Mathematica Aeterna, Vol. 5, 2015, no. 5, 667 - 672

Local Regularity for Minimizers of Obstacle Problems of Some Anisotropic Integral Functionals

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Abstract

A local regularity result is obtained for minimizers $u \in \mathcal{K}_\psi = \left\{u \in W^{1,(q_i)}_{loc}(\Omega) : u \geq \psi\right\}$, $q_i > 1, \forall i = 1, \cdots, N$, of anisotropic integral functionals of the type

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x, u, Du)dx,$$

where the Carathéodory function $f(x, u, Du) = f_0(x, u, Du) + f_1(x, u, Du)$, $f_0(x, s, z)$ grows like $\sum_{i=1}^N |z_i|^{q_i}$, and $f_1(x, s, z)$ satisfies some controllable growth condition.

Mathematics Subject Classification: 49N60, 35J60.

Keywords: Local regularity, minimizer, obstacle problem, anisotropic integral functional.

1 Introduction and Statement of Result.

Let $\Omega \subset \mathrm{R}^N$, $N \geq 2$, be a domain. Let $q_i > 1$, $i = 1, \ldots, N$. We denote

$$q = \max_{1\leq i\leq N} q_i, \quad p = \min_{1\leq i\leq N} q_i \quad \text{and} \quad \bar{q} : \frac{1}{\bar{q}} = \frac{1}{N}\sum_{i=1}^N \frac{1}{q_i}.$$

Throughout this paper we will make use of the anisotropic Sobolev space

$$W_{loc}^{1,(q_i)}(\Omega)=\left\{v\in L_{loc}^{1,q}(\Omega):\frac{\partial v}{\partial x_i}\in L_{loc}^{q_i}(\Omega),\forall i=1,\dots,N\right\}.$$

We consider integral functionals of the type

$$\mathcal{F}(u;\Omega)=\int_\Omega f(x,u,Du)dx, \tag{1.1}$$

where the Carathéodory function $f(x,s,z)$ satisfies the following assumptions:

(i) $f(x,s,z):\Omega\times \mathrm{R}\times \mathrm{R}^n\to \mathrm{R}$ can be written as

$$f(x,s,z)=f_0(x,s,z)+f_1(x,s,z);$$

(ii) $f_0(x,s,z)$ satisfies the growth condition

$$L^{-1}\sum_{i=1}^N|z_i|^{q_i}\le f_0(x,s,z)\le L\sum_{i=1}^N|z_i|^{q_i}+\varphi_1,$$

where $L>1, q_i>1, \forall i=1,\dots,N$, and $\varphi_1\in L_{loc}^r(\Omega)$, $1<r<\frac{N}{p}$;

(iii) there exist $0\le m_i<p_i$, $0\le h_i(x)\in L_{loc}^{\frac{q_ir}{q_i-m_i}}(\Omega)$, $i=1,\cdots,N$, such that

$$|f_1(x,s,z)|\le\sum_{i=1}^N h_i(x)|z_i|^{m_i}.$$

Let $\psi\in W_{loc}^{1,(q_ir)}(\Omega)$ and $\mathcal{K}_\psi=\left\{u\in W_{loc}^{1,(q_i)}(\Omega):u\ge\psi\right\}$. In the present paper we shall consider minimizers $u\in\mathcal{K}_\psi$ for obstacle problems of (1.1), that is,

$$\mathcal{F}(u,\mathrm{supp}(u-v))\le\mathcal{F}(v,\mathrm{supp}(u-v)) \tag{1.2}$$

for every $v\in K_\psi$. The main result is the following theorem.

Theorem 1.1 *Assume that the integral function (1.1) satisfies conditions (i), (ii) and (iii). Let* $\psi\in W_{loc}^{1,(q_ir)}(\Omega)$. *If* $u\in\mathcal{K}_\psi$ *satisfies (1.2), then it belongs to* $L_{loc}^s(\Omega)$, *where* $s=\frac{\bar q^*q}{q-\bar q^*(1-1/r)}$.

We refer the readers to [1-7] for some results related to local regularity properties for solutions of elliptic equations and minima of variational integrals.

2 Preliminary Lemmas.

For $x_0 \in \Omega$ and $t \in \mathrm{R}$, we denote by $B_t = B_t(x_0)$ the ball of radius t centred x_0. For $k > 0$, let

$$A_k = \{x \in \Omega : |u(x)| > k\} \text{ and } A_{k,t} = A_k \cap B_t. \tag{2.1}$$

Moreover, if $m < n$, m^* is the real number satisfying $m^* = \frac{nm}{n-m}$.

The following two lemmas will be used in the proof of Theorem 1.1. The first lemma comes from [1, Theorem 2.2].

