aligned} &(X+Y)(\lambda t_1 + (1-\lambda)t_2, \cdot) \\ &\quad = X(\lambda t_1 + (1-\lambda)t_2, \cdot) + Y(\lambda t_1 + (1-\lambda)t_2, \cdot) \\ &\quad \le h(\lambda)(X(t_1,\cdot) + Y(t_1,\cdot)) + h(1-\lambda)(X(t_2,\cdot) + Y(t_2,\cdot)) \\ &\quad = h(\lambda)(X+Y)(t_1,\cdot) + h(1-\lambda)(X+Y)(t_2,\cdot) \quad (a.e) \end{aligned}$$

Now, consider $\alpha > 0$. Then,

$$\begin{aligned} \alpha X(\lambda t_1 + (1-\lambda)t_2, \cdot) &\le \alpha h(\lambda)X(t_1,\cdot) + \alpha h(1-\lambda)X(t_2,\cdot) \\ &= h(\lambda)\alpha X(t_1,\cdot) + h(1-\lambda)\alpha X(t_2,\cdot) \quad (a.e) \end{aligned}$$

□

**Proposition 2.3.** *Let $h_1, h_2 : (0,1) \to \mathbb{R}$ be non negative functions and $X, Y : I \times \Omega \to \mathbb{R}$ non- negative stochastic processes such that*

$$(X(t_1,\cdot) - X(t_2,\cdot))(Y(t_1,\cdot) - Y(t_2,\cdot)) \ge 0,$$

*for all $t_1, t_2 \in I$. If $X$ is $h_1-$convex, $Y$ is $h_2-$convex and $h(\lambda) + h(1-\lambda) \le c$ for all $\lambda \in (0,1)$, where $h(\lambda) = max\{h_1(\lambda), h_2(\lambda)\}$ and $c$ is a fixed positive number, then the product $XY$ is a $ch-$convex stochastic process.*

*Proof.* Fix $t_1, t_2 \in I$ and $\lambda, \beta \in (0,1)$ such that $\lambda + \beta = 1$.

First, note that $(X(t_1,\cdot) - X(t_2,\cdot))(Y(t_1,\cdot) - Y(t_2,\cdot)) \geq 0$ implies

$$X(t_1,\cdot)Y(t_2,\cdot) + X(t_2,\cdot)Y(t_1,\cdot) \leq X(t_1,\cdot)Y(t_1,\cdot) + X(t_2,\cdot)Y(t_2,\cdot)$$

Hence,

$$\begin{aligned}
&XY(\lambda t_1 + \beta t_2,\cdot)\\
&\quad\leq (h_1(\lambda)X(t_1,\cdot) + h_1(\beta)X(t_2,\cdot))(h_2(\lambda)Y(t_1,\cdot) + h_2(\beta)Y(t_2,\cdot))\\
&\quad\leq (h(\lambda)X(t_1,\cdot) + h(\beta)X(t_2,\cdot))(h(\lambda)Y(t_1,\cdot) + h(\beta)Y(t_2,\cdot))\\
&\quad\leq h^2(\lambda)XY(t_1,\cdot)+h(\lambda)h(\beta)XY(t_1,\cdot)+h(\lambda)h(\beta)XY(t_2,\cdot)+h^2(\beta)XY(t_2,\cdot)\\
&\quad= (h(\lambda) + h(\beta))(h(\lambda)XY(t_1,\cdot) + h(\beta)XY(t_2,\cdot))\\
&\quad\leq ch(\lambda)(XY)(t_1,\cdot) + ch(\beta)(XY)(t_2,\cdot) \qquad (a.e)
\end{aligned}$$

□

Let $J \subseteq \mathbb{R}$ an interval. A function $h : J \to \mathbb{R}$ is a supermultiplicative function if $h(xy) \geq h(x)h(y)$ for all $x, y \in J$.

In the following theorem we present conditions under the inequality

$$X(\lambda t_1 + \beta t_2,\cdot) \leq h(\lambda)X(t_1,\cdot) + h(\beta)X(t_2,\cdot)$$

holds almost everywhere for all $\lambda, \beta > 0$ such that $\lambda + \beta \leq 1$.

