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Critical extinction exponents for a fast diffusive polytropic filtration equation with nonlocal source and inner absorption

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Abstract

In this paper, we deal with an initial boundary value problem for a fast diffusive polytropic filtration equation with nonlocal source and inner absorption in bounded domain. By using the super- and sub-solution method, we obtain some critical extinction exponents on whether occurrences the extinction phenomenon of nonnegative weak solutions or not.

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1 Introduction

We consider the following polytropic filtration equation with a nonlocal source and inner absorption

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \lambda u^r \int_\Omega u^s(x,t)dx - \alpha u^q, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

subjected to the homogeneous Dirichlet boundary and initial conditions

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $m > 0$, $0 < m(p-1) < 1$, $r+s > 0$, $q, \alpha, \lambda > 0$, and $\Omega \subset \mathbf{R}^N (N \geq 1)$ is a bounded domain with smooth boundary and the initial data u_0 is a nonnegative and bounded function with $u_0^m \in W_0^{1,p}(\Omega)$.

Many natural phenomena have been formulated as nonlocal diffusive equation (1.1), such as the model of non-Newton flux in the mechanics of fluid, the model of population, biological species, and so on(we refer to [1,2,3] and the references therein). For example, in the theory of nonlinear filtration, our model (1.1) may be used to describe the nonstationary flows in a porous medium of fluids with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, the equation (1.1) is called the non-Newtonian polytropic filtration equation. In the mathematical model for a heat conduction process, the function $u(x,t)$ represents the temperature, the term $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ represents the thermal diffusion, u^r is the local hot source, $\int_\Omega u^s(x,t)dx$ is the nonlocal hot source and $-u^q$ is the local cool source.

To motivate our work, let us recall some results of finite time extinction properties of solutions for nonlinear diffusion equations. Extinction is the phenomenon whereby there exists a finite time $T > 0$ such that the solution is nontrivial on $(0,T)$ and then $u(x,t) \equiv 0$ for all $(x,t) \in \Omega \times [T, +\infty)$. In this case, T is called the extinction time. It is also an important property of solutions for nonlinear parabolic equations which have been studied by many researchers. For example, Kalashnikov [4] studied the Cauchy problem of the semilinear parabolic equation with an absorption term

$$u_t = \triangle u - \lambda u^p \quad x \in \mathbf{R}^N, \quad t > 0,$$

and obtained extinctions as well as localization and finite propagation properties of the solutions. Evans and Knerr [5] investigated extinction behaviors of the solutions for the Cauchy problem of the semilinear parabolic equation with a fully nonlinear absorption term

$$u_t = \triangle u - \beta(u), \quad x \in \mathbf{R}^N, \quad t > 0.$$

Ferrieira and Vazquez [6] studied extinction phenomena of the solutions for the Cauchy problem of the porous medium equation with an absorption term

$$u_t = (u^m)_{xx} - u^p, \quad x \in \mathbf{R}^N, \quad t > 0,$$

by using the analysis of self-similar solution. By constructing a suitable comparison function, Li and Wu [7] considered the problem of the porous medium equation with a local source term

$$u_t = \triangle u^m + \lambda u^p, \quad x \in \Omega, \quad t > 0,$$

subject to the homogeneous Dirichlet boundary condition (1.2) and initial condition (1.3). They obtained some conditions for extinction and non-extinction of the solutions to the above equation and decay estimates. On extinctions of solutions to the p-Laplacian equation or the doubly degenerate equations, refer to [8,9] and the references therein.

For nonlocal parabolic equation (1.1) without inner absorption term, when $p = 2$, $r = 0$ and $N > 2$, Han and Gao [10] showed that $s = m$ is a critical exponent for occurrence of extinction or non-extinction. Thereafter, Fang and Wang [11] investigated the critical extinction exponents for equation (1.1) with $p = 2$, $r > 0$ and $q = 0$. Recently, Fang and Xu [12] considered equation (1.1) with $m = 1$, $r = 0$ and a linear absorption term in the whole dimensional space, and find the critical exponents. They also obtained the exponential decay estimates which depend on the initial data, coefficients, and domains. Thereafter, they obtained the same results for a class of nonlocal porous medium equations with strong absorption, see [13]. By using a similar argument in [9,12,13], one can show the following results for problem (1.1)-(1.3) with $0 < q \leq 1$:

Theorem. *Suppose that* $0 < m(p-1) < 1$ *and* $q = 1$.