Lemma 2.1 *Let $u \in W^{1,(q_i)}_{loc}(\Omega), \varphi_0 \in L^r_{loc}(\Omega)$, where $q, \bar{q}$ and r satisfies*

$$1 < r < \frac{N}{\bar{q}},\ q < \bar{q}^*,\ \bar{q} < N.$$

Assume that the following integral estimate holds

$$\int_{A_{k,\tau}} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \le c_0 \left[\int_{A_{k,t}} \varphi_0 dx + (t-\tau)^{-\alpha} \int_{A_{k,t}} \sum_{i=1}^{N} |u|^{q_i} dx \right] \tag{2.2}$$

for every $k \in \mathcal{N}$ and $R_0 \le \tau < t \le R_1$, where c_0 is a positive constant that depends only on N, q_i, r, R_0, R_1 and $|\Omega|$, and α is a real positive constant. Then $u \in L^s_{loc}(\Omega)$, where $s = \frac{\bar{q}^* q}{q - \bar{q}^*(1-1/r)}$.

The following lemma can be found in [8, p.161, Lemma 3.1].

Lemma 2.2 *Let $f(\tau)$ be a non-negative bounded function defined for $0 \le R_0 \le t \le R_1$. Suppose that for $R_0 \le \tau < t \le R_1$ we have*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \theta f(t), \tag{2.3}$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exist a constant c, depending only on α and θ such that for every $\rho, R, R_0 \le \rho < R \le R_1$, we have

$$f(\rho) \le c[A(R-\rho)^{-\alpha} + B]. \tag{2.4}$$

3 Proof of Theorem 1.1.

In the sequel the letter c will stands for a genetic constant which may vary from line to line. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1, R_1 - R_0 \le 1$, be arbitrarily fixed. Let

$$T_\psi = \max\{T_k(u), \psi\},$$

where

$$T_k(u) = \max\{-k, \min\{k, u\}\}$$

is the usual truncation of u at level $k > 0$. Choose $v = u - \eta(u - T_\psi)$ in (1.2), where η is a cut-off function such that

$$\eta \in C_0^\infty(B_t), 0 \le \eta \le 1, \eta = 1 \text{ in } B_\tau \text{ and } |D\eta| \le 2(t-\tau)^{-1}.$$

For $u \in \mathcal{K}_\psi$, from $\psi \in W_{loc}^{1,(q_i r)}(\Omega)$ and

$$v = u - \eta(u - T_\psi) = (1-\eta)u + \eta T_\psi \ge (1-\eta)\psi + \eta\psi = \psi,$$

we know that $v \in \mathcal{K}_\psi$. (1.2) implies

$$\begin{aligned} & \int_{B_t} f(x,u,Du)dx \le \int_{B_t} f(x,v,Dv)dx \\ = & \int_{A_{k,t}} f(x, u-\eta(u-T_\psi), Du - D(\eta(u-T_\psi)))dx + \int_{B_t \cap \{|u| \le k\}} f(x,u,Du)dx, \end{aligned} \tag{3.1}$$

from which we derive

$$\int_{A_{k,t}} f(x,u,Du)dx \le \int_{A_{k,t}} f(x, u-\eta(u-T_\psi), Du - D(\eta(u-T_\psi)))dx. \tag{3.2}$$

Using (i), (ii) in (3.2) we have

$$\begin{aligned} & L^{-1} \int_{A_{k,t}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ \le & \int_{A_{k,t}} f(x, u-\eta(u-T_\psi), Du - D(\eta(u-T_\psi)))dx - \int_{A_{k.t}} f_1(x,u,Du)dx \\ \le & \int_{A_{k,t} \setminus A_{k,\tau}} f(x, (1-\eta)u + \eta T_\psi, (1-\eta)Du - (u-T_\psi)D\eta + \eta DT_\psi)dx \\ & + \int_{A_{k,\tau}} f(x, T_\psi, DT_\psi)dx - \int_{A_{k.t}} f_1(x,u,Du)dx \\ \le & \int_{A_{k,t} \setminus A_{k,\tau}} f_0(x, (1-\eta)u + \eta T_\psi, (1-\eta)Du - (u-T_\psi)D\eta + \eta DT_\psi)dx \\ & + \int_{A_{k,t} \setminus A_{k,\tau}} f_1(x, (1-\eta)u + \eta T_\psi, (1-\eta)Du - (u-T_\psi)D\eta + \eta DT_\psi)dx \\ & + \int_{A_{k,\tau}} f_0(x, T_\psi, DT_\psi)dx + \int_{A_{k,\tau}} f_1(x, T_\psi, DT_\psi)dx - \int_{A_{k,t}} f_1(x,u,Du)dx \\ = & I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.3}$$

