

AN ANALYSIS OF A QUEUEING SYSTEM WITH
HETEROGENEOUS SERVERS SUBJECT TO CATASTROPHES

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Abstract

This study analyzed a queueing system with blocking and no waiting line. The customers arrive according to a Poisson process and the service times follow exponential distribution. There are two non-identical servers in the system.The queue discipline is FCFS, and the customers select the servers on fastest server first (FSF) basis. The service times are exponentially distributed with parameters μ_1 and μ_2 at servers I and II, respectively. Besides, the catastrophes occur in a Poisson manner with rate γ in the system. When server 1 is busy or blocked, the customer who arrives in the system leaves the system without being served. Such customers are called lost customers. The probability of losing a customer was computed for the system. The explicit time dependent probabilities of system size are obtained and a numerical example is presented in order to show the managerial insights of the model. Finally, the probability that arriving customer finds system busy and average number of server busy in steady state are obtained numerically.

Mathematics Subject Classification: 60K25, 90B22

Keywords: Stochastic Model, Queuing Model, Jockeying.

1 Introduction

Queueing models with catastrophes gained considerable interest during the last few decades due to their applications in the analysis of computer- and communication systems where catastrophes induced by external effects have an important influence on costs and performance from an economic viewpoint. Whenever a catastrophe occurs at the system, all the customers present are forced to abandon the system immediately, the server gets inoperative instantaneously, and the server is ready for service when a new

customer arrives. The modeling and analysis of queueing systems with catastrophes may be used to study the migration processes with catastrophes and computer networks with virus infections or a reset order.

Queueing systems with catastrophes have been investigated by many researchers [6, 7, 8]. The queueing models with blocking used to model the systems encountered in fields such as manufacturing systems, computer communications systems [3], The exact analytic solutions of such queueing models are only possible for the cases with a small number of servers [1]. Therefore, approximate solutions were obtained by using approximation methods for the systems of queues with blocking in many studies in the literature [2]. [1] dealt with the tandem queueing model with no waiting line and obtained the probability of lost customers in the system. [8, 9] examined the $M/M/2$ queueing model with two parallel servers also considering the fact that catastrophes fitting the Poisson distribution with rate might take place. As soon as a catastrophe occurs, both servers are inactivated momentarily and, immediately afterwards, the system returns to its initial state with probability 1 [5, 6]. Although tandem queueing models have been examined under many service disciplines in the studies in the literature, the case of catastrophe has not been considered in any studies. In real systems, a catastrophe might result either from outside the system/facility or from another service station. Computer networks with a virus infection might be considered examples of queueing models with catastrophes. Furthermore, the return of the system to the initial state either automatically or by the admin due to the busy condition of the Internet systems or any hitch might be considered examples of models with catastrophes [4, 12]. In this study, a queueing system with heterogeneous servers subject to catastrophes is examined and the mean number of customers in the system and the probability of lost customers are computed.

In this present paper, we study a standby system involving discrete distribution. We consider N system, one working and the other in standby, in repair or waiting for repair. There is a repairman. The standby system do not fail. The working machine is subject to internal(non-repairable) and accidental external repairs. When the non-repairable failure occurs the machine is removed from the system. When the working machine under goes a repairable failure, it goes to repair. In both cases, a standby machine becomes the working one instantaneously. When a unit is repaired, it re-enters the system and is new. The time of the internal failure has a general distribution and its PH representation is considered. We present a model where the reparability of the accidental failure can be independent of or dependent on the time system failure. The conditional probability is different types of failure are calculated in a matrix and algorithmic form. It is proved that the loss probability is minimum when the customer having

arrived in the system is first served at the fast server and then at the slow server, respectively. The results of this paper are organized as follows: the assumptions of the model are first described; followed by an analysis of the system with the Markov process and the steady-state probabilities of the system, the mean number of customers in the system and the loss probability are computed. Thereafter, the conditions under which the loss probability is optimum are determined. A numerical example of the model under consideration is subsequently provided, the obtained results are evaluated and the new studies likely to be made concerning the subject and recommendations are discussed. The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed.