(1) If $r + s = m(p-1)$*, then the solution vanishes in finite time for any nonnegative initial data* $u_0(x)$ *provided that* $|\Omega|$ *(or* λ*) is sufficiently small.*

(2) If $r + s > m(p-1)$*, then the solution of problem (1.1)-(1.3) vanishes in finite time provided that* $u_0(x)$*(or* $|\Omega|$ *or* λ*) is sufficiently small.*

(3) If $r + s < m(p-1)$*, then the solution of problem (1.1)-(1.3) can not vanish in finite time for any nonnegative initial data* $u_0(x)$*.*

In addition, suppose that $0 < m(p-1) = r + s < 1$ *and* $0 < q < 1$*, then the solution vanishes in finite time for any nonnegative initial data* $u_0(x)$ *provided that* $|\Omega|$ *(or* λ*) is sufficiently small.*

Note that, the above-mentioned Theorem holds under suitable L^p-integral norm sense. However, the critical case $r + s = m(p-1)$ with nonlocal source term does not depend on the first eigenvalue of the corresponding p-Laplacian operator, which is different from that of the local source case.

Motivated by the mentioned works above, instead of energy methods, we will use the super- and sub- solution method to obtain the critical extinction exponents for $(r+s,q)$ of problem (1.1)-(1.3), and improve the results of [10,11,12,13]. In fact, there exists a critical curvilinear line $(r+s)^* = \min(q, m(p-1))$ such that the $(r+s,q)$-parameter plane is divided into three parts, with the right part of the line corresponding to extinction for all solutions and the left part corresponding to at least one nonextinction solution. Moreover, there exists a critical point on this line such that the line is also divided into three parts, which exhibit different features of extinction. In our problem, the difficulty lies in finding the competitive relationship of diffusion, nonlocal source and inner absorption on whether determining the extinction of solutions or not and constructing corresponding super- and sub- solutions.

Throughout the paper, we need make some notations as follows.

Let $\psi(x)$ be the positive solution of the following elliptic problem

$$-\operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi) = 1, \text{ in } \Omega, \quad \psi|_{\partial\Omega} = 0, \tag{1.4}$$

and denote $M = \max_{x\in\overline{\Omega}} \psi(x)$. By the strong maximum principle, we know that $M > 0$.

Let λ_1 be the first eigenvalue of the problem

$$-\operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = \lambda|\varphi|^{p-2}\varphi, \text{ in } \Omega, \quad \varphi|_{\partial\Omega} = 0, \tag{1.5}$$

and denote the corresponding eigenfunction by $\varphi_1(x)$ with $\|\varphi_1\|_{L^\infty(\Omega)} = 1$.

Theorem 1.1 *If $r+s > m(p-1)$, then the solution of problem (1.1)-(1.3) vanishes in finite time for appropriately small initial data $u_0(x)$. If $r+s = m(p-1)$ with $\lambda|\Omega|M^{p-1} < 1$, then the solution of problem (1.1)-(1.3) vanishes in finite time for any nonnegative bounded initial data.*

Theorem 1.2 *If $q = r+s < 1$ with $\lambda|\Omega| < \alpha$ or $q < \min\{r+s, 1\}$, then the solution of problem (1.1)-(1.3) vanishes in finite time for appropriately small initial data $u_0(x)$.*

Remark 1.1 *The small condition on initial data $u_0(x)$ in Theorems 1 and 2 can be removed if λ is appropriately small.*

Theorem 1.3 *If $r+s < \min\{q, m(p-1)\}$ or $r+s = q < m(p-1)$ with $\lambda > \alpha$, then the problem (1.1)-(1.3) admits at least one nonextinction solution for any nonnegative initial data $u_0(x)$.*

Theorem 1.4 *If $r+s = m(p-1) < q$ with $\lambda\int_\Omega \varphi_1^{\frac{s}{m}}dx > \lambda_1$, then the problem (1.1)-(1.3) admits at least one nonextinction solution for any nonnegative initial data $u_0(x)$.*

Theorem 1.5 *If $r+s=m(p-1)<1\leq q$ with $\lambda\int_\Omega \varphi_1^{\frac{s}{m}}dx=\lambda_1$, then the problem (1.1)-(1.3) admits at least one nonextinction solution for any positive initial data $u_0(x)$.*

