Mathematica Aeterna, Vol. 5, 2015, no. 4, 599 - 603

Structure of 8-dimensional 3-Lie algebra J_{21}

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Abstract

In this paper, we study 3-Lie algebra J_{21} which is constructed by 2-cubic matrix. We give the multiplication in a special basis, and provide the concrete expression of all derivations and inner derivations.

2010 Mathematics Subject Classification: 17B05 17D30
Keywords: N-cubic matrix, 3-Lie algebra, derivation.

1 N-cubic matrix

We first introduce the cubic matrix which is discussed in paper [1]. Then according to the method given in the paper [2], we realized the 3-Lie algebra J_{21} [3] by the 2-cubic matrix, and study the structure of its inner derivation algebra adJ_{21} and derivation algebra $DerJ_{21}$.

An N-order cubic matrix $A = (a_{ijk})$ (see [1]) over a field F is an ordered object which the elements with 3 indices, and the element in the position (i, j, k) is $(A)_{ijk} = a_{ijk} \in F$, $1 \leq i, j, k \leq N$. Denote the set of all cubic matrix over a field F by Ω. Then Ω is an N^3-dimensional vector space over F with

$$A + B = (a_{ijk} + b_{ijk}) \in \Omega, \quad \lambda A = (\lambda a_{ijk}) \in \Omega,$$

for $\forall A=(a_{ijk}), B=(b_{ijk})\in\Omega,\ \lambda\in F$, that is, $(A+B)_{ijk}=a_{ijk}+b_{ijk}$, $(\lambda A)_{ijk}=\lambda a_{ijk}$.

Denote E_{ijk} a cubic matrix with the element in the position (i,j,k) is 1 and elsewhere are zero. Then $\{E_{ijk},\ 1\le i,j,k\le N\}$ is a basis of Ω, and for every $A=(a_{ijk})\in\Omega$, $A=\sum\limits_{1\le i,j,k\le N} a_{ijk}E_{ijk},\ a_{ijk}\in F$.

For all $A=(a_{ijk}), B=(b_{ijk})\in\Omega$, define the multiplication $*_{21}$ in Ω by

$$(A *_{21} B)_{ijk}=\sum_{p,q=1}^{N} a_{qjp}b_{ipk}, 1\le i,j,k\le N,$$

then $(\Omega, *_{21})$ is an associative algebra, and in the basis $\{E_{ijk}|1\le i,j,k\le N\}$, we have

$$E_{ijk} *_{21} E_{lmn}=\delta_{km}E_{ljn},\ 1\le i,j,k,l,m,n\le N,$$

where δ_{ij} is 1 in the cases $i=j$, and others are zero, $1\le i,j\le N$.

Define linear function $\langle\ \rangle_1:\Omega\to F$ by $\langle A\rangle_1=\sum\limits_{p,q=1}^{N} a_{pqq}$, Then we have

$$\langle A *_{21} B\rangle_1=\langle B *_{21} A\rangle_1. \tag{1}$$

So we define the multiplication $[,,]_{21}:\Omega\wedge\Omega\wedge\Omega\to\Omega$ as follows:

$$\begin{aligned}[A,B,C]_{21}=&\langle A\rangle_1(B *_{21} C-C *_{21} B)\\ &+\langle B\rangle_1(C *_{21} A-A *_{21} C)+\langle C\rangle_1(A *_{21} B-B *_{21} A).\end{aligned} \tag{2}$$

We obtain a 3-ary algebra $(\Omega,[,,]_{21})$.

2 The structure of J_{21}

First we give the following lemma.

Theorem 2.1[1] *The linear space Ω is a 3-Lie algebras [2] in the multiplication $[,,]_{21}$, which is denoted by J_{21}.*

In the following we suppose $N=2$. We have the following result.

Theorem 2.2 *The 3-Lie algebra J_{21} is a non-nilpotent indecomposable 3-Lie algebra with a basis $e_1=E_{111}, e_2=E_{112}, e_3=E_{121}, e_4=E_{111}-E_{122}, e_5=E_{211}-E_{111}, e_6=E_{212}-E_{112}, e_7=E_{221}-E_{121}, e_8=E_{122}-E_{222}$, and*

1) the multiplication in it is as follows:

$$\begin{cases}[e_1,e_2,e_3]=e_4,[e_1,e_4,e_2]=2e_2,[e_1,e_3,e_4]=2e_3,[e_1,e_7,e_4]=e_7,\\ [e_1,e_3,e_5]=e_7,[e_1,e_4,e_5]=e_5,\ [e_1,e_6,e_3]=e_8,\ \ [e_1,e_4,e_6]=e_6,\\ [e_1,e_2,e_7]=e_5,[e_1,e_8,e_2]=e_6,\ [e_1,e_4,e_8]=-e_8.\end{cases} \tag{3}$$

Then center of J_{21} is 0.