2 The Model Description and Its Assumptions

The queueing system with blocking, heterogeneous servers and no waiting line subject to catastrophes will be analyzed in this study. In this model, arrival times are Poisson distributed with parameter λ . There are two heterogeneous tandem servers in the system. Their mean service times are assumed to be different from each other. The service time of each customer at server m is random variable η_m and has an exponential distribution with parameter μ_m ($m = 1, 2..$) Catastrophe is exponentially distributed. That is, has an exponential distribution with parameter, that is. As soon as a catastrophe takes place in the system, all customers are immediately destroyed. Both servers are inactivated momentarily and when there is a new arrival, both servers get ready to serve. Briefly, when there is a catastrophe in the system, the system returns to its initial state with probability 1. Each customer arriving in the system is first served at server I and then at server II, respectively. Waiting line is not allowed in front of the servers. If server II is busy when the service time has been completed at server I, then server I is blocked until the service is completed at server II. If server I is busy or blocked at the time of arrival of a customer in the system, that customer leaves the system without being served at all. Such customers are called lost customers. Thus, the main problem herein is to compute the probability of lost customers in the system and minimize this probability.

3 The Model Analysing using markov process method

In this section, Let $\{X(t) = (u_t, v_t), t \geq 0\}$ be the number of customers in the system at time t . Let u_t and v_t be the states of servers I and II respectively. Let $P_{00}(t) = P(X(t) = 0)$ be the probability that the system is empty at time t . Let $P_{10}(t) = P(X(t) = 1)$ be the probability that there is one customer served by server I in the system at time t . Let $P_{01}(t) = P(X(t) = 1)$ be the probability that there is one served by server II in the system at time t . Let $P_{11}(t) = P(X(t) = 2)$ be the probability that there are two customers in the system at time t . and that both servers are busy. In addition, let $P_{b1}(t) = P(X(t) = 1)$ be the probability that there is one customer who served by server II in the system at time t and that server 1 is blocked or busy. It is clear that $\{u_t, v - t\}, t \geq 0$ is continuous parameter markov process with state spaces $S = \{(0, 0), (1, 0), (0, 1), (1, 1), (b, 1)\}$ and $P_{ij}(t) = P\{u_i = i, v_j, (i, j) \in S\}$.

$$\frac{dP_{00}(t)}{dt} = -\lambda P_{00}(t) + \mu_2 P_{01}(t) + \gamma [P_{00}(t)] \quad (1)$$

$$\frac{dP_{10}(t)}{dt} = -(\mu_1 + \gamma)P_{10}(t) + \lambda P_{00}(t) + \mu_2 P_{11}(t) \quad (2)$$

$$\frac{dP_{01}(t)}{dt} = -(\lambda + \mu_2 + \gamma)P_{01}(t) + \mu_1 P_{10}(t) + \mu_2 P_{b1}(t) \quad (3)$$

$$\frac{dP_{11}(t)}{dt} = -(\mu_1 + \mu_2 + \gamma)P_{11}(t) + \lambda P_{01}(t) \quad (4)$$

$$\frac{dP_{b1}(t)}{dt} = -(\mu_2 + \gamma)P_{b1}(t) + \mu_1 P_{11}(t) \quad (5)$$

The limits $P_{ij} = \lim_{x \rightarrow \infty} P\{u_t = i, v_t = j\}, (i, j) \in S$ exist and satisfy the system of linear equations. These equations are obtained from (1) to (5) on replacing the derivatives on the left by zero as follows.

$$\begin{aligned} -(\lambda P_{00} + \mu_2 P_{01} + \gamma - \gamma P_{00}) &= 0, \\ -(\mu_1 + \gamma)P_{10} + \lambda P_{00} + \mu_2 P_{11} &= 0, \\ -(\lambda + \mu_2 + \gamma)p_{0,1} + \mu_1 p_{0,1} + \mu_2 p_{b1} &= 0, \end{aligned} \quad (6)$$

$$\begin{aligned} -(\mu_1 + \mu_2 + \gamma)p_{11} + \lambda p_0 &= 0, \\ -(\mu_2 + \gamma)p_{b1} + \mu_1 P_{11} &= 0, \end{aligned}$$