Theorem 1.6 *If $r+s=m(p-1)<q<1$ with $\lambda|\Omega|\leq\lambda_1$, then the solution $u(x,t)$ of problem (1.1)-(1.3) vanishes in the sense that $\lim_{t\to\infty}\|u(\cdot,t)\|_{m+1}=0$. In addition, if $r+s=m(p-1)<q<1$ with $\lambda|\Omega|M^{p-1}=1$, then the problem (1.1)-(1.3) admits at least one extinction solution for any nonnegative and bounded initial data $u_0(x)$.*

Remark 1.2 *If $r=q$ and $s=0$, then the equation (1.1) becomes a local p-Laplacian equation. It is easy to see that, for $\lambda-\alpha<\lambda_1$ with $r=m(p-1)=q$, the solution of problem (1.1)-(1.3) vanishes in finite time, whereas for $\lambda-\alpha\geq\lambda_1$, the problem (1.1)-(1.3) admits at least one nonextinction solution.*

The rest of our paper is organized as follows. In Section 2, we give preliminary knowledges including the suitable definition of solutions of problem (1.1)-(1.3) and lemmas that are required in the proofs of our results and present the proofs for main results in Section 3.

2 Preliminaries

Throughout this paper

$$Q=\Omega\times(0,\infty),\quad Q_T=\Omega\times(0,T),$$

$$E=\{\xi:\xi\in L^{2q}(Q_T)\cap L^{2(r+s)}(Q_T),\xi_t\in L^2(Q_T),\nabla\xi^m\in L^p(Q_T)\},$$
$$\widetilde{E}=\{\xi:\xi\in L^2(Q_T),\xi_t\in L^2(Q_T),\nabla\xi\in L^p(Q_T),\xi\geq 0,\xi|_{\partial\Omega\times(0,T)}=0\}.$$

Due to the singularity of (1.1), problem (1.1)-(1.3) has no classical solutions in general, and hence, it is reasonable to find a weak solution of the problem. To this end, we first give the following definition of weak solution.

Definition 2.1 *A nonnegative measurable function $u\in E$ is called a weak super-solution of problem* (1.1)-(1.3) *in Q_T if the following conditions hold:*
a. $u(x,0)\geq u_0(x)$ in Ω;
b. $u(x,t)\geq 0$ on $\partial\Omega\times(0,T)$;
c. For any $T>0$ and $\xi\in\widetilde{E}$, we have

$$\int\int_{Q_T}\{u_t\xi+|\nabla u^m|^{p-2}\nabla u^m\nabla\xi-\lambda u^r\int_\Omega u^s dx\xi+\alpha u^q\xi\}dxdt\geq 0.$$

Replacing $\geq$ by $\leq$ in the aforementioned inequalities yields the definition of a sub-solution of (1.1)-(1.3). Furthermore, if u is a super-solution as well as a sub-solution, then it is called a solution of (1.1)-(1.3).

Before proving our main results, we give a modified comparison principle for problem (1.1)-(1.3), which can be proved by establishing suitable test function and Gronwall's inequality as in [14,15].

Proposition 2.2 (Comparison principle) *Let (v,w) be a pair of super- and sub-solution of problem (1.1)-(1.3). If either $r+s \geq 1$ and w is upper bounded or $0 < r+s < 1$ and v has a positive lower bound, then $w(x,t) \leq v(x,t)$ in Q_T.*

3 Proofs of the main results

In this section, we give detailed proofs of our main results to problem (1.1)-(1.3) by using the comparison principle and constructing suitable super- and sub- solutions.

Proof of Theorem 1.1: The proof is divided into two steps (i) $r+s = m(p-1)$ with $\lambda|\Omega|M^{p-1} < 1$ and (ii) $r+s > m(p-1)$.

(i) For any bounded smooth domain Ω' such that $\Omega' \supset\supset \Omega$, let $\phi(x)$ be the positive solution of the following elliptic problem

$$-\mathrm{div}(|\nabla\phi|^{p-2}\nabla\phi) = 1, \quad \text{in } \ \Omega', \quad \phi|_{\partial\Omega'} = 0.$$

By the comparison principle, we know $\psi(x) \leq \phi(x)$ in Ω. Set $M_1 = \max_{x\in\overline{\Omega}} \phi(x)$ and $\delta = \min_{x\in\overline{\Omega}} \phi(x)$. It is well known that $\delta > 0$ from the strong maximum principle.