2) The derived algebra $J_{21}^1=<e_2,e_3,e_4,e_5,e_6,e_7,e_8>$, and $M_1=<e_5,e_7>$, $M_2=<e_6,e_8>$ are minimal ideals of J_{21}.

3) J_{21} *is a non-2-solvable, but* 3*-solvable* 3*-Lie algebra with* $[J_{21}^1, J_{21}^1, J_{21}^1] = 0$.

Proof It is clear that $\{e_1, \cdots, e_8\}$ is a basis of Ω. By the definition of $*_{12}$, we obtain Eq.(3). Thanks to $ad(e_1, e_4)$ is non-nilpotent, the 3-Lie algebra J_{21} is non-nilpotent and the center is zero. By the multiplication, $\dim J_{21}^1 = 7$, and $J_{21}^1 =< e_2, \cdots, e_8 >$. Since $[J_{21}, M_1] = M_1$ and $[J_{21}, M_2] = M_2$, M_1 and M_2 are minimal ideals of J_{21}. Follows from $[J_{21}^1, J_{21}^1, J_{21}] = J_{21}^1$, and $[J_{21}^1, J_{21}^1, J_{21}^1] = 0$, we obtain the result. Then proof is completed.

Now we study the inner derivation algebra adJ_{21}. For $e_i, e_j \in \Omega$, denote

$$ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l, \text{ where } a_{kl}^{ij} = -a_{kl}^{ji} \in F.$$

Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl} are 8×8-matrix units.

Theorem 2.3 *Let* J_{21} *be a* 3*-Lie algebra in Theorem 2.2. Then we have*

1) $\dim adJ_{21} = 14$, *and* $\{X_1 = E_{34} - 2E_{42} + E_{75} - E_{86}, X_2 = -E_{24} + 2E_{43} + E_{57} - E_{68}, X_3 = 2E_{22} - 2E_{33} + E_{55} + E_{66} - E_{77} - E_{88}, X_4 = E_{37} + E_{45}, X_5 = E_{38} - E_{46}, X_6 = -E_{25} + E_{47}, X_7 = E_{26} + E_{48}, X_8 = E_{12}, X_9 = E_{13}, X_{10} = E_{14}, X_{11} = E_{15}, X_{12} = E_{16}, X_{13} = E_{17}, X_{14} = E_{18}\}$ *is a basis of* adJ_{21}, *the multiplication in it is*

$$\begin{cases} [X_2, X_1] = X_3, \ [X_3, X_2] = 2X_2, \ [X_1, X_3] = 2X_1, \ [X_{10}, X_2] = 2X_9, \\ [X_1, X_7] = X_5, \ [X_1, X_9] = -X_{10}, \ [X_{10}, X_1] = 2X_8, [X_1, X_{13}] = -X_{11}, \\ [X_1, X_{14}] = X_{12}, [X_3, X_5] = -X_5, \ [X_2, X_4] = X_6, \ [X_2, X_8] = X_{10}, \\ [X_1, X_6] = X_4, \ [X_2, X_{11}] = -X_{13}, [X_2, X_{12}] = X_{14}, [X_3, X_4] = -X_4, \\ [X_2, X_5] = X_7, \ [X_3, X_{12}] = -X_{12}, [X_3, X_6] = X_6, \ [X_3, X_7] = X_7, \\ [X_3, X_9] = 2X_9, [X_3, X_{11}] = -X_{11}, [X_3, X_{13}] = X_{13}, \ [X_3, X_8] = -2X_8, \\ [X_3, X_{14}] = X_{14}, [X_4, X_9] = -X_{13}, \ [X_4, X_{10}] = -X_{11}, [X_5, X_9] = -X_{14}, \\ [X_5, X_{10}] = X_{12}, \ [X_6, X_{10}] = -X_{13}, [X_6, X_8] = X_{11}, [X_7, X_8] = -X_{12}, \\ [X_7, X_{10}] = -X_{14}. \end{cases}$$

