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Extinction behavior of solutions for the polytropic filtration equation with nonlocal source and absorption

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Abstract

We investigate the extinction behavior of non-negative nontrivial weak solutions of the initial-boundary value problem for the fast diffusive polytropic filtration equation with nonlocal nonlinear source and interior absorption. We show that the effect of the absorption can change extinction behavior of solutions in the whole dimensional space, and decay estimates always depend on the choices of initial data, coefficients and domain.

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1 Introduction

In this paper, we consider the initial-boundary value problem of the fast diffusive polytropic filtration equation

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \lambda \int_\Omega u^q(x,t)dx - \beta u^k, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

with $1 < p < 2$, $0 < m(p-1) < 1$, $0 < k \leq 1$, λ, β, $q > 0$, $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary and $u_0^m(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a non-negative function. Symbols $\|\cdot\|_p$, $\|\cdot\|_{1,p}$ denote $L^p(\Omega)$, $W^{1,p}(\Omega)$ norms respectively (where $p \geq 1$) and $|\Omega|$ denotes the measure of Ω.

Nonlinear parabolic equation like (1.1) appears in various applications such as population dynamics, chemical reactions, combustion theory and so on (see [1-3]). In particular, equation (1.1) is a possible model for the diffusion system of some biological species with human-controlled distribution where $u(x,t)$ represents the density of the species at position x and time t, $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ portrays the mutation (which we view as a spreading of the characteristic), $-k$ measures here is the growth capacity of the species at location x and time t, while $\lambda \int_\Omega u^q dx$ denotes the human-controlled distribution. Nonlocal term is a way to express that the evolution of the species in a point of space depends not only on nearby density but also on the total amount of species due to the effects of spatial inhomogeneity (see [4-6]). And it has also been put forward that equation (1.1) may be used to describe the non-stationary flow in a porous medium of fluid with a power dependence of the tangential stress on the velocity of displacement under polytropic conditions. In this case, equation (1.1) is called the non-Newtonian polytropic filtration equation (see [7,8] and references therein).

In the last decades, many researchers devoted to the study of blow-up of solutions for nonlinear parabolic equations with nonlocal terms. For example, Q.L. Liu et al.[9] investigated the homogeneous Dirichlet boundary value problem for the semilinear parabolic equation with nonlocal source and weighted coefficient and proved that the solution blew up globally, and the uniform blow up rate was precisely determined. When $p = 2$, $m = \lambda = 1$ in (1.1) and the linear absorption term is replaced by a nonlinear power form term, the studies of the blow-up, blow-up rates and blow-up sets of solutions have been extensively studied (see [10-13]). However, extinction is also an important property of solutions for these equations and makes some progress. For instance, Evans and Knerr [14] investigated the extinction behavior of solution for the Cauchy problem of the semilinear parabolic equation

$$u_t(x,t) = \Delta u(x,t) - \beta(u(x,t)), \quad x \in R^n,\ t > 0, \tag{1.4}$$

by constructing a suitable comparison function. Y.G. Gu [15] studied the homogeneous Dirichlet boundary value problem for the semilinear heat conduction equation with absorption term

$$u_t = \Delta u - \lambda u^q, \quad x \in \Omega, \ t > 0, \tag{1.5}$$

with $\lambda > 0$ and proved that a solution of (1.5) vanished if and only if $0 < q < 1$ by using the L^p -integral norm estimate method. J.L. Vazquez [16] studied the extinction phenomenon of solutions for the Cauchy problem of the porous medium equations with absorption terms

$$u_t = (u^m)_{xx} - u^p, \quad x \in R, \ t > 0, \tag{1.6}$$

by using the analysis of self-similar solutions and demonstrated that the analysis of (1.6) could be extended to the p-Laplacian equation with absorption. W.J. Liu [17] considered the extinction properties of solutions for the homogeneous Dirichlet boundary value problem for the fast p-Laplacian equation with both local source and absorption term

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u^r - \beta u^q, \quad x \in \Omega, \ t > 0, \tag{1.7}$$

subject to (1.2) (1.3) and $r, \ \lambda, \ \beta > 0, \ q \leq 1$ by using the L^p -integral norm estimate method. For $\beta > 0$, he showed that $r = p - 1$ was still the critical extinction exponent when $q = 1$ and extinction could always occur when $0 < q \leq r < 1$. Moreover, there are some papers concerning the extinction for the following parabolic equation for special cases

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \lambda u^q - \beta u^k, \quad x \in \Omega, \ t > 0, \tag{1.8}$$

subject to (1.2) (1.3) and $q > 0, \ k = 1, \ 0 < m(p-1) < 1$. In case $\lambda = \beta = 0$, H.J. Yuan et al.[18] obtained sufficient conditions for the extinction of solution. For the case $\beta = 0$, J. Zhou and C.L. Mu [19] obtained sufficient conditions about the extinction of solutions by the upper and lower solutions methods. As a natural continuation, J.X. Yin et al.[20] investigated the case $m(p-1) \geq 1$ and showed the non-extinction property of nontrivial solutions. Lately, Z.B. Fang and G. Li [21] proved that the sufficient condition for the extinction of solutions for (1.8) to occur was $0 < m(p-1) \leq q < 1$. S.N. Antontsev et al.[22] studied the finite time extinction, space and time localization of solutions of elliptic and parabolic equations (but without nonlocal term) of a general view by using energy methods which are applied in many research fields, especially for those situations in which traditional methods based on maximum principles or comparison principles have failed (including equations with variable coefficients). Furthermore, for the extinction of the porous medium equations or the p-Laplacian equations, we refer to [23-25] and the references therein for details.

Recently, for (1.1), when $p = 2$, $m = k = 1$ and $q > 0$, the conditions about the extinction and non-extinction of solutions and the corresponding decay estimates under the assumption $N > 2$ have been obtained (see [26]). Then, for the case $1 < p < 2$, $m = k = 1$ and $q > 0$ in (1.1), we showed that $p = q + 1$ was the critical extinction exponent in the whole dimensional space and obtained precise decay estimates which depended on the choices of initial data, coefficients and domain (see [27]). As far as we know, no work has dealt with the extinction phenomenon for the fast diffusive polytropic filtration equation with coefficients and nonlocal source and absorption term like (1.1).

Motivated by the above works, the main goal of our work is to investigate whether the effect of the absorption can change extinction behavior of solutions for problem (1.1)-(1.3) in the whole dimensional space. When the linear absorption is contained in (1.1), we find that the critical exponent of extinction for the weak solution is determined by the competition of two nonlinear terms, and the critical case does not depend on the first eigenvalue of the corresponding operator, which is different from that of the local source case. Moreover, extinction can always occur when $0 < k \le q < 1$, and the decay estimates depend on the choices of initial data, coefficients and domain. The detailed results as follows.