By continuity, we can choose a suitable domain Ω' with $\Omega' \supset\supset \Omega$ such that $\lambda|\Omega|M_1^{p-1} < 1$. Let $v(x,t) = f(t)\phi^{\frac{1}{m}}(x)$, where $f(t)$ satisfies

$$f'(t)M_1^{\frac{1}{m}} + (1-\lambda|\Omega|M_1^{p-1})f^{r+s}(t) = 0, \quad t > 0,$$

$$f(0) = f_0 \geq \delta^{-\frac{1}{m}}\|u_0\|_{L_\infty(\Omega)}.$$

Since $r+s = m(p-1) < 1$, it follows from the ODE theory that $f(t)$ is nonincreasing and $f(t) = 0$ for all

$$t \geq T^* = \frac{M_1^{\frac{1}{m}}}{(1-\lambda|\Omega|M_1^{p-1})(1-r-s)} f_0^{1-r-s}.$$

Then it can be seen that $v(x,t)$ is the super-solution of problem (1.1)-(1.3). In fact, since $r+s = m(p-1)$ and $f'(t) \leq 0$, then we obtain for any $\xi \in \widetilde{E}$,

$$\int\int_{Q_T} \frac{\partial v}{\partial t}\xi + |\nabla v^m|^{p-2}\nabla v^m \nabla\xi - \lambda v^r \int_\Omega v^s dx\xi + \alpha v^q \xi dxdt$$

$$= \int\int_{Q_T} f'(t)\phi^{\frac{1}{m}}\xi + f^{m(p-1)}(t)|\nabla\phi|^{p-2}\nabla\phi\nabla\xi - \lambda f^{r+s}(t)\phi^{\frac{r}{m}}\int_\Omega \phi^{\frac{s}{m}}dx\xi + \alpha f^q(t)\phi^{\frac{q}{m}}\xi dxdt$$

$$= \int\int_{Q_T} \Big[f'(t)\phi^{\frac{1}{m}} + f^{m(p-1)}(t) - \lambda f^{r+s}(t)\phi^{\frac{r}{m}}\int_\Omega \phi^{\frac{s}{m}}dx + \alpha f^q(t)\phi^{\frac{q}{m}}\Big]\xi dxdt$$

$$\geq \int\int_{Q_T} \Big[f'(t)\phi^{\frac{1}{m}} + f^{m(p-1)}(t) - \lambda f^{r+s}(t)\phi^{\frac{r}{m}}\int_\Omega \phi^{\frac{s}{m}}dx\Big]\xi dxdt$$

$$\geq \int\int_{Q_T} \Big[f'(t)M_1^{\frac{1}{m}} + (1-\lambda|\Omega|M_1^{p-1})f^{r+s}(t)\Big]\xi dxdt$$

$$= 0.$$

Therefore, for any $T < T^*$, applying Proposition 2.2 to (1.1)-(1.3) in Q_T, we have $u(x,t) \leq v(x,t)$ for $(x,t) \in Q_T$, which implies $u(x,T) \leq v(x,T)$. Hence, $u(x,T^*) = 0$ by the arbitrariness of $T < T^*$ and $v(x,T^*) = 0$. Furthermore, let $\widetilde{u}(x,t) = u(x,t+T^*)$, then $\widetilde{u}(x,t)$ satisfies (1.1),(1.2) and the initial condition $\widetilde{u}(x,0) = 0$. Now, by the aforementioned proof, we can that $\widetilde{u}(x,t) \leq v(x,t)$ with any $f_0 > 0$. From the relation of the extinction time T^* of $v(x,t)$ to f_0, it follows that $\widetilde{u}(x,t) = 0$ for any $t > 0$, namely $u(x,t) = 0$ for all $t > T^*$.