Using (ii), (iii) and Young's inequality, $|I_i|$, $i = 1, 2, \cdots, 5$, can be estimated respectively as follows

$$\begin{aligned} |I_1| & \le L \sum_{i=1}^N \int_{A_{k,t} \setminus A_{k,\tau}} \left| (1-\eta)\frac{\partial u}{\partial x_i} - (u-T_\psi)\frac{\partial \eta}{\partial x_i} + \eta \frac{\partial T_\psi}{\partial x_i} \right|^{q_i} dx + \int_{A_{k,t} \setminus A_{k,\tau}} \varphi_1 dx \\ & \le c \sum_{i=1}^N \int_{A_{k,t} \setminus A_{k,\tau}} \left(\left| \frac{\partial u}{\partial x_i} \right|^{q_i} + (t-\tau)^{-q_i}|u|^{q_i} + |\frac{\partial \psi}{\partial x_i}|^{q_i} \right) dx + \int_{A_{k,t} \setminus A_{k,\tau}} \varphi_1 dx, \end{aligned}$$

$$\begin{aligned}|I_2| &\le \sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} h_i \left|(1-\eta)\frac{\partial u}{\partial x_i} - (u-T_\psi)\frac{\partial \eta}{\partial x_i} + \eta \frac{\partial T_\psi}{\partial x_i}\right|^{m_i} dx \\ &\le \varepsilon \sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} \left|(1-\eta)\frac{\partial u}{\partial x_i} - (u-T_\psi)\frac{\partial \eta}{\partial x_i} + \eta \frac{\partial T_\psi}{\partial x_i}\right|^{q_i} dx \\ &\quad + c(\varepsilon)\sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} h_i^{\frac{q_i}{q_i-m_i}} dx \\ &\le c\varepsilon \sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} (|\frac{\partial u}{\partial x_i}|^{q_i} + (t-\tau)^{-q_i}|u|^{q_i} + |\frac{\partial \psi}{\partial x_i}|^{q_i}) dx \\ &\quad + c(\varepsilon)\sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} h_i^{\frac{q_i}{q_i-m_i}} dx,\end{aligned}$$

$$|I_3| \le L\sum_{i=1}^N \int_{A_{k,\tau}} \left|\frac{\partial \psi}{\partial x_i}\right|^{q_i} dx + \int_{A_{k,\tau}} \varphi_1 dx,$$

$$|I_4| \le \sum_{i=1}^N \int_{A_{k,\tau}} h_i \left|\frac{\partial \psi}{\partial x_i}\right|^{m_i} dx \le \varepsilon \sum_{i=1}^N \int_{A_{k,\tau}} \left|\frac{\partial \psi}{\partial x_i}\right|^{q_i} dx + c(\varepsilon)\int_{A_{k,\tau}} \sum_{i=1}^N h_i^{\frac{q_i}{q_i-m_i}} dx,$$

$$|I_5| \le \sum_{i=1}^N \int_{A_{k,t}} h_i \left|\frac{\partial u}{\partial x_i}\right|^{m_i} dx \le \varepsilon \sum_{i=1}^N \int_{A_{k,t}} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx + c(\varepsilon)\sum_{i=1}^N \int_{A_{k,t}} h_i^{\frac{q_i}{q_i-m_i}} dx.$$

In the above estimates we have used the facts

$$|u - T_\psi| \le |u| \text{ and } |DT_\psi| \le |D\psi| \quad \text{in } A_{k,t}.$$

Substituting the above estimates into (3.3), we arrive at

$$\begin{aligned}\sum_{i=1}^N \int_{A_{k,\tau}} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx &\le c\sum_{i=1}^N \int_{A_{k,t}\setminus A_{k,\tau}} \left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i} + (t-\tau)^{-q_i}|u|^{q_i}\right) dx \\ &\quad + c\sum_{i=1}^N \int_{A_{k,t}} \left(\varphi_1 + h_i^{\frac{q_i}{q_i-m_i}} + |\frac{\partial \psi}{\partial x_i}|^{q_i}\right) dx.\end{aligned}$$

Adding to both sides c times the left-side and dividing by $1+c$ we get

$$\begin{aligned}\sum_{i=1}^N \int_{A_{k,\tau}} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx &\le \theta \sum_{i=1}^N \int_{A_{k,t}} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx + \frac{\theta}{(t-\tau)^q}\sum_{i=1}^N \int_{A_{k,t}} |u|^{q_i} dx \\ &\quad + c\sum_{i=1}^N \int_{A_{k,t}} \left(\varphi_1 + h_i^{\frac{q_i}{q_i-m_i}} + |\frac{\partial \psi}{\partial x_i}|^{q_i}\right) dx,\end{aligned}$$

where $\theta = \frac{c}{1+c} < 1$. Lemma 2.2 yields that for any ρ and R with $R_0 \le \rho \le \tau < t \le R \le R_1$, we have

$$\begin{aligned}&\sum_{i=1}^N \int_{A_{k,\rho}} \left|\frac{\partial u}{\partial x_i}\right|^{q_i} dx \\ \le\ & \frac{c}{(R-\rho)^q}\sum_{i=1}^N \int_{A_{k,R}} |u|^{q_i} dx + c\sum_{i=1}^N \int_{A_{k,R}} \left(\varphi_1 + h_i^{\frac{q_i}{q_i-m_i}} + |\frac{\partial \psi}{\partial x_i}|^{q_i}\right) dx.\end{aligned}$$

Theorem 1.1 follows from Lemma 2.1.

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Received: September, 2015