Theorem 1.1 *Assume that* $1 < p < 2$, $k = 1$, $0 < m(p-1) = q < 1$
(1) If $N = 1$ *or* 2, *the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data provided that* $|\Omega|$ *(or* λ*) is sufficiently small, and*

$$\|u(\cdot,t)\|_2 \le [(\|u_0\|_2^{1-m(p-1)} + \frac{C_1}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_1}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [0, T_1),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_1, +\infty),$$

where C_1, T_1 *are given by (3.4)(3.5) respectively.*
(2) If $N > 2$, *the non-negative nontrival weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data provided that* $|\Omega|$ *(or* λ*) is sufficiently small, and*
(a) If $\frac{N-2}{N+2} \le m(p-1) < 1$,

$$\|u(\cdot,t)\|_{d+1} \le [(\|u_0\|_{d+1}^{1-m(p-1)} + \frac{C_2}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_2}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [0, T_2),$$

$$\|u(\cdot,t)\|_{d+1} \equiv 0, \quad t \in [T_2, +\infty).$$

(b) If $0 < m(p-1) < \frac{N-2}{N+2}$,

$$\|u(\cdot,t)\|_{r+1} \le [(\|u_0\|_{r+1}^{1-m(p-1)} + \frac{C_3}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_3}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [0, T_3),$$

$$\|u(\cdot,t)\|_{r+1} \equiv 0, \ \ t \in [T_3, +\infty),$$

where $d = \frac{2m(p-1)+2}{p} - 1$, $r = \frac{N-p-Nm(p-1)}{p}$, C_2, C_3, T_2, T_3 *are given by (3.10)(3.14)(3.11)(3.15) respectively.*

Theorem 1.2 *Assume that* $1 < p < 2$, $k = 1$, $m(p-1) < q$
(1) If $N = 1$ *or* 2*, the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time provided that* u_0 *(or* $|\Omega|$ *or* λ*) is sufficiently small, and*

$$\|u(\cdot,t)\|_2 \le \|u_0\|_2 e^{-\alpha_1 t}, \ \ t \in [0, T_4),$$

$$\|u(\cdot,t)\|_2 \le [(\|u(\cdot,T_4)\|_2^{1-m(p-1)} + \frac{C_4}{\beta})e^{[m(p-1)-1]\beta(t-T_4)} - \frac{C_4}{\beta}]^{\frac{1}{1-m(p-1)}}, \ \ t \in [T_4, T_5),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \ \ t \in [T_5, +\infty),$$

where C_4, T_5 *are given by (3.19)(3.20) respectively.*
(2) If $N > 2$*, the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time provided that* u_0 *(or* $|\Omega|$ *or* λ*) is sufficiently small, and*
(a) If $\frac{N-2}{N+2} \le m(p-1) < 1$,

$$\|u(\cdot,t)\|_{d+1} \le \|u_0\|_{d+1} e^{-\alpha_2 t}, \ \ t \in [0, T_6),$$

$$\|u(\cdot,t)\|_{d+1} \le [(\|u(\cdot,T_6)\|_{d+1}^{1-m(p-1)} + \frac{C_5}{\beta})e^{[m(p-1)-1]\beta(t-T_6)} - \frac{C_5}{\beta}]^{\frac{1}{1-m(p-1)}}, \ \ t \in [T_6, T_7),$$

$$\|u(\cdot,t)\|_{d+1} \equiv 0, \ \ t \in [T_7, +\infty).$$

(b) If $0 < m(p-1) < \frac{N-2}{N+2}$,

$$\|u(\cdot,t)\|_{r+1} \le \|u_0\|_{r+1} e^{-\alpha_3 t}, \ \ t \in [0, T_8),$$

$$\|u(\cdot,t)\|_{r+1} \le [(\|u(\cdot,T_8)\|_{r+1}^{1-m(p-1)} + \frac{C_6}{\beta})e^{[m(p-1)-1]\beta(t-T_8)} - \frac{C_6}{\beta}]^{\frac{1}{1-m(p-1)}}, \ \ t \in [T_8, T_9),$$

$$\|u(\cdot,t)\|_{r+1} \equiv 0, \ \ t \in [T_9, +\infty),$$

where $d = \frac{2m(p-1)+2}{p} - 1$, $r = \frac{N-p-Nm(p-1)}{p}$, C_5, C_6, T_7, T_9 *are given by (3.23)(3.27)(3.24)(3.28) respectively.*

Theorem 1.3 *Assume* $1 < p < 2$, $k = 1$, $m(p-1) > q$*, then the non-negative weak solution of problem (1.1)-(1.3) can not vanish in finite time for any non-negative initial data.*

Remark 1.1 *According to Theorems 1.1-1.3, we observe that* $m(p-1) = q$ *is the critical exponent of extinction for the solution of (1.1)-(1.3) when* $k = 1$.

Remark 1.2 *We also use L^p- integral norm estimate method to prove our main results. However, in the critical case, for (1.1), we only need to deal with $\lambda \int_\Omega u^q(x,t)dx$ by Hölder inequality to predigest the original problem which does not depend on the first eigenvalue of the corresponding operator any longer.*

Remark 1.3 *If the coefficients of the nonlinear source term and linear absorption term change signs, the behavior of solution for problem(1.1)-(1.3) will also change. For instance, when $\lambda < 0, \beta > 0$, the non-negative weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data; when $\lambda < 0, \beta < 0$, the non-negative weak solution of problem (1.1)-(1.3) vanishes in finite time provided that u_0 is sufficiently small or β is sufficiently large, especially that when $N > 2$, $\frac{N-2}{N+2} \le m(p-1) < 1$, the non-negative weak solution of problem (1.1)-(1.3) also vanishes in finite time provided that $|\Omega|$ is sufficiently small; when $\lambda > 0, \beta < 0$, the non-negative weak solution of problem (1.1)-(1.3) blows up in infinite time for any non-negative initial data provided that β is sufficiently small.*

Theorem 1.4 *Assume that $1 < p < 2$, $0 < k < 1$, $0 < m(p-1) = q < 1$, then the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data provided that $|\Omega|$ (or λ)is sufficiently small.*

Theorem 1.5 *Assume that $1 < p < 2$, $0 < k < 1$, then the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time provided that u_0 (or $|\Omega|$ or λ) is sufficiently small and $q > \frac{pk(s+1)+N[m(p-1)-k]}{p(s+1)+N[m(p-1)-k]}$. (If $N = 1$ or 2, then $s = 1$; if $N > 2$, then $s > \max\{\frac{2m(p-1)+2}{p} - 1, \frac{N-p-Nm(p-1)}{p}\}$.)*

Remark 1.4 *If $k \ge m(p-1)$, the conditions in Theorem 1.5 imply that $q > m(p-1)$ (see the proof of Theorem 1.5 for details).*

Theorem 1.6 *Assume that $1 < p < 2$, $0 < k < 1$, $m(p-1) > q \ge k$, then the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time for any non-negative initial data provided that β is sufficiently large.*

Remark 1.5 *One can see from Theorems 1.4-1.6 that extinction can always occur when $0 < k \le q < 1$.*

Remark 1.6 *Theorems 1.1-1.6 all require that $|\Omega|$ or λ or u_0 should be sufficiently small or β should be sufficiently large, and we will give more concrete conditions which they satisfy in the later proofs.*

The outline of the paper is as follows. In Section 2, we firstly give the definition of weak solutions for problem (1.1)-(1.3), and then show some preliminary lemmas. In Section 3, we mainly prove Theorems 1.1-1.3 which deal with the case $k = 1$. Finally, the proofs of Theorems 1.4-1.6 in the case $0 < k < 1$ are the subject of Section 4.

2 Preliminary knowledge

Due to the singularity of the equation that we consider with, the problem of (1.1)-(1.3) has no classical solutions in general. So we consider its weak solutions in the following sense.