(ii) Let ϕ and M_1 be the same as case (i) and set $v(x,t) = k\phi^{\frac{1}{m}}(x)$ with

$$k = \Big[\frac{1}{2\lambda|\Omega|M_1^{\frac{r+s}{m}}}\Big]^{\frac{1}{r+s-m(p-1)}}.$$

Then it is easy to verify $v(x,t)$ is a super-solution of problem (1.1)-(1.3) provided that $u_0(x) \leq k\phi^{\frac{1}{m}}$ in Ω. Applying Proposition 2.2 to (1.1)-(1.3) in Q_T, we have $u(x,t) \leq v(x,t)$ for $(x,t) \in Q_T$, which implies that $u(x,t) \leq kM_1^{\frac{1}{m}}$. Therefore, $u(x,t)$ satisfies

$$u_t - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) + \alpha u^q \leq \lambda(kM_1^{\frac{1}{m}})^{r+s-m(p-1)}u^{m(p-1)-s}\int_\Omega u^s dy, \quad x \in \Omega,\ t > 0.$$

By the choice of k, we can easily prove that $\lambda(kM_1^{\frac{1}{m}})^{r+s-m(p-1)}|\Omega|M_1^{p-1} = \frac{1}{2} < 1$. Thus, by the result of case (i), we can conclude that the solution $u(x,t)$ of (1.1)-(1.3) vanishes in finite time if the initial data $u_0(x)$ is appropriately small.

Proof of Theorem 1.2: (i) For $q = r+s < 1$ with $\lambda|\Omega| < \alpha$, we set $v(x,t) = f(t)$, where $f(t)$ satisfies the following problem

$$f'(t) + (\alpha - \lambda|\Omega|)f^{r+s} = 0, \quad t > 0,$$

$$f(0) = \|u_0\|_{L^\infty(\Omega)}.$$

Since $q = r+s < 1$ and $\lambda|\Omega| < \alpha$, it is easily verify that $f(t)$ vanishes at some finite time T^*. Moreover, as in the proof of Theorem 1.1, we can prove

$v(x,t)=f(t)$ is a super-solution of (1.1)-(1.3). Hence, by applying Proposition 2.2 to $u(x,t)$ and $f(t)$ for any $0<T<T^*$, then $u(x,t)$ also vanishes at T^*.

(ii) For $q<\min\{r+s,1\}$, we set $v(x,t)=f(t)$, where $f(t)$ satisfies the following problem

$$f'(t)+(\alpha-\lambda|\Omega|f^{r+s-q}(t))f^q=0,\quad t>0,$$

$$f(0)=f_0,$$

where $0<f_0<(\frac{\alpha}{\lambda|\Omega|})^{\frac{1}{r+s-q}}$. Since $0<q<1$, similar to the case (i), it is well known that $f(t)$ vanishes in finite time and $f(t)$ is a super-solution of (1.1)-(1.3) provided that $u_0(x)$ is small enough such that $\|u_0\|_{L^\infty(\Omega)}\le f_0$. Applying Proposition 2.2 to $u(x,t)$ and $f(t)$ guarantees the finite time extinction of $u(x,t)$.

Proof of Theorem 1.3: (i) For $r+s<\min\{q,m(p-1)\}$, we we shall prove that problem (1.1)-(1.3) admits at least one nonextinction solution for any nonnegative initial data by constructing a suitable pair of super-and sub-solution of (1.1)-(1.3).

We firstly consider case $r+s<q\le m(p-1)$. Let $w(x,t)=\mu g(t)\varphi_1^{\frac{1}{m}}(x)$, where $\varphi_1(x)$ is the first eigenfunction corresponding to the eigenvalue λ_1 of problem (1.5) with $\|\varphi_1\|_{L^\infty(\Omega)}=1$, $\mu>0$ is to be determined later, and $g(t)$ satisfies the ODE problem

$$\begin{aligned}&g'(t)=-\lambda_1 g^{m(p-1)}(t)+\lambda g^{r+s}(t)-\alpha g^q(t),\quad t>0,\\ &g(t)>0,\quad t>0,\\ &g(0)=0.\end{aligned}\tag{3.4}$$

It is easy to check that $g(t)$ is a nondecreasing and bounded function. In fact,

$$g(t)\le\min\{(\frac{\lambda}{\alpha})^{\frac{1}{q-r-s}},(\frac{\lambda}{\lambda_1})^{\frac{1}{m(p-1)-r-s}}\}.$$