**Theorem 2.4.** *Let $I$ be an interval such that $0 \in I$ and $h : (0,1) \to \mathbb{R}$ a non negative function. If $h$ is supermultiplicative and $X : I \times \Omega \to \mathbb{R}$ is an $h-$convex stochastic process such that $X(0,\cdot) = 0$, then the inequality*

$$X(\lambda t_1 + \beta t_2,\cdot) \leq h(\lambda)X(t_1,\cdot) + h(\beta)X(t_2,\cdot)$$

*holds almost everywhere for all $t_1, t_2 \in I$ and all $\lambda, \beta > 0$ such that $\lambda + \beta \leq 1$.*

*Proof.* If $\lambda + \beta = 1$, the inequality holds because of definition of $h-$convexity in stochastic processes. Let $\lambda, \beta > 0$ be numbers such that $\lambda + \beta = \gamma$ with $\gamma < 1$. Let us define numbers $a := \frac{\lambda}{\gamma}$ and $b := \frac{\beta}{\gamma}$. Then, $a + b = 1$ and we have the following:

$$\begin{aligned}
&X(\lambda t_1 + \beta t_2,\cdot)\\
&\quad= X(a\gamma t_1 + b\gamma t_2,\cdot)\\
&\quad\leq h(a)X(\gamma t_1,\cdot) + h(b)X(\gamma t_2,\cdot)\\
&\quad= h(a)X(\gamma t_1 + (1-\gamma)0,\cdot) + h(b)X(\gamma t_2 + (1-\gamma)0,\cdot))\\
&\quad\leq h(a)h(\gamma)X(t_1,\cdot) + h(a)h(1-\gamma)X(0,\cdot) + h(b)h(\gamma)X(t_2,\cdot)\\
&\qquad +h(b)h(1-\gamma)X(0,\cdot)\\
&\quad= h(a)h(\gamma)X(t_1,\cdot) + h(b)h(\gamma)X(t_2,\cdot))\\
&\quad\leq h(a\gamma)X(t_1,\cdot) + h(b\gamma)X(t_2,\cdot)\\
&\quad= h(\lambda)X(t_1,\cdot) + h(\beta)X(t_2,\cdot)) \qquad (a.e)
\end{aligned}$$

□

**Theorem 2.5.** *Let $h$ be a non-negative function with $h(\lambda) < \frac{1}{2}$ for some $\lambda \in (0, \frac{1}{2})$. If $X : I \times \Omega \to \mathbb{R}$ is a non negative stochastic process such that*

$$X(\lambda t_1 + \beta t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot) \tag{1}$$

*holds almost everywhere for any $t_1, t_2 \in I$ and $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, then $X(0, \cdot) = 0$.*

*Proof.* Let us suppose that exists $w \in \Omega$ with $X(0, \omega) \neq 0$. Then, $X(0, \omega) > 0$ and putting $t_1 = t_2 = 0$ in the inequality (1), we get

$$X(0, \omega) \leq h(\lambda)X(0, \omega) + h(\beta)X(0, \omega) \quad (a.e.)$$

for $\lambda, \beta > 0$ such that $\lambda + \beta \leq 1$. Considering $\lambda = \beta$, $\lambda \in (0, \frac{1}{2})$ and dividing by $X(0, \omega)$, we obtain $1 \leq h(\lambda) + h(\lambda) = 2h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$. That is, $\frac{1}{2} \leq h(\lambda)$ for all $\lambda \in (0, \frac{1}{2})$, what is a contradiction with the assumption of theorem.

□

**Corollary 2.6.** *Fixed $s > 0$, let consider the function $h_s : (0, \infty) \to \mathbb{R}$ defined by $h_s(x) = x^s$. If $I$ is an interval such that $0 \in I$ and $X : I \times \Omega \to \mathbb{R}$ is a non negative $h_s-$convex stochastic process, then the inequality*

$$X(\lambda t_1 + \beta t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(\beta)X(t_2, \cdot)$$

*holds almost everywhere for all $\lambda, \beta > 0$, $\lambda + \beta \leq 1$ if and only if $X(0, \cdot) = 0$.*

*Proof.* ($\Rightarrow$) For $s \geq 1$, is enough to note that $h_s$ is non-negative and $h_s(\frac{1}{3}) < \frac{1}{2}$. The proof for $s < 1$ can be read in [2].

($\Leftarrow$) Note that $h_s$ is non-negative and supermultiplicative. Then use Theorem (2.4).