Definition 2.1 *Assume that $u(x,t)$ satisfies the following conditions*

(1)$u \in L^{2q}(Q_T)\cap L^2(Q_T), u_t \in L^2(Q_T), \nabla u^m \in L^p(Q_T)$,

(2) $\int\int_{Q_T}(u_t\varphi+|\nabla u^m|^{p-2}\nabla u^m\nabla\varphi+\beta u^k\varphi)dxdt = \lambda\int\int_{Q_T}\varphi(\int_\Omega u^q(y,t)dy)dxdt$,

where $\varphi \geq 0$, $\varphi \in L^2(Q_T)$, $\varphi_t \in L^2(Q_T)$, $\nabla\varphi \in L^p(Q_T)$, $\varphi|\partial\Omega = 0$ *and* $Q_T = \Omega\times(0,T)$, $T > 0$,

(3)$u(x,0) = u_0(x), u|_{\partial\Omega\times(0,T)} = 0$,

then $u(x,t)$ is called the weak solution of problem (1.1)-(1.3).

We can also define the weak lower solution and upper solution of problem (1.1)-(1.3) in the same way except that the "=" in Definition 1 is replaced by "$\leq$" and "$\geq$" respectively. The existence and regularity of non-negative solution of problem (1.1)-(1.3) can be studied as in [2,8,28].

Before proving our main results, we show some preliminary lemmas which are very important in the following proofs. For convenience, we only give these lemmas (the detail proofs can been seen in [26,29-31]).

Lemma 2.2 *Let $y(t)$ be a non-negative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt} + \alpha y^k \leq 0, \ \ t \geq 0; \ \ y(0) \geq 0,$$

where $\alpha > 0$ is a constant and $k \in (0,1)$, then we have decay estimate

$$y(t) \leq [(y^{1-k}(0) - \alpha(1-k)t]^{\frac{1}{1-k}}, \ \ t \in [0, T_*),$$

$$y(t) \equiv 0, \ \ t \in [T_*, +\infty),$$

where $T_* = \frac{y^{1-k}(0)}{\alpha(1-k)}$.

Lemma 2.3 ([29]) *Let $y(t)$ be a non-negative absolutely continuous function on $[0,+\infty)$ satisfying*

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq 0, \ \ t \geq T_0; \ \ y(T_0) \geq 0,$$

where $\alpha, \beta > 0$ are constants and $k \in (0,1)$, then we have decay estimate

$$y(t) \leq [(y^{1-k}(T_0) + \frac{\alpha}{\beta})e^{(k-1)\beta(t-T_0)} - \frac{\alpha}{\beta}]^{\frac{1}{1-k}}, \ \ t \in [T_0, T_*),$$

$$y(t) \equiv 0, \ \ t \in [T_*, +\infty),$$

where $T_* = \frac{1}{(1-k)\beta} \ln(1 + \frac{\beta}{\alpha} y^{1-k}(T_0)) + T_0$.

Lemma 2.4 ([30]) *Let $0 < k < m \leq 1, y(t) \geq 0$ be a solution of the differential inequality*

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq \gamma y^m, \ \ t \geq 0; \ \ y(0) = y_0 > 0,$$

where $\alpha, \beta > 0, \gamma$ is a positive constant such that $\gamma < \alpha y_0^{k-m}$, then there exists $\eta > \beta$, such that

$$0 \leq y(t) \leq y_0 e^{-\eta t}, \ \ t \geq 0.$$

Lemma 2.5 ([26]) *Let $\alpha, \beta, \gamma > 0$ and $0 < m < k < 1$, then exists at least one non-constant solution of the ODE problem*

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq \gamma y^m, \ \ t \geq 0; \ \ y(0) = y_0 > 0, \ \ y(t) > 0, t > 0.$$

Lemma 2.6 ([31]) (Gagliardo-Nirenberg inequality) *Suppose that $u \in W_0^{k,m}(\Omega)$, $1 \leq m \leq +\infty$, $0 \leq j < k$, $1 \geq \frac{1}{r} \geq \frac{1}{m} - \frac{k}{N}$, then we have*

$$\|D^j u\|_q \leq C \|D^k u\|_m^\theta \|u\|_r^{1-\theta},$$

where C is a constant depending only on N, m, r, j, k, q and $\frac{1}{q} = \frac{j}{N} + \theta(\frac{1}{m} - \frac{k}{N}) + \frac{1-\theta}{r}$. While if $m < \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, \frac{Nm}{N-(k-j)m}]$, if $m \geq \frac{N}{k-j}$, then $q \in [\frac{Nr}{N+rj}, +\infty]$.

3 The case $1 < p < 2$, $0 < m(p-1) < 1$, $k = 1$: proofs of Theorems 1.1-1.3

3.1 proof of Theorem 1.1

(1)If $N = 1$ or 2, multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_2^2 = \lambda \int_\Omega u^{m(p-1)}dx \int_\Omega u dx. \tag{3.1}$$

By the Hölder inequality, we have

$$\int_\Omega u^{m(p-1)}dx \int_\Omega udx \leq |\Omega|^{\frac{2s_1-m(p-1)-1}{s_1}} \|u\|_{s_1}^{m(p-1)+1},$$

where $s_1 \geq 1$ will be determined later. Setting $s_1 = 2$, one can get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_2^2 \leq \lambda|\Omega|^{\frac{3-m(p-1)}{2}}\|u\|_2^{m(p-1)+1}. \tag{3.2}$$

By the Sobolev embedding inequality, there exists an embedding constant $\gamma(N, \Omega) > 0$ such that

$$\|u^{\frac{m(p-1)+1}{p}}\|_{s_2} \leq \gamma(N, \Omega)\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p,$$

where $s_2 \geq p$ will be determined later.
i.e.

$$\gamma^{-p}(N, \Omega)\|u\|_{\frac{[m(p-1)+1]s_2}{p}}^{m(p-1)+1} \leq \|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p.$$

Here we set $s_2 = \frac{2p}{m(p-1)+1}$, then the above inequality turns to

$$\gamma^{-p}(N, \Omega)\|u\|_2^{m(p-1)+1} \leq \|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p. \tag{3.3}$$

So we have

$$\frac{d}{dt}\|u\|_2 + C_1\|u\|_2^{m(p-1)} + \beta\|u\|_2 \leq 0,$$

where

$$C_1 = \frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p} - \lambda|\Omega|^{\frac{3-m(p-1)}{2}}. \tag{3.4}$$

By Lemma 2.3, we have

$$\|u(\cdot, t)\|_2 \leq [(u_0\|_2^{1-m(p-1)} + \frac{C_1}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_1}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [0, T_1),$$

$$\|u(\cdot, t)\|_2 \equiv 0, \quad t \in [T_1, +\infty),$$

provided that

$$|\Omega| < \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p\lambda}\}^{\frac{2}{3-m(p-1)}},$$

where

$$T_1 = \frac{1}{[1-m(p-1)]\beta}\ln(1 + \frac{\beta}{C_1}\|u_0\|_2^{1-m(p-1)}). \tag{3.5}$$

(2) If $N > 2$,
(a) If $\frac{N-2}{N+2} \le m(p-1) < 1$, multiplying (1.1) by u^d (here $d = \frac{2m(p-1)+2}{p} - 1 \ge 1$) and integrating over Ω, we have

$$\frac{1}{d+1}\frac{d}{dt}\|u\|_{d+1}^{d+1} + \frac{dm^{p-1}p^p}{[m(p-1)+d]^p}\|\nabla u^{\frac{m(p-1)+d}{p}}\|_p^p + \beta\|u\|_{d+1}^{d+1} = \lambda\int_\Omega u^{m(p-1)}dx\int_\Omega u^d dx. \tag{3.6}$$

