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A second expansion of positive large solutions for quasilinear elliptic equations

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Abstract

By means of the Karamata theory, we establish a second expansion of large solutions to the quasilinear elliptic problem $\Delta_p u = b(x)f(u)$ with a singular boundary condition $u|_{\partial\Omega} = \infty$, where the domain $\Omega \subset R^N$ is a bounded region with C^4-smooth boundary. The weight function b, which may vanish on the boundary, is nonnegative and nontrivial, and the function f is nonlinear and normalized regularly varying at infinity with index m.

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1 Introduction and main results

We consider a boundary behavior of the blow-up positive weak solutions to the following quasilinear elliptic problem with a singular boundary condition:

$$\Delta_p u = b(x)f(u), \quad u > 0, \quad x \in \Omega, \quad u\mid_{\partial\Omega} = \infty, \tag{1.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplacian operator with $p > 1$ and the domain $\Omega \subset R^N (N \geq 1)$ is a bounded region with C^4-smooth boundary. We assume that the weight function b in (1.1) satisfies the properties

(b_1) b belongs to $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ and is positive in Ω,

(b_2) There exist a function $k \in \Lambda$ and a constant $B_0 \in R$ such that $b(x) = k^p(d(x))(1 + B_0 d(x) + o(d(x)))$ near $\partial\Omega$, where Λ denotes the set of all positive nondecreasing functions $k \in C^2(0, \delta_0)$, where $d(x)=$ dist$(x, \partial\Omega)$ and δ_0 is a positive constant, such that

$$\lim_{t\longrightarrow 0+} \frac{K(t)}{k(t)} = 0, \quad K(t) = \int_0^t k(s)ds,$$
$$\lim_{t\longrightarrow 0+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := C_k \in (0,1],$$
$$\lim_{t\longrightarrow 0+} \frac{1}{t}\left(\frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) - C_k\right) := G_k \in R,$$

and that the function f in (1.1) satisfies the following properties:

(f_1) $f \in C^1[0,\infty)$, $f(0) = 0$, and f is increasing in $(0,\infty)$,

(f_2) There exist a constant $m > p-1$ and a function $h \in C^1[S_0,\infty)$ such that $\frac{sf'(s)}{f(s)} = m + h(s)$, $s \geq S_0$ for S_0 large enough and $\lim_{s\to\infty} h(s) = 0$, i.e., $f(s) = c_0 s^p \exp\left(\int_{S_0}^s \frac{h(\mu)}{\mu} d\mu\right)$ for all $s \geq S_0$ and $c_0 > 0$,

(f_3) There exists a positive constant q such that $\lim_{s\to\infty} \dfrac{sh'(s)}{h(s)} = -q$.

Some basic examples of $k \in \Lambda$ are

(1) $k(t) = t^{\frac{\alpha}{2}}$ with $\alpha > 0$ ($C_k = 2(2+\alpha)^{-1}$ and $G_k = 0$),

(2) $k(t) = e^{-t^{-\alpha}}$ with $\alpha > 0$ ($C_k = 0$ and $G_k = 0$),

(3) $k(t) = \ln(1 + t^\alpha)$ with $\alpha > 1$ ($C_k = (1+\alpha)^{-1}$ and $G_k = 0$),

(4) $k(t) = (\ln(1+t))^\alpha$ with $\alpha > 0$ ($C_k = (1+\alpha)^{-1}$ and $G_k = \frac{\alpha}{2(1+\alpha)(2+\alpha)}$).

By a solution of (1.1), we mean there exists a function $u \in W^{1,p}_{loc}(\Omega) \bigcap L^{\infty}_{loc}(\Omega)$ such that $\Delta_p u = b(x)f(u)$ in the weak sense and u satisfies the singular boundary condition, i.e., $u(x) \to \infty$ as $d(x)$=dist$(x, \partial\Omega) \to 0$. This solution is also called a large solution, an explosive solution or a boundary blow-up solution.

Problem (1.1) appears in the study of a steady-state for the non-Newtonian fluids through porous media, combustion theory, and the turbulent flow of a gas in porous media. In the non-Newtonian fluid theory, the quantity p characterizes the media. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo plastics. If $p = 2$, they are Newtonian fluids. The p-Laplacian operator also appears in the study of torsional creep; for example, elastic for $p = 2$ and plastic for $p < 2$, see [1], flow through porous media ($p = \frac{3}{2}$, see [2]) or glacial sliding ($p \in (1, \frac{4}{3}]$, see [3]).