□

**Proposition 2.7.** *Let $h : (0, 1) \to \mathbb{R}$ be a non-negative supermultiplicative function and let $X : I \times \Omega \to \mathbb{R}$ be an $h-$convex stochastic process. Then, for $t_1, t_2, t_3 \in I$, with $t_1 < t_2 < t_3$ such that $t_3 - t_1, t_3 - t_2, t_2 - t_1 \in J$ the following inequality holds almost everywhere,*

$$h(t_3 - t_2)X(t_1, \cdot) - h(t_3 - t_1)X(t_2, \cdot) + h(t_2 - t_1)X(t_3, \cdot) \geq 0$$

*Proof.* Let $X : I \times \Omega \to \mathbb{R}$ be an $h-$convex stochastic process and $t_1, t_2, t_3 \in I$ be numbers wich satisfy assumptions of the proposition. Then, we have $\frac{t_3-t_2}{t_3-t_1}, \frac{t_2-t_1}{t_3-t_1} \in (0, 1)$ and $\frac{t_3-t_2}{t_3-t_1} + \frac{t_2-t_1}{t_3-t_1} = 1$. Also, since $h$ is supermultiplicative and non-negative,

$$h(t_3 - t_2) = h\left(\frac{t_3 - t_2}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) h(t_3 - t_1),$$

$$h(t_2 - t_1) = h\left(\frac{t_2 - t_1}{t_3 - t_1}.(t_3 - t_1)\right) \geq h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) h(t_3 - t_1),$$

Let $h(t_3 - t_1) > 0$. Because of the $h-$convexity, $X$ satisfies

$$X(\lambda z_1 + (1-\lambda)z_2, \cdot) \leq h(\lambda)X(z_1, \cdot) + h(1-\lambda)X(z_2, \cdot) \quad (a.e)$$

for all $z_1, z_2 \in I, \lambda \in (0,1)$. In particular, for $\lambda = \frac{t_3 - t_2}{t_3 - t_1}$, $z_1 = t_1, z_2 = t_3$, we have $t_2 = \lambda z_1 + (1-\lambda)z_2$ and

$$\begin{aligned} X(t_2, \cdot) &\leq h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) X(t_1, \cdot) + h\left(\frac{t_2 - t_1}{t_3 - t_1}\right) X(t_3, \cdot) \\ &\leq \frac{h(t_3 - t_2)}{h(t_3 - t_1)} X(t_1, \cdot) + \frac{h(t_2 - t_1)}{h(t_3 - t_1)} X(t_3, \cdot) \qquad (a.e) \end{aligned}$$

Finally, multiplying by $h(t_3 - t_1)$ we obtain the following

$$h(t_3 - t_1)X(t_2, \cdot) \leq h(t_3 - t_2)X(t_1, \cdot) + h(t_2 - t_1)X(t_3, \cdot) \quad (a.e)$$

That is,

$$0 \leq h(t_3 - t_2)X(t_1, \cdot) - h(t_3 - t_1)X(t_2, \cdot) + h(t_2 - t_1)X(t_3, \cdot) \quad (a.e)$$

□

Now, we will present a Jensen-type inequality for $h-$convex stochastic processes.

**Jensen-type inequality**

**Theorem 2.8.** *Let $\lambda_1, ..., \lambda_n$ be positive real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$, $n \geq 2$. If $h$ is a non-negative supermultiplicative function and $X : I \times \Omega \to \mathbb{R}$ is a non-negative $h-$convex stochastic process, then the inequality*

$$X\left(\sum_{i=1}^{n} \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^{n} h(\lambda_i) X(t_i, \cdot)$$

*holds almost everywhere for every $t_1, t_2, ..., t_n \in I$.*

*Proof.* The proof is by induction. If $n = 2$, the inequality is satisfied because of definition of $h-$convexity in stochastic processes. Let us suppose that the inequality holds for $n-1$ and consider $t_1, t_2, ..., t_n \in I$ and $\lambda_1, \lambda_2, ..., \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. For every $n > 1$ define $L_n = \sum_{i=1}^{n} \lambda_i$. Then, it follows

$$
\begin{aligned}
X\left(\sum_{i=1}^{n}\lambda_i t_i,\cdot\right) &= X\left(\lambda_n t_n+\sum_{i=1}^{n-1}\lambda_i t_i,\cdot\right)\\
&= X\left(\lambda_n t_n+L_{n-1}\sum_{i=1}^{n}\tfrac{\lambda_i}{L_{n-1}}t_i,\cdot\right)\\
&\leq h(\lambda_n)X(t_n,\cdot)+h(L_{n-1})X\left(\sum_{i=1}^{n-1}\tfrac{\lambda_i}{L_{n-1}}t_i,\cdot\right) \quad (a.e)
\end{aligned}
$$