By the Hölder inequality, we have

$$\int_\Omega u^{m(p-1)}dx\int_\Omega u^d dx \le |\Omega|^{\frac{2s_3-m(p-1)-d}{s_3}}\|u\|_{s_3}^{m(p-1)+d},$$

where $s_3 \ge 1$ will be determined later. Setting $s_3 = d+1$, one can get

$$\frac{1}{d+1}\frac{d}{dt}\|u\|_{d+1}^{d+1} + \frac{dm^{p-1}p^p}{[m(p-1)+d]^p}\|\nabla u^{\frac{m(p-1)+d}{p}}\|_p^p + \beta\|u\|_{d+1}^{d+1}$$
$$\le \lambda|\Omega|^{\frac{d-m(p-1)+2}{d+1}}\|u\|_{d+1}^{m(p-1)+d}. \tag{3.7}$$

By the Sobolev embedding inequality, there exists an embedding constant $C_0 > 0$ such that

$$\|u^{\frac{m(p-1)+d}{p}}\|_{\frac{Np}{N-p}}^p \le C_0^p\|\nabla u^{\frac{m(p-1)+d}{p}}\|_p^p. \tag{3.8}$$

By the Hölder inequality, we have

$$\|u\|_{d+1}^{m(p-1)+d} \le |\Omega|^{\frac{m(p-1)+d}{d+1}-\frac{N-p}{N}}\|u^{\frac{m(p-1)+d}{p}}\|_{\frac{Np}{N-p}}^p. \tag{3.9}$$

So we have

$$\frac{d}{dt}\|u\|_{d+1} + C_2\|u\|_{d+1}^{m(p-1)} + \beta\|u\|_{d+1} \le 0,$$

where

$$C_2 = \frac{dm^{p-1}p^p}{[m(p-1)+d]^p}C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{m(p-1)+d}{d+1}} - \lambda|\Omega|^{\frac{d-m(p-1)+2}{d+1}}. \tag{3.10}$$

By Lemma 2.3, we have

$$\|u(\cdot,t)\|_{d+1} \le [(\|u_0\|_{d+1}^{1-m(p-1)} + \frac{C_2}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_2}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [0, T_2),$$

$$\|u(\cdot,t)\|_{d+1} \equiv 0, \quad t \in [T_2, +\infty),$$

provided that

$$|\Omega| < \{\frac{dm^{p-1}p^p}{[m(p-1)+d]^pC_0^p\lambda}\}^{\frac{N}{N+p}},$$

where

$$T_2 = \frac{1}{[1-m(p-1)]\beta}\ln(1+\frac{\beta}{C_2}\|u_0\|_{d+1}^{1-m(p-1)}). \tag{3.11}$$

(b) If $0 < m(p-1) < \frac{N-2}{N+2}$, multiplying (1.1) by u^r (here $r = \frac{N-p-Nm(p-1)}{p}$) and integrating over Ω, we have

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1}+\frac{dm^{p-1}p^p}{[m(p-1)+r]^p}\|\nabla u^{\frac{m(p-1)+r}{p}}\|_p^p+\beta\|u\|_{r+1}^{r+1} = \lambda\int_\Omega u^{m(p-1)}dx\int_\Omega u^r dx. \tag{3.12}$$

By the embedding theorem and the specific choice of r, there exists an embedding constant $C_{00} > 0$ such that

$$\|u\|_{\frac{N[m(p-1)+r]}{N-p}}^{m(p-1)+r} \le C_{00}^p\|\nabla u^{\frac{m(p-1)+r}{p}}\|_p^p.$$

i.e.

$$C_{00}^{-p}\|u\|_{r+1}^{m(p-1)+r} \le \|u^{\frac{m(p-1)+r}{p}}\|_p^p. \tag{3.13}$$

By the Hölder inequality, we have

$$\int_\Omega u^{m(p-1)}dx\int_\Omega u^r dx \le |\Omega|^{\frac{2s_4-m(p-1)-r}{s_4}}\|u\|_{s_4}^{m(p-1)+r},$$

where $s_4 \ge 1$ will be determined later. Here we set $s_4 = r+1 \ge 1$, and obtain

$$\frac{d}{dt}\|u\|_{r+1} + C_3\|u\|_{r+1}^{m(p-1)} + \beta\|u\|_{r+1} \le 0,$$

where

$$C_2 = \frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p} - \lambda|\Omega|^{\frac{r-m(p-1)+2}{r+1}}. \tag{3.14}$$

By Lemma 2.3, we have

$$\|u(\cdot,t)\|_{r+1} \le [(\|u_0\|_{r+1}^{1-m(p-1)} + \frac{C_3}{\beta})e^{[m(p-1)-1]\beta t} - \frac{C_3}{\beta}]^{\frac{1}{1-m(p-1)}}, \;\; t\in[0,T_3),$$

$$\|u(\cdot,t)\|_{r+1} \equiv 0, \;\; t\in[T_3,+\infty),$$

provided that

$$|\Omega| < \{\frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p\lambda}\}^{\frac{r+1}{r-m(p-1)+2}},$$

where

$$T_3 = \frac{1}{[1-m(p-1)]\beta}\ln(1+\frac{\beta}{C_3}\|u_0\|_{r+1}^{1-m(p-1)}). \tag{3.15}$$

3.2 proof of Theorem 1.2

Firstly, we consider the case $q \le 1$.

(1)If $N = 1$ or 2, multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_2^2 = \lambda\int_\Omega u^q dx\int_\Omega u dx. \quad (3.16)$$

By the Hölder inequality, we have

$$\int_\Omega u^q dx\int_\Omega u dx \le |\Omega|^{\frac{2s_5-q-1}{s_5}}\|u\|_{s_5}^{q+1},$$

where $s_5 \ge 1$ will be determined later. Here we set $s_5 = 2$, and obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_2^2 \le \lambda|\Omega|^{\frac{3-q}{2}}\|u\|_2^{q+1}. \quad (3.17)$$

We substitute (3.3) into (3.17), and set $s_2 = \frac{2p}{m(p-1)+1}$, so we have

$$\frac{d}{dt}\|u\|_2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p}\|u\|_2^{m(p-1)} + \beta\|u\|_2 \le \lambda|\Omega|^{\frac{3-q}{2}}\|u\|_2^q. \quad (3.18)$$

By Lemma 2.4, there exists $\alpha_1 > \beta$, such that

$$0 \le \|u(\cdot,t)\|_2 \le \|u_0\|_2 e^{-\alpha_1 t},\ t \ge 0,$$

provided that

$$\|u_0\|_2 < \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p\lambda|\Omega|^{\frac{3-q}{2}}}\}^{\frac{1}{q-m(p-1)}}.$$

