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Structure of the 3-Lie algebra J_{11}

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Abstract

The paper main concerns the structure of 8-dimensional 3-Lie algebra J_{11} which is constructed by 2-cubic matrix. The multiplication of J_{11} is discussed and the decomposition of J_{11} associate with a Cartan subalgebra is provided. The structure of derivation algebra and inner derivation algebra of J_{11} are also studied.

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1 Introduction

n-Lie algebras [1-2], especially, 3-Lie algebras, have wide applications in mathematics and mathematical physics [3-4]. Researchers try to construct n-Lie algebras by algebras which we know well. For example, by means of one and two dimensional extensions, people constructed n-Lie algebras from $(n-1)$-Lie algebras. In papers [5-6], 3-Lie algebras are constructed by Lie algebras, associative algebras, pre-Lie algebras and commutative associative algebras and their derivations and involutions. In paper [7], fifteen kinds of multiplications of N-cubic matrix are provided, and four non-isomorphic N^3-dimensional 3-Lie algebras are constructed. In this paper, we pay our main attention to

8-dimensional 3-Lie algebras which are constructed by 2-cubic matrix, we suppose that 3-Lie algebras over a field F of characteristic of zero, and the subspace generated by a subset S of a vector space V is denoted by $< S >$.

2 Structure of 3-Lie algebras J_{11}

An N-order cubic matrix $A = (a_{ijk})$ (see [7]) over a field F is an ordered object which the elements with 3 indices, and the element in the position (i, j, k) is $(A)_{ijk} = a_{ijk}$, $1 \le i, j, k \le N$. Denote the set of all cubic matrix over a field F by Ω. Then Ω is an N^3-dimensional vector space over F with $A + B = (a_{ijk} + b_{ijk}) \in \Omega$, $\lambda A = (\lambda a_{ijk}) \in \Omega$, for $\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega$, $\lambda \in F$, that is, $(A + B)_{ijk} = a_{ijk} + b_{ijk}$, $(\lambda A)_{ijk} = \lambda a_{ijk}$.

Denote E_{ijk} a cubic matrix with the element in the position (i, j, k) is 1 and elsewhere are zero. Then $\{E_{ijk},\ 1 \le i, j, k \le N\}$ is a basis of Ω, and for every $A = (a_{ijk}) \in \Omega$, $A = \sum\limits_{1\le i,j,k\le N} a_{ijk}E_{ijk}$, $a_{ijk} \in F$.

For all $A = (a_{ijk}), B = (b_{ijk}) \in \Omega$, define the multiplication $*_{11}$ in Ω by

$$(A *_{11} B)_{ijk} = \sum_{p=1}^{N} a_{ijp}b_{ipk},$$

then $(\Omega, *_{11})$ is associative algebra.

Denote $\langle A\rangle_1 = \sum\limits_{p,q=1}^{N} a_{pqq}$. Then $\langle\ \rangle_1$ is linear functions from Ω to F and satisfies $\langle A *_{11} B\rangle_1 = \langle B *_{11} A\rangle_1$.

Define the multiplication $[, ,]_{11} : \Omega \wedge \Omega \wedge \Omega \to \Omega$ as follows:

$$[A, B, C]_{11} = \langle A\rangle_1(B *_{11} C - C *_{11} B) + \langle B\rangle_1(C *_{11} A - A *_{11} C) + \langle C\rangle_1(A *_{11} B - B *_{11} A). \quad (1)$$

We obtain the following lemma.

Theorem 2.1[7] *The linear space Ω is a 3-Lie algebra in the multiplication $[, ,]_{11}$, which is denoted by J_{11}.*

In the following we suppose $N = 2$. We have the following result.

Theorem 2.2 *The 3-Lie algebra J_{11} is a non-nilpotent indecomposable 3-Lie algebra with a basis $e_1 = E_{111}$, $e_2 = E_{112}$, $e_3 = E_{121}$, $e_4 = E_{111} - E_{122}$, $e_5 = E_{211} - E_{111}$, $e_6 = E_{212}$, $e_7 = E_{221}$, $e_8 = E_{211} - E_{222}$, and the multiplication in it is as follows:*

$$\begin{cases} [e_1, e_2, e_3] = e_4, [e_1, e_2, e_4] = -2e_2, [e_1, e_3, e_4] = 2e_3, \\ [e_1, e_6, e_7] = e_8, [e_1, e_6, e_8] = -2e_6, [e_1, e_7, e_8] = 2e_7, \\ [e_1, e_2, e_5] = e_2, [e_1, e_3, e_5] = -e_3, [e_1, e_5, e_6] = e_6, [e_1, e_5, e_7] = -e_7. \end{cases} \quad (2)$$

Then center of J_{11} is $< e_4 + 2e_5 - e_8 >$.