Since $\sum_{i=1}^{n-1}\frac{\lambda_i}{L_{n-1}}=1$ and $h\geq 0$, using the inductive hypothesis, we have

$$
\begin{aligned}
h(L_{n-1})X\left(\sum_{i=1}^{n-1}\tfrac{\lambda_i}{L_{n-1}}t_i,\cdot\right) &\leq h(L_{n-1})\sum_{i=1}^{n-1}h\left(\tfrac{\lambda_i}{L_{n-1}}\right)X(t_i,\cdot)\\
&= \sum_{i=1}^{n-1}h(L_{n-1})h\left(\tfrac{\lambda_i}{L_{n-1}}\right)X(t_i,\cdot) \quad (a.e)
\end{aligned}
$$

Due to $h$ and $X$ are non-negative and h is supermultiplicative,

$$
\sum_{i=1}^{n-1}h(L_{n-1})h\left(\frac{\lambda_i}{L_{n-1}}\right)X(t_i,\cdot)\leq\sum_{i=1}^{n-1}h(\lambda_i)X(t_i,\cdot) \quad (a.e)
$$

Then,

$$
\begin{aligned}
X\left(\sum_{i=1}^{n}\lambda_i t_i\right) &\leq h(\lambda_n)X(t_n,\cdot)+\sum_{i=1}^{n-1}h(\lambda_i)X(t_i,\cdot)\\
&= \sum_{i=1}^{n}h(\lambda_i)X(t_i,\cdot) \qquad (a.e)
\end{aligned}
$$

□

The following theorem is a conversion of Jensen's inequality.

**Theorem 2.9.** *Let $\lambda_1,...,\lambda_n$ be positive real numbers such that $\sum_{i=1}^{n}\lambda_i=1$ and $(m,M)\subseteq I$. If $h:(0,1)\to\mathbb{R}$ is a non negative supermultiplicative function and $X:I\times\Omega\to\mathbb{R}$ is an $h-$convex stochastic process, then for any $t_1,...,t_n\in(m,M)$, the following inequality holds almost everywhere*

$$
\sum_{i=1}^{n}h\left(\tfrac{\lambda_i}{L_n}X(t_i,\cdot)\right)\leq X(m,\cdot)\sum_{i=1}^{n}h\left(\tfrac{\lambda_i}{L_n}\right)h\left(\tfrac{M-t_i}{M-m}\right)+X(M,\cdot)\sum_{i=1}^{n}h\left(\tfrac{\lambda_i}{L_n}\right)h\left(\tfrac{t_i-m}{M-m}\right)
$$

*Proof.* Fix $i \in \{1, ..., n\}$. Putting $t_1 = m, t_2 = t_i, t_3 = M$ in the inequality 2, we get

$$X(t_i, \cdot) \leq h\left(\tfrac{M-t_i}{M-m}\right) X(m, \cdot) + h\left(\tfrac{t_i-m}{M-m}\right) X(M, \cdot)$$

Since $h$ is non negative, we have

$$h\left(\tfrac{\lambda_i}{L_n}\right) X(t_i, \cdot) \leq h\left(\tfrac{\lambda_i}{L_n}\right) h\left(\tfrac{M-t_i}{M-m}\right) X(m, \cdot) + h\left(\tfrac{\lambda_i}{L_n}\right) h\left(\tfrac{t_i-m}{M-m}\right) X(M, \cdot)$$

Adding all inequalities for $i = 1, ..., n$, we complete the proof.

□

In [1], Angulo et al. proved a Hermite-Hadamard type inequality for $h-$convex functions. Now, we will present an analogous result for $h-$convex stochastic processes.

### Hermite-Hadamard type inequality

**Theorem 2.10.** *Let be $h : (0, 1) \to \mathbb{R}$ a non-negative function, $h \not\equiv 0$ and $X : I \times \Omega \to \mathbb{R}$ a non negative, $h-$convex, mean square integrable stochastic process. For every $a, b \in I$, $(a < b)$, the following inequality is satisfied almost everywhere*

$$\frac{1}{2h\left(\frac{1}{2}\right)} X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{b-a} \int_a^b X(t, \cdot) dt \leq (X(a, \cdot) + X(b, \cdot)) \int_0^1 h(z) dz$$

*Proof.* Fix $a, b \in I$, $a < b$ and take $u = za + (1-z)b$, $v = (1-z)a + zb$. Then, $\frac{u+v}{2} = \frac{a+b}{2}$. The $h-$convexity of $X$ implies that

$$\begin{aligned} X\left(\frac{a+b}{2}, \cdot\right) &= X\left(\frac{u+v}{2}, \cdot\right) \\ &\leq h\left(\frac{1}{2}\right) [X(u, \cdot) + X(v, \cdot)] \\ &= h\left(\frac{1}{2}\right) [X(za + (1-z)b, \cdot) + X((1-z)a + zb, \cdot)] \quad (a.e) \end{aligned}$$