2) adJ_{21} *is an indecomposable Lie algebra, and*

$$adJ_{21} = L \dot{+} M = ad^1 J_{21}, \ \textit{where } L =< X_1, X_2, X_3 > \cong sl(2),$$

$M = M_1 \dot{+} M_2 \dot{+} M_3 \dot{+} M_4 \dot{+} M_5$ *is a maximal nilpotent ideal of* adJ_{21}, *and* M_i *are irreducible* $sl(2)$*-modules,* $M_1 =< X_6, X_4 >$, $M_2 =< X_7, X_5 >$, $M_3 =< X_9, X_{10}, X_8 >$, $M_4 =< X_{13}, X_{11} >$, $M_5 =< X_{14}, X_{12} >$.

Proof By a direct computation according to Eq.(3) we have

$$ad(e_1, e_2) = E_{34} - 2E_{42} + E_{75} - E_{86}, ad(e_1, e_3) = -E_{24} + 2E_{43} + E_{57} - E_{68},$$

$$ad(e_1, e_4) = 2E_{22} - 2E_{33} + E_{55} + E_{66} - E_{77} - E_{88}, ad(e_1, e_5) = -E_{37} - E_{45},$$

$$ad(e_1,e_6)=E_{38}-E_{46}, ad(e_1,e_7)=-E_{25}+E_{47}, ad(e_1,e_8)=E_{26}+E_{48},$$
$$ad(e_2,e_4)=-2E_{12}, ad(e_3,e_4)=2E_{13}, ad(e_2,e_3)=E_{14}, ad(e_2,e_7)=E_{15},$$
$$ad(e_2,e_8)=-E_{16}, ad(e_3,e_5)=E_{17}, ad(e_3,e_6)=E_{18}.$$

Denote $X_1=E_{34}-2E_{42}+E_{75}-E_{86}, X_2=-E_{24}+2E_{43}+E_{57}-E_{68}, X_3=2E_{22}-2E_{33}+E_{55}+E_{66}-E_{77}-E_{88}, X_4=E_{37}+E_{45}, X_5=E_{38}-E_{46}, X_6=-E_{25}+E_{47}, X_7=E_{26}+E_{48}, X_8=E_{12}, X_9=E_{13}, X_{10}=E_{14}, X_{11}=E_{15}, X_{12}=E_{16}, X_{13}=E_{17}, X_{14}=E_{18}$.
We obtain that $\{X_1,\cdots,X_{14}\}$ is a basis of adJ_{21}. From

$[ad(e_i,e_j),ad(e_k,e_l)]= ad([e_i,e_j,e_k],e_l)+ad(e_k,[e_i,e_je_l])$,

we have the result 1).

Let $L=<X_1,X_2,X_3>$, $M_1=<X_6,X_4>$, $M_2=<X_7,X_5>$, $M_3=<X_9,X_{10},X_8>$, $M_4=<X_{13},X_{11}>$, $M_5=<X_{14},X_{12}>$. From the above discussion, $L=<X_1,X_2,X_3>\cong sl(2)$, and M_i for $1\le i\le 5$ are irreducible L-modules, and $adJ_{21}=ad^1J_{21}$, $[M,M]\subseteq M$. The proof is completed.

Theorem 2.4 *Let J_{21} be a 3-Lie algebra in Theorem 2.2. Then we have*

1) The dimension of $DerJ_{21}$ is 19, and $DerJ_{21}$ with a basis $\{X_1,\cdots,X_{19}\}$, where $X_{15}=E_{11}-2E_{33}-E_{44}-E_{77}-E_{88}, X_{16}=E_{66}+E_{88}, X_{17}=E_{55}+E_{77}-E_{66}-E_{88}, X_{18}=E_{56}-E_{78}, X_{19}=E_{65}-E_{87}$, and X_i for $1\le i\le 14$ is in Theorem 2.3. And the multiplication in the basis is