Proof It is clear that $\{e_1, \cdots, e_8\}$ is a basis of Ω. By the definition of $[, ,]_{11}$, we obtain Eq.(2). Thank to $ad(e_1, e_4)$ is non-nilpotent, the 3-Lie algebra J_{11}

is non-nilpotent. By a direct computation, $[e_4 + 2e_5 - e_8, x, y] = 0$ for all $x, y \in J_{11}$. Then proof is completed.

Theorem 2.3 *The subalgebra* $H =< e_1, e_4, e_5, e_8 >$ *is a Cartan subalgebra of the* 3*-Lie algebra* J_{11}*. And the decomposition of* J_{11} *associate to* H *is*

$$J_{11} = H \dot{+} J_\alpha \dot{+} J_{-\alpha}, \text{ where } J_\alpha =< e_2, e_6 >, \ J_{-\alpha} =< e_3, e_7 >,$$

where the linear function $\alpha : H \wedge H \to F$ *defined by* $\alpha(1,4) = 2, \alpha(1,8) = 2, \alpha(1,5) = -1$, *and others are zero.*

Proof Define linear function $\alpha : H \wedge H \to F$ by $\alpha(1,4) = 2, \alpha(1,8) = 2, \alpha(1,5) = -1$, and others are zero. By the multiplication (2) we have $[e_i, e_j, e_2] = \alpha(e_i, e_j)e_2$, $[e_i, e_j, e_6] = \alpha(e_i, e_j)e_6$, $[e_i, e_j, e_3] = -\alpha(e_i, e_j)e_3$, $[e_i, e_j, e_7] = -\alpha(e_i, e_j)e_7$, for all $e_i,\ e_j \in H$. Then we have $J_\alpha =< e_2, e_6 >$, $J_{-\alpha} =< e_3, e_7 >$, and $J_{11} = H \dot{+} J_\alpha \dot{+} J_{-\alpha}$. The proof is completed.

Now we study the inner derivation algebra adJ_{11}. For $e_i, e_j \in \Omega$, denote

$$ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l, \text{ where } a_{kl}^{ij} = -a_{kl}^{ji} \in F.$$

Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl} are the matrix units.

Theorem 2.4 Let J_{11} be a 3-Lie algebra in Theorem 2.2. Then we have

1) $\dim adJ_{11} = 12$, and $X_1 = E_{34} - 2E_{42} + E_{52}$, $X_2 = -E_{24} + 2E_{43} - E_{53}$, $X_3 = 2E_{22} - 2E_{33}$, $X_4 = -E_{56} + E_{78} - 2E_{86}$, $X_5 = E_{57} - E_{68} + 2E_{87}$, $X_6 = 2E_{66} - 2E_{77}$, $X_7 = E_{14}$, $X_8 = E_{12}$, $X_9 = E_{13}$, $X_{10} = E_{16}$, $X_{11} = E_{17}$, $X_{12} = E_{18}$ is a basis of adJ_{11}. And the multiplication in it is

$[X_2, X_1] = X_3, [X_3, X_2] = 2X_2$, $[X_3, X_1] = -2X_1$, $[X_6, X_4] = -2X_4$, $[X_5, X_4] = X_6$, $[X_6, X_5] = 2X_5$, $[X_1, X_7] = 2X_8$, $[X_1, X_9] = -X_7$, $[X_2, X_7] = -2X_9$, $[X_3, X_9] = 2X_9$, $[X_4, X_{11}] = -X_{12}$, $[X_4, X_{12}] = 2X_{10}$, $[X_5, X_{10}] = X_{12}$, $[X_5, X_{12}] = -2X_{11}$, $[X_6, X_{10}] = -2X_{10}$, $[X_6, X_{11}] = 2X_{11}$, $[X_2, X_8] = X_7$, $[X_3, X_8] = -2X_8$.

2) adJ_{11} is a decomposable Lie algebra, and

$$adJ_{11} = L_1 \dot{+} L_2, \ [L_1, L_1] = L_1, [L_2, L_2] = L_2, [L_1, L_2] = 0,$$

where $L_1 =< X_1, X_2, X_3, X_7, X_8, X_9 >$, $L_2 =< X_4, X_5, X_6, X_{10}, X_{11}, X_{12} >$, $< X_1, X_2, X_3 > \cong < X_4, X_5, X_6 > \cong sl_2$, and $I_1 =< X_7, X_8, X_9 >$, $I_2 =< X_{10}, X_{11}, X_{12} >$ are minimal ideals of adJ_{11}.