Simple calculations show that

$$w_t=\mu(-\lambda_1 g^{m(p-1)}(t)+\lambda g^{r+s}(t)-\alpha g^q(t))\varphi_1^{\frac{1}{m}},$$

and

$$\begin{aligned}&\operatorname{div}(|\nabla w^m|^{p-2}\nabla w^m)-\alpha w^q+\lambda w^r\int_\Omega w^s dx\\ &=-\lambda_1\mu^{m(p-1)}g^{m(p-1)}(t)\varphi_1^{\frac{m(p-1)}{m}}(x)-\alpha\mu^q g^q(t)\varphi_1^{\frac{q}{m}}(x)+\lambda\mu^{r+s}g^{r+s}\varphi_1^{\frac{r}{m}}\int_\Omega\varphi_1^{\frac{s}{m}}dx.\end{aligned}$$

We can choose $\mu>0$ small enough such that

$$\lambda_1 g^{m(p-1)}(\mu^{m(p-1)}\varphi_1^{p-1}-\mu\varphi_1^{\frac{1}{m}})+\alpha g^q(t)(\mu^q\varphi_1^{\frac{q}{m}}-\mu\varphi_1^{\frac{1}{m}})$$

$$\leq \lambda g^{r+s}(t)(\mu^{r+s}\varphi_1^{\frac{r}{m}}\int_\Omega \varphi_1^{\frac{s}{m}}dx - \mu\varphi_1^{\frac{1}{m}}(x)), \tag{3.5}$$

which implies that $w(x,t)$ is a sub-solution of (1.1)-(1.3). Indeed, let

$$F_1(x) = \frac{x^{m(p-1)} - x}{x^{r+s} - x},$$

and

$$F_2 = \frac{x^q - x}{x^{r+s} - x},$$

then because of $r+s < m(p-1)$ and $r+s < q$, $\lim_{x\to 0^+} F_i(x) = 0 (i = 1,2)$, which guarantee that (3.5) holds for sufficiently small $\mu > 0$.

Next, we turn our attention to construct a super-solution of (1.1)-(1.3). Set

$$v(x,t) = \max\Big\{\|u_0\|_{L^\infty(\Omega)}, (\frac{\lambda|\Omega|}{\alpha})^{\frac{1}{q-r-s}}, \mu \max_{t\geq 0} g(t)\Big\},$$

then it can be verified that $v(x,t)$ is the super-solution of (1.1)-(1.3) and $w(x,t) \leq v(x,t)$. Therefore, by an iteration process, we can obtain s solution $u(x,t)$ of (1.1)-(1.3), which satisfies $w(x,t) \leq u(x,t) \leq v(x,t)$. Indeed, define $u_1(x,t) = v(x,t)$ and $\{u_k(x,t)\}_{k=2}^\infty$ iteratively to be a solution of the problem

$$u_{kt} - \mathrm{div}(|\nabla u_k^m|^{p-2}\nabla u_k^m) + \alpha u_k^q = \lambda u_{k-1}^r \int_\Omega u_{k-1}^s dx,$$

subject to the boundary value condition (1.2) and initial condition (1.3). Then, by the comparison principle and regularity of p-Laplacian equation, the function $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ for every $x \in \overline{\Omega}$ and $t > 0$, is the solution of (1.1)-(1.3). Because $w(x,t)$ does not vanish, neither does $u(x,t)$.

Similar to that mentioned above, we can prove that the solution $u(x,t)$ of (1.1)-(1.3) does not vanish in finite time for the case $r+s < m(p-1) < q$.

(ii) For $r+s = q < m(p-1)$ with $\lambda > \alpha$, we let $w(x,t) = \mu g(t)\varphi_1^{\frac{1}{m}}(x)$,where $\varphi_1(x)$ is the same as case (i), $\mu > 0$ is to be determined later, and $g(t)$ satisfies the ODE problem

$$g'(t) = -\lambda_1 g^{m(p-1)}(t) + (\lambda - \alpha)g^q(t), \quad t > 0,$$
$$g(t) > 0, \quad t > 0,$$
$$g(0) = 0.$$

Then $g(t)$ is nondecreasing and satisfies $g(t) \leq (\frac{\lambda-\alpha}{\lambda_1})^{\frac{1}{m(p-1)-q}}$. Similar to (i) we can see that $w(x,t)$ is a sub-solution of (1.1)-(1.3) provided that $\mu > 0$ is sufficiently small.