Problem (1.1) with $p = 2$, $N = 2$, $b(x) = 1$, and $f(u) = e^u$ was first considered by Bieberbach [4] in early 1916, and the author showed that there exists a unique solution $u \in C^2(\Omega)$ such that $u(x) - \log(d^{-2}(x)) = O(1)$ as $d(x) \to 0$ in two-dimensional space. Problems of this type arise frequently in Riemannian geometry. More precisely, if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has the constant Gaussian curvature $-b^2$, then $\triangle u = b^2 e^{2u}$. Many scholars showed existence and uniqueness of blow-up solutions and gave accurate estimates and more perfect results on the boundary behaviors of the large solutions to the semilinear elliptic, logistic, and elliptic equations, see [5-13] and references therein. Recently, when $p > 1$ and the functions b and f satisfy some proper conditions, many researchers proved existence and uniqueness of large solutions to the equations and gave blow up rates of the solutions by using the perturbed, the lower and upper solutions methods and radial solutions and by constructing comparison functions, see [14-21] and references therein.

Cirstea and Rădulescu [8,9] first adopted the Karamata regular variation theory to study the boundary blow-up problem of elliptic equations and opened up a new efficient method which can deal with the uniqueness of boundary blow-up solutions and boundary behavior in a general framework that removes previous restrictions, and they expanded the existing results. In particular, this setting becomes a powerful tool in describing the asymptotic behavior of solutions for large classes of nonlinear elliptic equations, and singular solutions with blow-up boundary and stationary problems with either degenerate or singular nonlinearity as well. For many researches used the methods to deal with elliptic problems, see [7-13, 21] and references therein. Recently, Zhang and Huang et al. [11,13] established the second expansion of large solutions for problem (1.1) when $p = 2$ by using the Karamata regular variation theory, the perturbed, and the lower and upper solutions methods. Huang et al. [21] investigated the asymptotic behavior of boundary blow-up solutions to the quasilinear elliptic problem (1.1), when the weight function b is nonnegative and nontrivial, which may vanish on the boundary, and the nonlinear function f is a Γ-varying function at infinity, whose variation at infinity is not regular.

Motivated by the mentioned works above, we will analyze an influence of mean curvature in the boundary behavior of solutions to problem (1.1), where the weight b covers a broad class of functions and the absorption f covers a large class of Keller-Osserman type nonlinearities; that is, we will establish a second expansion of the singular solutions near the boundary via the Karamata theory.

Our more detailed results can be summarized as follows:

Theorem 1.1 *Suppose that f satisfies (f_1)-(f_3) and that b satisfies (b_1) and (b_2) with $C_k \in (0,1)$. If q is a constant in hypothesis (f_3) such that $q \in \left(\frac{m+1-p}{p}, m+1\right)$, then the unique solution of problem (1.1) can be written as*

$$u(x) = \xi_0 \phi(K(d(x)))\Big(1 + C_1 d(x) + C_2 H(\overline{x}) d(x) + o(d(x))\Big) \text{ as } d(x) \to 0, \quad (1.2)$$

where for all $x \in \Omega$ in a neighborhood of $\partial\Omega$, $\overline{x} \in \partial\Omega$ is the unique point such that $d(x) = |x - \overline{x}|$ and $H(\overline{x})$ is the mean curvature of $\partial\Omega$ at the point $\overline{x}$, and ϕ is uniquely defined by

$$\int_{\phi(t)}^{\infty} (p' f(\mu))^{-\frac{1}{p}} d\mu = t, \quad \frac{1}{p'} + \frac{1}{p} = 1, \quad \forall t > 0, \quad (1.3)$$

and

$$\xi_0 = \left((p-1)\frac{p + C_k(m+1-p)}{m+1}\right)^{\frac{1}{m+1-p}}, \quad (1.4)$$

$$C_1 = \frac{(m+1-p)G_k - B_0(p + (m+1-p)C_k)}{((m+1-p)C_k + p)(m-1)}, \quad (1.5)$$

$$C_2 = \frac{(N-1)(m+1-p)C_k}{(m+1-p)C_k + m^2 + m - p}. \quad (1.6)$$

For the following theorem, we now give an additional condition on the function f as follows:

(f_4) There exist constants $m > p-1$ and $c_0 > 0$ and a function $h_1 \in C^1[S_0, \infty)$ such that $f(s) = c_0 s^m(1 + h_1(s))$, where $s \geq S_0$ for S_0 large enough, and $\lim_{s\to\infty} h_1(s) = 0$.