$$\begin{cases}
[X_2,X_1]=X_3,[X_3,X_2]=2X_2,[X_3,X_1]=-2X_1,[X_1,X_6]=X_4,\\
[X_1,X_7]=X_5,[X_1,X_9]=-X_{10},[X_1,X_{10}]=-2X_8,[X_1,X_{13}]=-X_{11},\\
[X_2,X_5]=X_7,[X_1,X_{14}]=X_{12},[X_2,X_4]=X_6,[X_3,X_{14}]=X_{14},\\
[X_2,X_8]=X_{10},[X_2,X_{10}]=-2X_9,[X_2,X_{11}]=-X_{13},[X_2,X_{12}]=X_{14}\\
[X_4,X_3]=X_4,[X_3,X_5]=-X_5,[X_3,X_6]=X_6,[X_3,X_7]=X_7,\\
[X_3,X_9]=2X_9,[X_3,X_{11}]=-X_{11},[X_3,X_{12}]=-X_{12},[X_3,X_{13}]=X_{13},\\
[X_9,X_4]=X_{13},[X_4,X_{10}]=-X_{11},[X_5,X_9]=-X_{14},[X_5,X_{10}]=X_{12},\\
[X_6,X_8]=X_{11},[X_{10},X_6]=X_{13},[X_7,X_8]=-X_{12},[X_7,X_{10}]=-X_{14},\\
[X_1,X_{15}]=X_1,[X_2,X_{15}]=-X_2,[X_4,X_{15}]=X_4,[X_5,X_{15}]=X_5,\\
[X_{15},X_8]=X_8,[X_{15},X_9]=3X_9,[X_{15},X_{10}]=2X_{10},[X_{15},X_{14}]=2X_{14},\\
[X_8,X_3]=2X_8,[X_{15},X_{11}]=X_{11},[X_{15},X_{12}]=X_{12},[X_{15},X_{13}]=2X_{13},\\
[X_6,X_{17}]=X_6,[X_{11},X_{17}]=X_{11},[X_{13},X_{17}]=X_{13},[X_4,X_{18}]=X_5,\\
[X_{11},X_{18}]=X_{12},[X_{18},X_{13}]=X_{14},[X_{19},X_5]=X_4,[X_{19},X_7]=X_6,\\
[X_{19},X_{14}]=X_{13},[X_5,X_{16}]=X_5,[X_7,X_{16}]=X_7,[X_{12},X_{16}]=X_{12},\\
[X_{17},X_{18}]=2X_{18},[X_{17},X_{19}]=-2X_{19},[X_{18},X_{19}]=X_{17},\\
[X_{18},X_{16}]=X_{18},[X_{19},X_{16}]=-X_{19},[X_4,X_{17}]=X_4,[X_6,X_{18}]=-X_7,\\
[X_{12},X_{19}]=X_{11},[X_{14},X_{16}]=X_{14},[X_5,X_{17}]=-X_5,[X_7,X_{17}]=-X_7,\\
[X_{12},X_{17}]=-X_{12},[X_{14},X_{17}]=-X_{14}.
\end{cases}$$

2) $DerJ_{21}$ is an indecomposable Lie algebra, and

$$DerJ_{21}=adJ_{21}\dot{+}B,\ \ Der^1J_{21}=adJ^1_{21}\dot{+}<X_{17},X_{18},X_{19}>,$$

where $B =< X_{15}, X_{16}, X_{17}, X_{18}, X_{19} >$, $[B, B] =< X_{17}, X_{18}, X_{19} >\cong sl(2)$, $< X_{15}, X_{17} + X_{16} >$ *is contained in the center of* B.

Proof For all $D \in Der J_{21}$, suppose $D(e_i) = \sum\limits_{j=1}^{8} a_{ij}e_j$, $1 \leq i \leq 8$, then the matrix form of D in the basis $\{e_1, \cdots, e_8\}$ is $A = (a_{ij})_{i,j=1}^{8} = \sum\limits_{i,j=1}^{8} a_{ij}E_{ij}$, where E_{ij} are (8×8) matrix units, $1 \leq i, j \leq 8$. By a direct computation according to the multiplication (3), we have the result 1).

Thanks to Theorem 2.3, $B =< X_{15}, X_{16}, X_{17}, X_{18}, X_{19} >$ are exterior derivations. Then we have $Der J_{21} = adJ_{21} \dot{+} B$, and $[B, B] =< X_{17}, X_{18}, X_{19} >\cong sl(2)$, $< X_{15}, X_{17} + X_{16} >$ is contained in the center of B. By a direct computation, $Der^1 J_{21} =< X_2, \cdots, X_{19} >$. The proof is completed.

Acknowledgements

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

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Received: August, 2015