The steady- state probabilities P_{ij} are expressed in terms of P_{00} .

$$P_{01} = \frac{(\lambda + \gamma)P_{00} - \gamma}{\mu_2} \quad (7)$$

$$P_{11} = \frac{\lambda [(\lambda + \mu + 2\gamma)P_{00} - \gamma]}{(\mu_1 + \gamma)(\mu + \gamma)} \quad (8)$$

$$P_{10} = \frac{\lambda [(\lambda + \gamma)P_{00} - \gamma]}{\mu_1(\mu + \gamma)} \quad (9)$$

$$P_{b1} = \frac{\lambda\mu_1 [(\lambda + \gamma)P_{00} - \gamma]}{\mu_2(\mu_2 + \gamma)(\mu + \gamma)} \quad (10)$$

$$P_{00} = \frac{\mu_2 + \gamma + \frac{\lambda\gamma}{\mu + \gamma} \left(\frac{\mu_1}{\mu_2 + \gamma} + \frac{\mu + \gamma}{\mu_1 + \gamma} \right)}{\lambda + \gamma + \frac{\mu_2(\lambda + \mu_1 + \gamma)}{\mu_1 + \gamma} + \frac{\lambda(\lambda + \gamma)}{\mu + \gamma} \left(\frac{\mu_1}{\mu_2 + \lambda} + \frac{\mu + \lambda}{\mu_1 + \lambda} \right)} \quad (11)$$

Where $\mu = \mu_1 + \mu_2$. Unknown probability P_{00} is determined using the condition $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$ as follows. Where $\mu = \mu_1 + \mu_2$. By using equations (7), (8), (9), (10) and (11) the expected number of customers in the system L_s and the probability of lost customers in the system P_L are obtained as follows.

$$\begin{aligned} E_s &= P_{10} + P_{01} + 2(P_{11} + P_{b1}) \\ &= 1 - P_{00} + P_{11} + P_{b1} \\ &= 1 + \frac{[\lambda(\lambda + \gamma) - \mu_2(\mu_2 + \gamma)] P_{00} - \gamma}{\mu_2(\mu_2 + \gamma)} \end{aligned} \quad (12)$$

$$\begin{aligned} E_L &= P_{10} + P_{11} + P_{b1} \\ &= 1 - P_{00} - P_{01} \\ &= \frac{\mu_2 + \lambda(\lambda + \mu_2 + \gamma)P_{00} - \gamma}{\mu_2} \end{aligned} \quad (13)$$

4 Performance Measure

Having the probability vector P been computed, we are able to calculate performance measures of the considered model. The main performance measure in the case of a finite is the probability loss P that an arbitrary customer will be lost. As no waiting line is available in the system, some of the customers arriving in the system have to leave the system without being served. Such customers are called lost customers. The probability of lost customers is easily obtained from formula (13). In such systems, the

minimization of loss probability is a serious problem. In the model considered in this study, the customer arriving in the system first enters the first server and then the second server, respectively. we like to minimize both the blocking probability as well as the dropping probability. The decision variables are the number of guard channels, and the number of channels, . In a simpler version of the problem, we fix and consider only as the decision variable. Given the two objectives, there are several different ways we can set up the optimization problem. We can pick either or as the objective function to be minimized and we impose a constraint on the other one. Thus, we consider two representative optimization problems below.

Theorem 4.1 *The loss probability P_L is calculated satisfying inequality $\mu_1 \geq \mu_2$ under the condition $\mu_1 + \mu_2 = \mu$.*

The proof follows, Loss probability P_L takes the minimum value when servers ordered according to the values of μ_1 and μ_2 satisfying inequality $\mu_1 \geq \mu_2$ under the condition $\mu_1 + \mu_2 = \mu$. When the customers are first served at the fast server and then at the slow server, the system becomes optimal in terms of the probability of customers leaving without being served.