To construct a super-solution of (1.1)-(1.3), we consider the following eigenvalue problem

$$-\mathrm{div}(|\nabla\varphi|^{p-2}\nabla\phi) = \lambda|\varphi|^{p-2}\varphi, \text{ in } \widetilde{\Omega}, \quad \varphi|_{\partial\widetilde{\Omega}} = 0,$$

where $\widetilde{\Omega} \supset\supset \Omega$ is a bounded domain with smooth boundary $\partial\widetilde{\Omega}$. Let $\widetilde{\lambda_1}$ and $\widetilde{\varphi_1}(x) > 0$ $(x \in \widetilde{\Omega})$ be its first eigenvalue and the corresponding eigenfunction, respectively. We may normalize $\widetilde{\varphi_1}(x)$ such that $\|\widetilde{\varphi_1}\|_{L^\infty(\widetilde{\Omega})} = 1$. Denote $\widetilde{\delta} = \min_{x\in\overline{\Omega}} \widetilde{\varphi_1}(x) > 0$ and set $v(x,t) = k\widetilde{\varphi_1}^{\frac{1}{m}}(x)$ with

$$k = \max\Big\{\big(\frac{\lambda|\Omega|}{\widetilde{\lambda_1}\widetilde{\delta}^{p-1}}\big)^{\frac{1}{m(p-1)-q}}, \widetilde{\delta}^{-\frac{1}{m}}\|u_0\|_{L^\infty(\Omega)}, \mu\big(\frac{\lambda-\alpha}{\lambda_1}\big)^{\frac{1}{m(p-1)-q}}\Big\},$$

then it is easy to verify that $v(x,t)$ is a super-solution of (1.1)-(1.3) and $w(x,t) \le v(x,t)$. Therefore, by applying the monotonicity iteration process we can obtain a nonextinction solution $u(x,t)$ satisfying $w(x,t) \le u(x,t) \le v(x,t)$.

Proof of Theorem 1.4: The proof is similar to that of Theorem 1.3, so we sketch it briefly here. Set $w(x,t) = \mu g(t)\varphi_1^{\frac{1}{m}}(x)$, where $\varphi_1(x)$ is defined in (1.5) and $g(t)$ satisfies the following ODE problem

$$g'(t) = (\lambda \int_\Omega \varphi_1^{\frac{s}{m}} dy - \lambda_1) g^{m(p-1)}(t) - \alpha g^q(t), \quad t > 0,$$

$$g(t) > 0, \quad t > 0,$$

$$g(0) = 0.$$

Since $\lambda \int_\Omega \varphi_1^{\frac{s}{m}} dy > \lambda_1$ and $m(p-1) < q$, it is well known that $g(t)$ is nondecreasing and bounded by $(\frac{\lambda\int_\Omega \varphi_1^{\frac{s}{m}} dy - \lambda_1}{\alpha})^{\frac{1}{q-m(p-1)}}$. Then $w(x,t)$ is a sub-solution of (1.1)-(1.3) provided that $\mu > 0$ is sufficiently small. On the other hand,

$$v(x,t) = \max\Big\{\|u_0\|_{L^\infty(\Omega)}, \big(\frac{\lambda|\Omega|}{\alpha}\big)^{\frac{1}{q-r-s}}, \mu \max_{t\ge 0} g(t)\Big\},$$

can be chosen to be the super-solution of (1.1)-(1.3) and satisfy $w(x,t) \le v(x,t)$. Therefore, by monotonicity iteration, we know that (1.1)-(1.3) admits at least one solution $u(x,t)$ such that $w(x,t) \le u(x,t) \le v(x,t)$. Since $w(x,t) > 0$ in $\Omega \times (0,+\infty)$, $u(x,t)$ can not vanish at any finite time.