Furthermore, there exists $T_4 > 0$, such that

$$\frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p} - \lambda|\Omega|^{\frac{3-q}{2}}\|u\|_2^{q-m(p-1)}$$

$$\ge \frac{m^{p-1}p^p}{[m(p-1)+1]^p\gamma^p} - \lambda|\Omega|^{\frac{3-q}{2}}(\|u_0\|_2 e^{-\alpha_1 T_4})^{q-m(p-1)} = C_4 > 0, \quad (3.19)$$

holds for $t \in [T_4, +\infty)$. Therefore, when $t \in [T_4, +\infty)$, we have

$$\frac{d}{dt}\|u\|_2 + C_4\|u\|_2^{m(p-1)} + \beta\|u\|_2 \le 0.$$

By Lemma 2.3, we have

$$\|u(\cdot,t)\|_2 \le [(\|u(\cdot,T_4)\|_2^{1-m(p-1)} + \frac{C_4}{\beta})e^{[m(p-1)-1]\beta(t-T_4)} - \frac{C_4}{\beta}]^{\frac{1}{1-m(p-1)}},\ \ t \in [T_4, T_5),$$

$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_5, +\infty),$$

where

$$T_5 = \frac{1}{[1-m(p-1)]\beta} \ln(1 + \frac{\beta}{C_4}\|u(\cdot,T_4)\|_2^{1-m(p-1)}) + T_4. \tag{3.20}$$

(2) If $N > 2$,

(a) If $\frac{N-2}{N+2} \le m(p-1) < 1$, multiplying (1.1) by u^d (here $d = \frac{2m(p-1)+2}{p} - 1$) and integrating over Ω, and then using the *Hölder* inequality and the embedding theorem, we have

$$\frac{d}{dt}\|u\|_{d+1} + \frac{dm^{p-1}p^p}{[m(p-1)+d]^p} C_0^{-p} |\Omega|^{\frac{N-p}{N} - \frac{m(p-1)+d}{d+1}} \|u\|_{d+1}^{m(p-1)} + \beta\|u\|_{d+1}$$

$$\le \lambda|\Omega|^{\frac{d-q+2}{d+1}} \|u\|_{d+1}^q. \tag{3.21}$$

By Lemma 2.4, there exists $\alpha_2 > \beta$, such that

$$0 \le \|u(\cdot,t)\|_{d+1} \le \|u_0\|_{d+1} e^{-\alpha_2 t}, \; t \ge 0,$$

provided that

$$\|u_0\|_{d+1} < \{\frac{dm^{p-1}p^p}{[m(p-1)+d]^p C_0^p \lambda |\Omega|^{\frac{m(p-1)-q}{d+1} + \frac{N+p}{N}}}\}^{\frac{1}{q-m(p-1)}}. \tag{3.22}$$

Furthermore, there exists $T_6 > 0$, such that

$$\frac{dm^{p-1}p^p}{[m(p-1)+d]^p} C_0^{-p} |\Omega|^{\frac{N-p}{N} - \frac{m(p-1)+d}{d+1}} - \lambda|\Omega|^{\frac{d-q+2}{d+1}} \|u\|_{d+1}^{q-m(p-1)}$$

$$\ge \frac{dm^{p-1}p^p}{[m(p-1)+d]^p} C_0^{-p} |\Omega|^{\frac{N-p}{N} - \frac{m(p-1)+d}{d+1}} - \lambda|\Omega|^{\frac{d-q+2}{d+1}} (\|u_0\|_{d+1} e^{-\alpha_2 T_6})^{q-m(p-1)} = C_5 > 0, \tag{3.23}$$

holds for $t \in [T_6, +\infty)$. Therefore, when $t \in [T_6, +\infty)$, we have

$$\frac{d}{dt}\|u\|_{d+1} + C_5\|u\|_{d+1}^{m(p-1)} + \beta\|u\|_{d+1} \le 0.$$

By Lemma 2.3, we have

$$\|u(\cdot,t)\|_{d+1} \le [(\|u(\cdot,T_6)\|_{d+1}^{1-m(p-1)} + \frac{C_5}{\beta}) e^{[m(p-1)-1]\beta(t-T_6)} - \frac{C_5}{\beta}]^{\frac{1}{1-m(p-1)}}, \quad t \in [T_6, T_7),$$

$$\|u(\cdot,t)\|_{d+1} \equiv 0, \quad t \in [T_7, +\infty),$$

where

$$T_7 = \frac{1}{[1-m(p-1)]\beta} \ln(1 + \frac{\beta}{C_5}\|u(\cdot,T_6)\|_{d+1}^{1-m(p-1)}) + T_6. \tag{3.24}$$

(b) If $0 < m(p-1) < \frac{N-2}{N+2}$, multiplying (1.1) by u^r (here $r = \frac{N-p-Nm(p-1)}{p}$) and integrating over Ω, then using the *Hölder* inequality and the embedding theorem, we have

$$\frac{d}{dt}\|u\|_{r+1}+\frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p}\|u\|_{r+1}^{m(p-1)}+\beta\|u\|_{r+1} \leq \lambda|\Omega|^{\frac{r-q+2}{r+1}}\|u\|_{r+1}^q. \quad (3.25)$$

By Lemma 2.4, there exists $\alpha_3 > \beta$, such that

$$0 \leq \|u(\cdot,t)\|_{r+1} \leq \|u_0\|_{r+1}e^{-\alpha_3 t},\ t \geq 0,$$

provided that

$$\|u_0\|_{r+1} < \{\frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p\lambda|\Omega|^{\frac{r-q+2}{r+1}}}\}^{\frac{1}{q-m(p-1)}}. \quad (3.26)$$

Furthermore, there exists $T_8 > 0$, such that

$$\frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p} - \lambda|\Omega|^{\frac{r-q+2}{r+1}}\|u\|_{r+1}^{q-m(p-1)}$$

$$\geq \frac{rm^{p-1}p^p}{[m(p-1)+r]^pC_{00}^p} - \lambda|\Omega|^{\frac{r-q+2}{r+1}}(\|u_0\|_{r+1}e^{-\alpha_3 T_8})^{q-m(p-1)} = C_6 > 0, \quad (3.27)$$

holds for $t \in [T_8, +\infty)$. Therefore, when $t \in [T_8, +\infty)$, we have

$$\frac{d}{dt}\|u\|_{r+1} + C_6\|u\|_{r+1}^{m(p-1)} + \beta\|u\|_{r+1} \leq 0.$$

By Lemma 2.3, we have

$$\|u(\cdot,t)\|_{r+1} \leq [(\|u(\cdot,T_8)\|_{r+1}^{1-m(p-1)}+\frac{C_6}{\beta})e^{[m(p-1)-1]\beta(t-T_8)}-\frac{C_6}{\beta}]^{\frac{1}{1-m(p-1)}},\quad t \in [T_8, T_9),$$

$$\|u(\cdot,t)\|_{r+1} \equiv 0,\quad t \in [T_9, +\infty),$$

where

$$T_9 = \frac{1}{[1-m(p-1)]\beta}\ln(1+\frac{\beta}{C_6}\|u(\cdot,T_8)\|_r^{1-m(p-1)}) + T_8. \quad (3.28)$$

For the case $q > 1$.

Assume that λ_1 is the first eigenvalue of

$$-div(|\nabla\phi|^{p-2}\nabla\phi) = \lambda|\phi|^{p-2}\phi,\quad x \in \Omega;\quad \phi(x) = 0,\quad x \in \partial\Omega, \quad (3.29)$$

and $\phi(x) \geq 0$, $\|\phi(x)\|_\infty = 1$ is the eigenfunction corresponding to the eigenvalue λ_1.