We performed several experiments with both modifications to demonstrate solvability of the presented model and to obtain some graphical dependencies. Applied values of the model parameters are summarized in Table 1. Substituting the values summarized in Table 1 into the model rewritten in Matlab we are able to compute the steady-state probabilities of the system states and on the basis of them we get the performance measures P_{00} , E_s and E_L using equations (11), (12) and (13). The P_{00} , E_s and E_L values obtained in Table 1 were computed by using the equations (11), (12) and (13), respectively. As clearly seen from this table, the loss probability E_L takes its minimum value if the customer who has arrived in the system is first served at the fast server and then at the slow server. On the other hand, the mean number of customers in the system (E_s) increases as the loss probability in the system decreases.

5 Table.1 Calculations of the performance measures using the given values of parameters

Parameter	Value	Parameter	value
λ	1.000	λ	1.0000
γ	0.2000	γ	0.2000
μ_1	1.5200	μ_1	2.1100
μ_2	2.1100	μ_2	1.5200
Performance Measure	Calculation	Performance Measure	Calculation
P_{00}	0.4542	P_{00}	0.4368
E_s	0.6165	E_s	0.6871
E_L	0.7280	E_s	0.3500

for the $\mu_1 \leq \mu_2$

under the condition and $\mu_1 + \mu_2 = \mu$. Let us reorganize the loss probability provided with equation (13) for $\gamma = 0$ in the following way.

$$E_L = \frac{\lambda\mu^2\mu_2 + \lambda^2\mu\mu_2 + \lambda^2\mu_1^2 - \lambda\mu\mu_1\mu_2}{\lambda\mu^2\mu_2 + \lambda^2\mu\mu_2 + \lambda^2\mu_1^2 + \mu\mu_1\mu_2^2} \tag{14}$$

If we multiply both the numerator and denominator of the equation (14) by μ_1 the equation (15) is obtained as follows.

$$E_L = \frac{(\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2) + \lambda^2\mu_1^3 - \lambda\mu\mu_1^2\mu_2}{(\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2) + \lambda^2\mu_1^3 + \mu\mu_1^2\mu_2^2} \tag{15}$$

Let $\theta = (\lambda\mu^2\mu_1\mu_2 + \lambda^2\mu\mu_1\mu_2)$ in equation (15). then we consider that $\mu = \mu_1 + \mu_2$. proof of the equation (15) is analogue, so we will prove only (15) according to the formula of the total probability. The probability loss is calculated is obtained as follows.

$$E_L = \frac{\theta + \lambda\mu_1^3(\lambda - \mu_2) - \lambda\mu_1^2\mu_2^2}{\theta + \lambda^2\mu_1^3 + \mu\mu_1^2\mu_2^2} \tag{16}$$

Hence the value the denominator of equation (16) takes for $\mu_1 \geq \mu_2$ is either equal to or greater than the value it takes for $\mu_1 \leq \mu_2$ under this condition $\mu_1 + \mu_2 = \mu$.

Let the numerator of equation (16) be examined. It will be $(\lambda - \mu_2) > 0$ and $(\lambda - \mu_2) = 0$ for $\lambda > \mu_2$, $\lambda < \mu_2$ and $\lambda = \mu_2$. For $(\lambda - \mu_2) < 0$ and $(\lambda - \mu_2) = 0$ the denominator of equation (16) always decreases under the condition $\mu_1 + \mu_2 = \mu$. Depending on the above mentioned the value that the numerator of equation (16) takes for $\mu_1 \geq \mu_2$ is either equal to smaller than the value takes for $\mu_1 \leq \mu_2$ under the conditions $(\lambda - \mu_2) > 0$,

$(\lambda - \mu_2) = 0$ and $\mu_1 + \mu_2 = \mu$. Therefore The proof has been completed.

Conclusion: This paper discussed that the performance analysis of a single-server markovian queueing system with heterogeneous servers subject to catastrophes. The system size probabilities are calculated explicitly. In conclusion loss probability E_L obtained for $\mu_1 \geq \mu_2$ is either equal to or smaller than the loss probability obtained for $\mu_1 \leq \mu_2$ under this condition $\mu_1 + \mu_2 = \mu$. When the customers are first served at the fast server and then at the slow server, the system becomes optimal in terms of the probability of customers leaving without being served.

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Received: August, 2015