Because of the monotonicity and linearity of the mean-square integral (see [12]), we have (a.e.)

$$\begin{aligned} X\left(\frac{a+b}{2}, \cdot\right) &= \int_0^1 X\left(\frac{a+b}{2}, \cdot\right) dz \\ &\leq h\left(\frac{1}{2}\right) \left[\int_0^1 X(za + (1-z)b, \cdot) dz + \int_0^1 X((1-z)a + zb, \cdot) dz\right] \end{aligned}$$

Changing variables in the mean square integral (see [12]), we obtain

$$\int_0^1 X\left(za + (1-z)b, \cdot\right) dz = \int_b^a \frac{1}{a-b} X(u, \cdot) du = \frac{1}{a-b} \int_b^a X(u, \cdot) du \quad (a.e)$$

and

$$\int_0^1 X((1-z)a+zb,\cdot)dz = \frac{1}{b-a}\int_a^b X(v,\cdot)dv$$

Hence, we have the following

$$\begin{aligned} X\left(\frac{a+b}{2},\cdot\right) &\leq h\left(\frac{1}{2}\right)\left[\frac{1}{a-b}\int_b^a X(u,\cdot)du + \frac{1}{b-a}\int_a^b X(v,\cdot)dv\right] \\ &= h\left(\frac{1}{2}\right)\frac{2}{b-a}\int_a^b X(t,\cdot)dt \end{aligned}$$

That is,

$$\frac{1}{2h\left(\frac{1}{2}\right)}X\left(\frac{a+b}{2},\cdot\right) \leq \frac{1}{b-a}\int_a^b X(t,\cdot)dt$$

In the other hand side, we have

$$X((1-z)a+zb,\cdot) \leq h(1-z)X(a,\cdot)+h(z)X(b,\cdot)$$

Using basic properties of mean square integral,

$$\begin{aligned} \frac{1}{b-a}\int_a^b X(t,\cdot)dt &= \int_0^1 X((1-z)a+zb,\cdot)dz \\ &\leq X(a,\cdot)\int_0^1 h(1-z)dz + X(b,\cdot)\int_0^1 h(z)dz \\ &= X(a,\cdot)\int_0^1 h(t)dt + X(b,\cdot)\int_0^1 h(t)dt \\ &= (X(a,\cdot)+X(b,\cdot))\int_0^1 h(z)dz \quad (a.e.) \end{aligned}$$

□

**ACKNOWLEDGEMENTS.** This research has been partially supported by the Central Bank of Venezuela (BCV). We want to give thanks to the library staff of B.C.V for compiling the references.
Also, we want to express our acknowledgement to Dr. K. Nikodem for all his suggestions and comments.

# References

[1] H. Angulo, J. Gimenez, A. M. Moros, K. Nikodem, *On strongly $h-$convex functions*, Ann. Funct. Anal. **2**, 2011, pp. 87-93.

[2] H. Hudzik, L. Maligranda, *Some remarks on $s-$convex functions*, Aequationes Math. **48**, 1994, pp. 100-111.

[3] D. Kotrys, *Hermite-Hadamard inequality for convex stochastic processes*, Aequationes Math. **83**, 2012, pp. 143-151.

[4] D. Kotrys, *Remarks on strongly convex stochastic processes*, The article is published with open access at Springerlink.com Aequationes Math., 2012.

[5] D. Kotrys, *Some characterizations of strongly convex stochastic processes*, Mathematica Aeterna **4**, 2014, pp. 855-861, no. 8.

[6] K. Nikodem, *On convex stochastic processes*, Aequationes Math. **20**, 1980, pp. 184-197.

[7] K. Nikodem, *On quadratic stochastic processes*, Aequationes Math. **21**, 1980, pp. 192-199.

[8] S. Varošanec, *On $h-$convexity*, J. Math. Appl. **326**, 2007, pp. 303-311.

[9] M. Bombardelli, S. Varošanec, *Properties of $h-$convex functions related to the Hermite-Hadamard-Fejér inequalities*, Computers and Mathematics with Appilcations **58**, 2009, pp. 1869-1877.

[10] A. Skowroński, *On some properties of $J$-convex stochastic processes*, Ann. Polonici. Mathematici L., 1989, pp. 145-151.

[11] A. Skowroński, *On Wright-convex stochastic processes*, Ann. Math. Sil. **9**, 1995, pp. 29-32.

[12] K. Sobczyk, *Stochastic differential equations with applications to physics and engineering*, Kluwer, Dordrecht, 1991.

**Received: July, 2015**