Theorem 1.2 *Suppose that b satisfies (b_1) and (b_2) and that f satisfies (f_1) and (f_4). If h_1 is a function in hypothesis (f_4) such that $\lim_{s\to\infty} \frac{s h_1'(s)}{h_1(s)} = -q_1$*

and $2q_1 > (p-1)C_k$*, then the unique solution of problem (1.1) can be written as*

$$u(x) = \xi_1(K(d(x)))^{-\frac{p}{m+1-p}}\Big(1 + C_3 d(x) + C_4 H(\overline{x})d(x) + o(d(x))\Big) \text{ as } d(x) \to 0, \tag{1.2}$$

where for all $x \in \Omega$ *in a neighborhood of* $\partial\Omega$*,* $\overline{x} \in \partial\Omega$ *is the unique point such that* $d(x) = |x - \overline{x}|$ *and* $H(\overline{x})$ *is the mean curvature of* $\partial\Omega$ *at the point* $\overline{x}$*, and*

$$\xi_1 = \left(\frac{(p-1)p^{p-1}(p + C_k(m+1-p))}{c_0(m+1-p)^p}\right)^{\frac{1}{m+1-p}}, \tag{1.7}$$

$$C_3 = \frac{(m+1-p)G_k - B_0(p + (m+1-p)C_k)}{((m+1-p)(m+1)C_k) - p(m-1)}, \tag{1.8}$$

$$C_4 = \frac{(N-1)(m+1-p)C_k}{(m+1-p)(m+1)C_k - p(m-1)}. \tag{1.9}$$

Remark 1.1 *By a direct calculation, one can see that* $h(s) = \frac{s(1+h_1'(s))}{1+h_1(s)}$ *for* $s \in [S_0, \infty)$*. Moreover, it will be seen that* $q = q_1$ *in Section 2.*

Remark 1.2 *From Lemma 2.13 given in Section 2, one can see that the function* f *in (1.1) satisfies the Keller-Osserman condition, and hence, the solution for problem (1.1) exists, see [17].*

Our paper is organized as follows: In Section 2, some notions and results from regular variation theory are given. Theorems 1 and 2 will be proved in Sections 3 and 4, respectively.

2 Preliminary results

2.1 Properties of regularly varying function

The Karamata regular variation theory was established by Karamata in 1930, which is a basic tool in the stochastic process, and in 1970 Haan improved the results, which have been applied to the stochastic process, the analytical function theory, integral functions, integral transforms, and asymptotic estimation of integral sequences, see [22-24].

We present some definitions and basic properties of regularly varying functions.

Definition 2.1 *A positive measurable function* f *defined on* $[a, \infty)$ *is said to be regularly varying at infinity with index* ρ*, written as* $f \in RV_\rho$*, if for each* $\xi > 0$ *and some* $\rho \in R$

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho, \tag{2.1}$$

where a is a positive constant. In particular, when $\rho = 0$, the function f is said to be slowly varying at infinity.

Clearly, if $f \in RV_\rho$, then the function $L(s) := \frac{f(s)}{s^\rho}$ is slowly varying at infinity.

Definition 2.2 *A positive measurable function f defined on $[a, \infty)$ is said to be rapidly varying at infinity if for each $\rho > 1$*

$$\lim_{s\to\infty} \frac{f(s)}{s^\rho} = \infty, \tag{2.2}$$

where a is a positive constant.

Some basic examples of slowly varying functions at infinity are

(1) every positive measurable function on $[a, \infty)$ which has a positive limit at infinity,

(2) $(\ln t)^s$ and $(\ln(\ln t))^s$ for $s \in R$,

(3) $e^{(\ln t)^s}$ for $0 < s < 1$,

and some basic examples of rapidly varying functions at infinity are

(1) e^t and e^{e^t} for $t \in R$,

(2) $e^{e^{(\ln t)^s}}$, $e^{e^{t^s}}$, and $e^{e^{t^s}}$ for $s > 0$,

(3) $t^\gamma e^{(\ln t)^q}$ and $(\ln t)^\gamma e^{(\ln t)^q}$, where q and $\gamma > 1$,

(4) $(\ln t)^\gamma e^{t^q}$ and $t^\gamma e^{t^\gamma}$, where $q > 0$ and $\gamma \in R$.

Definition 2.3 *We say that a positive measurable function g defined on $(0, a)$ is regularly varying at zero with index σ, written as $g \in RVZ_\sigma$, if the mapping $t \mapsto g(\frac{1}{t})$ belongs to $RV_{-\sigma}$, where $a > 0$ is a constant. Similarly, a function g is said to be rapidly varying at zero if the mapping $t \mapsto g(\frac{1}{t})$ is rapidly varying at infinity.*

Proposition 2.4 (Uniform Convergence Theorem)
If $f \in RV_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then the uniform convergence holds on all intervals (a_1, ∞) with $a_1 > 0$, and if $\rho > 0$ and f is bounded on $(0, a_1]$ for all $a_1 > 0$, then the uniform convergence holds on all intervals $(0, a_1]$.