For sufficiently small $a > 0$, it can be easily verified that $a\phi^{\frac{1}{m}}(x)$ is a upper solution of (1.1)-(1.3) if $u_0(x) \le a\phi^{\frac{1}{m}}(x), x \in \Omega$. Then $u(x,t) \le a\phi^{\frac{1}{m}}(x), x \in \Omega, t > 0$ by the comparison principle. Therefore we can rewrite (3.18)(3.21)(3.25) as (e.g.(3.21))

$$\frac{d}{dt}\|u\|_{d+1} + \frac{dm^{p-1}p^p}{[m(p-1)+d]^p}C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{m(p-1)+d}{d+1}}\|u\|_{d+1}^{m(p-1)} + \beta\|u\|_{d+1}$$
$$\le \lambda|\Omega|a^{q-1}\|u\|_{d+1}. \tag{3.30}$$

The above argument can also be applied and hence we omit it.

Remark 3.1 *If $q > 1$, the non-negative nontrivial weak solution of problem (1.1)-(1.3) vanishes in finite time, and it still has exponential decay estimates. But u_0 and decay estimates should be changed accordingly. Here we only give the concrete decay estimates under the condition of $q \le 1$ in Theorem 1.2.*

Remark 3.2 *For the other properties of the first eigenvalue and the corresponding function for problem (3.29), we refer the reader to [24] and the references therein.*

3.3 proof of Theorem 1.3

Let $v(x,t) = g(t)\phi^{\frac{1}{m}}(x)$, where $\phi(x)$ is still the first eigenfunction corresponding to the eigenvalue λ_1 for problem (3.29), while $g(t)$ satisfies the ODE problem

$$g'(t) + \lambda_1 g^{m(p-1)}(t) + \beta g(t) = \lambda\int_\Omega \phi^{\frac{q}{m}}(x)dx g^q(t), \quad t \ge 0; \quad g(0) = 0.$$

Then we have

$$\int\int_{Q_T}\{v_t\varphi - |\nabla v^m|^{p-2}\nabla v^m\nabla\varphi + \beta v\varphi - \lambda\varphi\int_\Omega v^q(x,t)dx\}dxdt$$
$$= \int\int_{Q_T}\{g'(t)\phi^{\frac{1}{m}}(x) + \lambda_1\phi^{p-1}(x)g^{m(p-1)}(t) + \beta g(t)\phi^{\frac{1}{m}}(x) - \lambda g^q(t)\int_\Omega \phi^{\frac{q}{m}}(x)dx\}\varphi dxdt$$
$$\le \int\int_{Q_T}\{g'(t) + \lambda_1 g^{m(p-1)}(t) + \beta g(t) - \lambda g^q(t)\int_\Omega \phi^{\frac{q}{m}}(x)dx\}\varphi dxdt = 0.$$

Moreover, $v(x,0) = g(0)\phi^{\frac{1}{m}}(x) = 0 \le u_0(x), \ x \in \Omega; \quad v(x,t) = 0, x \in \partial\Omega, \ t > 0$.
Therefore, we have

$$u(x,t) \ge v(x,t) > 0, \quad x \in \Omega, \ t > 0.$$

i.e. $v(x,t)$ is a non-extinction lower solution of problem (1.1)-(1.3).

4 The case $1 < p < 2$, $0 < m(p-1) < 1$, $0 < k < 1$: proofs of Theorems 1.4-1.6

4.1 proof of Theorem 1.4

(1)If $N = 1$ or 2, applying the same computation as for(3.1)-(3.2), one can get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \lambda|\Omega|^{\frac{3-m(p-1)}{2}}\gamma^p\}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_{k+1}^{k+1} \leq 0. \tag{4.1}$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_2 \leq C(N,p,k)\|u\|_{k+1}^{1-\theta_1}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^{\frac{\theta_1 p}{m(p-1)+1}}, \tag{4.2}$$

where $\theta_1 = \frac{m(p-1)+1}{p}(\frac{1}{k+1} - \frac{1}{2})[\frac{1}{N} - \frac{1}{p} + \frac{m(p-1)+1}{p}\frac{1}{k+1}]^{-1} = \frac{N(1-k)[m(p-1)+1]}{2\{p(k+1)+N[m(p-1)-k]\}}$. Since $1 < p < 2$, $0 < m(p-1) < 1$ and $0 < k < 1$, we can easily get $0 < \theta_1 < 1$. It follows from (4.2) and the Young's inequality that

$$\begin{aligned}\|u\|_2^{k_1} &\leq C(N,p,k)^{k_1}\|u\|_{k+1}^{k_1(1-\theta_1)}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^{\frac{k_1\theta_1 p}{[m(p-1)+1]}}\\ &\leq C(N,p,k)^{k_1}(\eta_1\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + C(\eta_1)\|u\|_{k+1}^{\frac{k_1(1-\theta_1)[m(p-1)+1]}{m(p-1)+1-k_1\theta_1}}),\end{aligned} \tag{4.3}$$

where $k_1 > 1$ and $\eta_1 > 0$ will be determined later. Here we choose $k_1 = \frac{(k+1)[m(p-1)+1]}{(1-\theta_1)[m(p-1)+1]+\theta_1(k+1)} = \frac{2(k+1)\{p(k+1)+N[m(p-1)-k]\}}{2\{p(k+1)+N[m(p-1)-k]\}+N(1-k)[k-m(p-1)]}$, then $1 < k_1 < 2$ and $\frac{k_1(1-\theta_1)[m(p-1)+1]}{m(p-1)+1-k_1\theta_1} = k+1$. Thus, (4.3) becomes

$$\frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1} \leq \frac{\eta_1\beta}{C(\eta_1)}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_{k+1}^{k+1}. \tag{4.4}$$

We substitute (4.4) into (4.1) to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \lambda|\Omega|^{\frac{3-m(p-1)}{2}}\gamma^p - \frac{\eta_1\beta}{C(\eta_1)}\}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p$$

$$+\frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1} \leq 0.$$

Here we can choose η_1 and λ or $|\Omega|$ small enough such that $\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \lambda|\Omega|^{\frac{3-m(p-1)}{2}}\gamma^p - \frac{\eta_1\beta}{C(\eta_1)} \geq 0$. Setting $C_{01} = \frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)}$, we have

$$\frac{d}{dt}\|u\|_2 + C_{01}\|u\|_2^{k_1-1} \leq 0.$$

By Lemma 2.2, we have

$$\|u\|_2 \le [\|u_0\|_2^{2-k_1} - C_{01}(2-k_1)t]^{\frac{1}{2-k_1}}, \ \ t \in [0, T_{01}),$$

$$\|u\|_2 \equiv 0, \ \ t \in [T_{01}, +\infty),$$

where $T_{01} = \frac{\|u_0\|_2^{2-k_1}}{C_{01}(2-k_1)}$.