Proof of Theorem 1.5: For $r + s = m(p-1) < 1 \le q$ with $\lambda \int_\Omega \varphi_1^{\frac{s}{m}} dy = \lambda_1$, set $w(x,t) = h_0 e^{-\beta t}\varphi_1^{\frac{1}{m}}(x)$, where h_0 and β are two positive constants to be determined later. It is easy to verify that when $q = 1$, $w(x,t)$ is a sub-solution of (1.1)-(1.3) if $\beta \ge \alpha$ and h_0 is so small that $h_0\varphi_1^{\frac{1}{m}}(x) \le u_0(x)$. When $q > 1$, $w(x,t)$ is a sub-solution of (1.1)-(1.3) if $\beta \ge \alpha h_0^{q-1}$ and h_0 is so small that $h_0\varphi_1^{\frac{1}{m}}(x) \le u_0(x)$. In addition, since $w(x,t)$ is bounded, we can choose a sufficiently large constant $C > 0$ such that $C > w(x,t)$ to be a super-solution of (1.1)-(1.3). Therefore, by monotonicity iteration, we can obtain a solution $u(x,t)$ satisfying $w(x,t) \le u(x,t) \le L$. Since $w(x,t)$ does not vanish at any finite time, neither does $u(x,t)$.

Proof of Theorem 1.6: (i) For $r+s=m(p-1)<q<1$ with $\lambda|\Omega|\le\lambda_1$, we let $u(x,t)$ be the solution of (1.1)-(1.3) with the bounded initial datum u_0. Then, by De Giorgi method, it is easily to show that

$$\|u(\cdot,t)\|_{L^\infty}\le l=\max\{\|u_0\|_{L^\infty(\Omega)},(\frac{\lambda}{\alpha})^{\frac{1}{q-r-s}}\}.$$

Multiplying Equation (1.1) by u^m and integrating over Ω produces the identity

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}dx+\int_\Omega|\nabla u^m|^p dx+\alpha\int_\Omega u^{q+m}dx=\lambda\int_\Omega u^{r+m}dx\int_\Omega u^s dx. \tag{3.6}$$

Recall the hölder inequality and Sobolev embedding theorem, we have

$$\int_\Omega u^{r+m}dx\le|\Omega|^{\frac{s}{r+m+s}}(\int_\Omega u^{r+m+s}dx)^{\frac{r+m}{r+m+s}},$$

$$\int_\Omega u^s dx\le|\Omega|^{\frac{r+m}{r+m+s}}(\int_\Omega u^{r+m+s}dx)^{\frac{s}{r+m+s}},$$

and

$$\int_\Omega|\nabla u^m|^p dx\ge\lambda_1\int_\Omega u^{mp}dx.$$

Combining together with (3.6) and noting that $r+s=m(p-1)$, we have

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}dx+\alpha\int_\Omega u^{q+m}dx\le(\lambda|\Omega|-\lambda_1)\int_\Omega u^{r+m+s}dx.$$

Setting $\lambda|\Omega|\le\lambda_1$, we can obtain

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}dx+\alpha\int_\Omega u^{q+m}dx\le 0$$

Because of $q<1$, we obtain

$$\int_\Omega u^{m+q}dx\ge l^{q-1}\int_\Omega u^{m+1}dx. \tag{3.7}$$

Substituting above inequality into (3.7) yields

$$\frac{1}{m+1}\frac{d}{dt}\int_\Omega u^{m+1}dx+\alpha l^{q-1}\int_\Omega u^{m+1}dx\le,$$

and thus

$$\int_\Omega u^{m+1}dx\le e^{-\frac{\alpha(m+1)t}{l^{1-q}}}\int_\Omega u_0^{m+1}dx,$$

which implies that $\int_\Omega u^{m+1}dx\to 0$ as $t\to\infty$.

(ii) For $r+s=m(p-1)<q<1$ with $\lambda|\Omega|M^{p-1}=1$, we let $v(x,t)=f(t)\psi^{\frac{1}{m}}(x)$, where $\psi(x)$ is defined in (1.4) and $f(t)$ satisfies

$$f'(t)+\alpha M^{-\frac{1-q}{m}}f^q(t)=0,\ \ t>0$$

$$f(0) = f_0 > 0$$

Since $0 < q < 1$, then $f(t)$ is nonincreasing and $f(t) = 0$ for $t \geq T^* = \frac{f_0^{1-q}}{\alpha M^{-\frac{1-q}{m}}(1-q)}$. Noting that $r + s = m(p-1)$ and $\lambda|\Omega|M^{p-1} = 1$, one can see that $v(x,t)$ is the super-solution of (1.1)-(1.3) provided that $u_0(x) \leq f_0\psi^{\frac{1}{m}}(x)$ in Ω. By using the arguments similar to the proof of case (i) of Theorem 1.1, we can show that any solution $u(x,t)$ of (1.1)-(1.3) vanishes in finite time.

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