Proof By a direct computation according to Eq.(2) we have
$ad(e_1, e_2) = E_{34} - 2E_{42} + E_{52}$, $ad(e_1, e_3) = -E_{24} + 2E_{43} - E_{53}$, $ad(e_1, e_4) = 2E_{22} - 2E_{33}$, $ad(e_1, e_6) = -E_{56} + E_{78} - 2E_{86}$; $ad(e_1, e_7) = E_{57} - E_{68} + 2E_{87}$, $ad(e_1, e_8) = 2E_{66} - 2E_{77}$, $ad(e_2, e_3) = E_{14}, ad(e_2, e_5) = E_{12}$, $ad(e_3, e_5) = -E_{13}$, $ad(e_5, e_6) = E_{16}$, $ad(e_5, e_7) = -E_{17}$, $ad(e_6, e_7) = E_{18}$. Then $\{X_1, \cdots, X_{12}\}$ is a basis of adJ_{11}. From

$[ad(e_i, e_j), ad(e_k, e_l)] = ad([e_i, e_j, e_k], e_l) + ad(e_k, [e_i, e_j e_l])$,
we have the result.

At the last of the paper, we discuss the derivation algebra $Der J_{11}$.

Theorem 2.5 The derivation algebra $Der J_{11}$ satisfies:

1) The dimension of $Der J_{11}$ is 15, and $Der J_{11}$ with a basis $\{X_1, \cdots, X_{15}\}$, where $X_{13} = E_{11} - 2E_{33} - E_{44} - E_{55} - 2E_{77} - E_{88}$, $X_{14} = E_{54} + 2E_{55} - E_{58}$, $X_{15} = E_{15}$, X_i is in Theorem 2.4 for $1 \leq i \leq 12$. And the multiplication in the basis is

$$\begin{cases} [X_2, X_1] = X_3, [X_{10}, X_{13}] = -X_{10}, [X_5, X_{12}] = -2X_{11}, \\ [X_6, X_5] = 2X_5, [X_6, X_4] = -2X_4, [X_1, X_7] = 2X_8, [X_1, X_9] = -X_7, \\ [X_2, X_7] = -2X_9, [X_2, X_8] = X_7, [X_3, X_8] = -2X_8, [X_3, X_9] = 2X_9, \\ [X_4, X_{11}] = -X_{12}, [X_4, X_{12}] = 2X_{10}, [X_5, X_{10}] = X_{12}, \\ [X_3, X_2] = 2X_2, [X_6, X_{10}] = -2X_{10}, [X_6, X_{11}] = 2X_{11} \\ [X_1, X_{13}] = X_1, [X_2, X_{13}] = -X_2, [X_4, X_{13}] = X_4, [X_5, X_{13}] = -X_5 \\ [X_7, X_{13}] = -2X_7, [X_8, X_{13}] = -X_8, [X_9, X_{13}] = -3X_9, \\ [X_3, X_1] = -2X_1, [X_{11}, X_{13}] = -3X_{11}, [X_{12}, X_{13}] = -2X_{12}, \\ [X_2, X_{15}] = X_9, [X_4, X_{15}] = X_{10}, [X_5, X_{15}] = -X_{11}, [X_{13}, X_{15}] = 2X_{15}, \\ [X_5, X_4] = X_6, [X_{14}, X_{15}] = -X_7 - 2X_{15} + X_{12}, [X_1, X_{15}] = -X_8. \end{cases}$$

2) $Der J_{11}$ is an indecomposable Lie algebra, and

$$Der J_{11} = ad J_{11} \dot{+} W,$$

where $W =< X_{13}, X_{14}, X_{15} >$.

3) Derived algebra $Der^1 J_{11} = < X_1, \cdots, X_{12}, X_{15} >$, I_1, I_2 are minimal ideals of $Der J_{11}$, L_1, L_2 are ideals of $Der J_{11}$ and $[W, L_1] \subseteq L_1$, $[W, L_2] \subseteq L_2$.

Proof For all $D \in Der J_{11}$, suppose $D(e_i) = \sum\limits_{j=1}^{8} a_{ij} e_j$, $1 \leq i \leq 8$, then the matrix of D in the basis $\{e_1, \cdots, e_8\}$ is $A = (a_{ij})_{i,j=1}^{8} = \sum\limits_{i,j=1}^{8} a_{ij} E_{ij}$, where E_{ij} are (8×8) matrix units, $1 \leq i, j \leq 8$. By s direct computation according to the multiplication (2), we have the result 1).

Thanks to Theorem 2.5, $W =< X_{13}, X_{14}, X_{15} >$ are exterior derivations. Then we have $Der J_{11} = ad J_{11} \dot{+} W$.

By a direct computation, $Der^1 J_{11} =< X_2, \cdots, X_{12}, X_{15} >$ and L_1, L_2 defined in Theorem 2.5 are ideals of $Der J_{11}$, and I_1, I_2 are minimal ideals.

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