(2) If $N > 2$,

(a) If $\frac{N-2}{N+2} \le m(p-1) < 1$, multiplying (1.1) by u^s ($s > d \ge 1$) and integrating over Ω, one can get

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p} - \lambda C_0^p|\Omega|^{1+\frac{p}{N}}\}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p + \beta\|u\|_{k+s}^{k+s} \le 0. \tag{4.5}$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{s+1} \le C(N,p,k,s)\|u\|_{k+s}^{1-\theta_2}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^{\frac{\theta_2 p}{m(p-1)+s}}, \tag{4.6}$$

where $\theta_2 = \frac{m(p-1)+s}{p}(\frac{1}{k+s} - \frac{1}{s+1})[\frac{1}{N} - \frac{1}{p} + \frac{m(p-1)+s}{p}\frac{1}{k+s}]^{-1} = \frac{N(1-k)[m(p-1)+s]}{(s+1)\{p(k+s)+N[m(p-1)-k]\}}$. Since $1 < p < 2$, $\frac{N-2}{N+2} \le m(p-1) < 1$ and $0 < k < 1$, we can easily get $0 < \theta_2 < 1$. It follows from (4.6) and the Young's inequality that

$$\|u\|_{s+1}^{k_2} \le C(N,p,k,s)^{k_2}\|u\|_{k+s}^{k_2(1-\theta_2)}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^{\frac{k_2\theta_2 p}{m(p-1)+s}}$$

$$\le C(N,p,k,s)^{k_2}(\eta_2\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p + C(\eta_2)\|u\|_{k+s}^{\frac{k_2(1-\theta_2)[m(p-1)+s]}{m(p-1)+s-k_2\theta_2}}, \tag{4.7}$$

where $k_2 > 0$ and $\eta_2 > 0$ will be determined later. Here we choose $k_2 = \frac{(k+s)[m(p-1)+s]}{(1-\theta_2)[m(p-1)+s]+\theta_2(k+s)} = \frac{p(s+1)(k+s)+N(s+1)[m(p-1)-k]}{p(s+1)+N[m(p-1)-k]}$, then $s < k_2 < s+1$ and $\frac{k_2(1-\theta_2)[m(p-1)+s]}{m(p-1)+s-k_2\theta_2} = k+s$. Thus, (4.7) becomes

$$\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2} \le \frac{\eta_2\beta}{C(\eta_2)}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p + \beta\|u\|_{k+s}^{k+s}. \tag{4.8}$$

We substitute (4.8) into (4.5) to get

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p} - \lambda C_0^p|\Omega|^{1+\frac{p}{N}} - \frac{\eta_2\beta}{C(\eta_2)}\}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p$$

$$+\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2} \le 0.$$

Here we can choose η_2 and λ or $|\Omega|$ small enough such that $\frac{sm^{p-1}p^p}{[m(p-1)+s]^p} - \lambda C_0^p|\Omega|^{1+\frac{p}{N}} - \frac{\eta_2\beta}{C(\eta_2)} \ge 0$. Setting $C_{02} = \frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2}$, we have

$$\frac{d}{dt}\|u\|_{s+1} + C_{02}\|u\|_{s+1}^{k_2-s} \le 0.$$

By Lemma 2.2, we have

$$\|u\|_{s+1} \leq [\|u_0\|_{s+1}^{s+1-k_2} - C_{02}(s+1-k_2)t]^{\frac{1}{s+1-k_2}}, t \in [0, T_{02}),$$

$$\|u\|_{s+1} \equiv 0, t \in [T_{02}, +\infty),$$

where $T_{02} = \frac{\|u_0\|_{s+1}^{s+1-k_2}}{C_{02}(s+1-k_2)}$.

(b)If $0 < m(p-1) < \frac{N-2}{N+2}$, the proof of (a) can also be applied and hence we omit it here.

4.2 proof of Theorem 1.5

(1)If $N = 1$ or 2, multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{m^{p-1}p^p}{[m(p-1)+1]^p}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_{k+1}^{k+1} = \lambda\int_\Omega u^q dx \int_\Omega u dx. \quad (4.9)$$

Substituting (4.4) into (4.9) and using the *Hölder* inequality, one can get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \frac{\eta_1\beta}{C(\eta_1)}\}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)}\|u\|_2^{k_1}$$
$$\leq \lambda|\Omega|^{\frac{3-q}{2}}\|u\|_2^{q+1}.$$

By choosing η_1 small enough such that $\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \frac{\eta_1\beta}{C(\eta_1)} \geq 0$, we get

$$\frac{d}{dt}\|u\|_2 + \|u\|_2^{k_1-1}[\frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)} - \lambda|\Omega|^{\frac{3-q}{2}}\|u\|_2^{q-k_1+1}] \leq 0.$$

Therefore,

$$\frac{d}{dt}\|u\|_2 + C_{03}\|u\|_2^{k_1-1} \leq 0,$$

provided that

$$\|u_0\|_2 < [\frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)\lambda|\Omega|^{\frac{3-q}{2}}}]^{\frac{1}{q-k_1}+1},$$

and

$$q > k_1 - 1 = \frac{2kp + N[m(p-1)-k]}{2p + N[m(p-1)-k]},$$

where $C_{03} = \frac{C(N,p,k)^{-k_1}\beta}{C(\eta_1)} - \lambda|\Omega|^{\frac{3-q}{2}}\|u_0\|_2^{q-k_1+1} > 0$.

(2) If $N > 2$,

(a) If $\frac{N-2}{N+2} \leq m(p-1) < 1$, multiplying (1.1) by u^s ($s > d \geq 1$) and integrating over Ω, we have

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1} + \frac{sm^{p-1}p^p}{[m(p-1)+s]^p}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p + \beta\|u\|_{k+s}^{k+s} = \lambda\int_\Omega u^q dx \int_\Omega u^s dx. \quad (4.10)$$

Substituting (4.8) into (4.10) and using the *Hölder* inequality, one can get

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1}+\{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p}-\frac{\eta_2\beta}{C(\eta_2)}\}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p$$

$$+\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}\|u\|_{s+1}^{k_2}\le\lambda|\Omega|^{\frac{s-q+2}{s+1}}\|u\|_{s+1}^{q+s}.$$

By choosing η_2 small enough such that $\frac{sm^{p-1}p^p}{[m(p-1)+s]^p}-\frac{\eta_2\beta}{C(\eta_2)}\ge 0$, we get

$$\frac{d}{dt}\|u\|_{s+1}+\|u\|_{s+1}^{k_2-s}[\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}-\lambda|\Omega|^{\frac{s-q+2}{s+1}}\|u\|_{s+1}^{q-k_2+s}]\le 0.$$

Therefore,

$$\frac{d}{dt}\|u\|_{s+1}+C_{04}\|u\|_{s+1}^{k_2-s}\le 0,$$

provided that

$$\|u_0\|_{s+1}<[\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)\lambda|\Omega|^{\frac{s-q+2}{s+1}}}]^{\frac{1}{q-k_2+s}},$$

and

$$q>k_2-s=\frac{pk(s+1)+N[m(p-1)-k]}{p(s+1)+N[m(p-1)-k]},$$

where $C_{04}=\frac{C(N,p,k,s)^{-k_2}\beta}{C(\eta_2)}-\lambda|\Omega|^{\frac{s-q+2}{s+1}}\|u_0\|_{s+1}^{q-k_2+s}>0$.

Since $s>d$, we have $p(s+1)>2m(p-1)+2$. Therefore, if $k\ge m(p-1)$, then $q>k_2-s\ge m(p-1)$. For the case $q>1$, we can rewrite (4.9) and (4.10) as (3.30), so the above argument can also be applied and we omit it here.

(b)If $0<m(p-1)<\frac{N-2}{N+2}$, the proof will be similar to (a) and hence we omit it.

4.3 proof of Theorem 1.6

(1)If $N=1$ or 2, multiplying (1.1) by u and integrating over Ω, and then using the *Hölder* inequality, we can get

$$\int_\Omega u^q dx\int_\Omega u dx\le|\Omega|\|u\|_{q+1}^{q+1}.$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{q+1}\le C(N,p,k,q)\|u\|_{k+1}^{1-\theta_3}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^{\frac{\theta_3 p}{m(p-1)+1}}, \tag{4.11}$$

where $\theta_3=\frac{m(p-1)+1}{p}(\frac{1}{k+1}-\frac{1}{q+1})[\frac{1}{N}-\frac{1}{p}+\frac{m(p-1)+1}{p}\frac{1}{k+1}]^{-1}=\frac{N(q-k)[m(p-1)+1]}{(q+1)\{[p(k+1)+N[m(p-1)-k]\}}\in[0,1)$. Since $q<m(p-1)$, we have $m(p-1)+1-(q+1)\theta_3>0$. Therefore, it follows from (4.11) and the Young's inequality that

$$\lambda|\Omega|\|u\|_{q+1}^{q+1}\le\lambda|\Omega|C(N,p,k,q)^{q+1}\|u\|_{k+1}^{(q+1)(1-\theta_3)}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^{\frac{(q+1)\theta_3 p}{m(p-1)+1}}$$

$$\leq \lambda|\Omega|C(N,p,k,q)^{q+1}(\eta_3\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + C(\eta_3)\|u\|_{k+1}^{\frac{(q+1)(1-\theta_3)[m(p-1)+1]}{m(p-1)+1-(q+1)\theta_3}}), \quad (4.12)$$

where η_3 will be determined later. Substituting (4.12) into (4.9), one can get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \eta_3\lambda|\Omega|C(N,p,k,q)^{q+1}\}\|\nabla u^{\frac{m(p-1)+1}{p}}\|_p^p + \beta\|u\|_{k+1}^{k+1}$$

$$\leq C(\eta_3)\lambda|\Omega|C(N,p,k,q)^{q+1}\|u\|_{k+1}^{\frac{(q+1)(1-\theta_3)[m(p-1)+1]}{m(p-1)+1-(q+1)\theta_3}}.$$

We then substitute (3.3) into the above inequality to get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \eta_3\lambda|\Omega|C(N,p,k,q)^{q+1}\}\gamma^{-p}\|u\|_2^{m(p-1)+1}$$

$$+\|u\|_{k+1}^{k+1}[\beta - C(\eta_3)\lambda|\Omega|C(N,p,k,q)^{q+1}\|u\|_{k+1}^{\alpha_1}] \leq 0,$$

where $\alpha_1 = \frac{(q+1)(1-\theta_3)[m(p-1)+1]}{m(p-1)+1-(q+1)\theta_3} - (k+1) = \frac{p(q-k)(k+1)}{p(k+1)+N[m(p-1)-q]} \geq 0$. We choose η_3 small enough such that $C_{05} = \{\frac{m^{p-1}p^p}{[m(p-1)+1]^p} - \eta_3\lambda|\Omega|C(N,p,k,q)^{q+1}\}\gamma^{-p} > 0$. Once η_3 is fixed, we can choose β large enough such that

$$\beta - C(\eta_3)\lambda|\Omega|C(N,p,k,q)^{q+1}\|u\|_{k+1}^{\alpha_1} \geq 0.$$

Thus, we have

$$\frac{d}{dt}\|u\|_2 + C_{05}\|u\|_2^{m(p-1)} \leq 0,$$

which implies the result.

(2) If $N > 2$,

(a) If $\frac{N-2}{N+2} \leq m(p-1) < 1$, multiplying (1.1) by u^s and integrating over Ω, and then using the *Hölder* inequality, we can get

$$\int_\Omega u^q dx \int_\Omega u^s dx \leq |\Omega|\|u\|_{q+s}^{q+s}.$$

By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{q+s} \leq C(N,p,k,q,s)\|u\|_{k+s}^{1-\theta_4}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^{\frac{\theta_4 p}{m(p-1)+s}}, \quad (4.13)$$

where $\theta_4 = \frac{m(p-1)+s}{p}(\frac{1}{k+s} - \frac{1}{q+s})[\frac{1}{N} - \frac{1}{p} + \frac{m(p-1)+s}{p}\frac{1}{k+s}]^{-1} = \frac{N(q-k)[m(p-1)+s]}{(q+s)\{[p(k+s)+N[m(p-1)-k]\}} \in [0,1)$. Since $q < m(p-1)$, we have $m(p-1)+s-(q+s)\theta_4 > 0$. Therefore, it follows from (4.12) and the Young's inequality that

$$\lambda|\Omega|\|u\|_{q+s}^{q+s} \leq \lambda|\Omega|C(N,p,k,q,s)^{q+s}\|u\|_{k+s}^{(q+s)(1-\theta_4)}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^{\frac{(q+s)\theta_4 p}{m(p-1)+s}}$$

$$\leq \lambda|\Omega|C(N,p,k,q,s)^{q+s}(\eta_4\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p + C(\eta_4)\|u\|_{k+s}^{\frac{(q+s)(1-\theta_4)[m(p-1)+s]}{m(p-1)+s-(q+s)\theta_4}}), \quad (4.14)$$

where η_4 will be determined later. Substituting (4.14) into (4.10), one can get

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1}+\{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p}-\eta_4\lambda|\Omega|C(N,p,k,q,s)^{q+s}\}\|\nabla u^{\frac{m(p-1)+s}{p}}\|_p^p$$

$$+\beta\|u\|_{k+s}^{k+s}\le C(\eta_4)\lambda|\Omega|C(N,p,k,q,s)^{q+s}\|u\|_{k+s}^{\frac{(q+s)(1-\theta_4)[m(p-1)+s]}{m(p-1)+s-(q+s)\theta_4}}.$$

We then substitute (3.8) (3.9) into the above inequality to get

$$\frac{1}{s+1}\frac{d}{dt}\|u\|_{s+1}^{s+1}+\{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p}-\eta_4\lambda|\Omega|C(N,p,k,q,s)^{q+s}\}C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{m(p-1)+s}{s+1}}$$

$$\cdot\|u\|_{s+1}^{m(p-1)+s}+\|u\|_{k+s}^{k+s}[\beta-C(\eta_4)\lambda|\Omega|C(N,p,k,q,s)^{q+s}\|u\|_{k+s}^{\alpha_2}]\le 0,$$

where $\alpha_2=\frac{(q+s)(1-\theta_4)[m(p-1)+s]}{m(p-1)+s-(q+s)\theta_4}-(k+s)=\frac{p(q-k)(k+s)}{p(k+s)+N[m(p-1)-q]}\ge 0$. We can choose η_4 small enough such that

$$C_{06}=\{\frac{sm^{p-1}p^p}{[m(p-1)+s]^p}-\eta_4\lambda|\Omega|C(N,p,k,q,s)^{q+s}\}C_0^{-p}|\Omega|^{\frac{N-p}{N}-\frac{m(p-1)+s}{s+1}}>0.$$

Once η_4 is fixed, we choose β large enough that

$$\beta-C(\eta_4)\lambda|\Omega|C(N,p,k,q,s)^{q+s}\|u\|_{k+s}^{\alpha_2}\ge 0.$$

Thus, we have

$$\frac{d}{dt}\|u\|_{s+1}+C_{06}\|u\|_{s+1}^{m(p-1)}\le 0,$$

which implies the result.

(b)If $0<m(p-1)<\frac{N-2}{N+2}$, the proof will be similar to the proof of (a), hence we omit it here.

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