Definition 2.5 *A function f defined on (a, ∞) is said to be normalized regularly varying at infinity with index ρ, written as $f \in NRV_\rho$, if it is a continuously differentiable function such that*

$$\lim_{s \to \infty} \frac{sf'(s)}{f(s)} = \rho.$$

When the index $\rho = 0$, the function f is said to be normalized slowly varying at infinity.

Definition 2.6 *A function g is said to be normalized regularly varying at zero with index σ, written as $g \in NRVZ_\sigma$, if the mapping $t \mapsto g(\frac{1}{t})$ belongs to $NRV_{-\sigma}$.*

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty) \ \text{for some} \ a_1 > 0 \ \text{and} \ \lim_{s \to \infty} \frac{sf'(s)}{f(s)} = \rho.$$

Proposition 2.7 (Representation Theorem)
A function L is slowly varying at infinity if and only if L can be written as

$$L(s) = \varphi(s) \exp\left(\int_{a_1}^{s} \frac{y(t)}{t} dt\right), \quad s \geq a_1, \tag{2.3}$$

for some $a_1 > 0$, where the functions φ and y are measurable, and $y(s) \to 0$ and $\varphi(s) \to c_0$ with $c_0 > 0$ as $s \to \infty$.

The function

$$\widehat{L}(s) := c_0 \exp\left(\int_{a_1}^{s} \frac{y(t)}{t} dt\right), \quad s \geq a_1, \tag{2.4}$$

is normalized slowly varying at infinity and

$$f(s) := c_0 s^\rho \widehat{L}(s), \quad s \geq a_1, \tag{2.5}$$

is a normalized regularly varying function at infinity with index ρ.

Proposition 2.8 *If the functions L_1 and L_2 are slowly varying at infinity, then*

(1) *L^σ with $\sigma \in R$, $c_1 L + c_2 L_1$, and $L \circ L_1$ are also slowly varying at infinity, where c_1 and c_2 are nonnegative constants such that $c_1 + c_2 > 0$ and $L_1(t) \to \infty$ as $t \to \infty$,*

(2) *for all $\theta > 0$ we have $t^\theta L(t) \to \infty$ and $t^{-\theta} L(t) \to 0$ as $t \to \infty$,*

(3) *for all* $\rho \in R$ *we have* $\frac{\ln(L(t))}{\ln t} \to 0$ *and* $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$ *as* $t \to \infty$.

Proposition 2.9 *If* $f_1 \in RV_{\rho_1}$ *and* $f_2 \in RV_{\rho_2}$ *with* $\lim_{t\to\infty} f_2(t) = \infty$, *then* $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.

Proposition 2.10 (Asymptotic Behavior)
If a function L *is slowly varying at infinity, then for all* $a \geq 0$ *we have*

(1) $\int_a^t s^\beta L(s)ds \cong (\beta+1)^{-1}t^{1+\beta}L(t)$ *for* $\beta > -1$,

(2) $\int_t^\infty s^\beta L(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}L(t)$ *for* $\beta < -1$,

as $t \to \infty$.

Proposition 2.11 (Asymptotic Behavior)
If a function H *is slowly varying at infinity, then for all* $a > 0$ *we have*

(1) $\int_0^t s^\beta H(s)ds \cong (\beta+1)^{-1}t^{1+\beta}H(t)$ *for* $\beta > -1$,

(2) $\int_t^\infty s^\beta H(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}H(t)$ *for* $\beta < -1$,

as $t \to 0^+$.

2.2 Auxiliary results

In this section, we will give some auxiliary results, which will be used in the proofs of Theorems 1 and 2.

Lemma 2.12 (cf. [10])
If $k \in \Lambda$, *then we have*

(1) $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$ *and* $\lim_{t\to 0^+} \frac{tk(t)}{K(t)} = C_k^{-1}$, *i.e.,* $K \in NRVZ_{C_k^{-1}}$,

(2) $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \frac{1-C_k}{C_k}$, *i.e.,* $K \in NRVZ_{\frac{1-C_k}{C_k}}$, *and* $\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$,

(3) $\lim_{t\to 0^+} \frac{1}{t}\left(\frac{K(t)k'(t)}{k^2(t)} - (1-C_k)\right) = -G_k$.

Set

$$\Theta(t) = \int_t^\infty \frac{ds}{(p'F(s))^{\frac{1}{p}}}, \quad t > 0. \tag{2.6}$$

We then have

$$\Theta'(t) = -\frac{1}{(p'F(t))^{\frac{1}{p}}}, \quad t > 0. \tag{2.7}$$

Lemma 2.13 *If a function f satisfies (f_1) and (f_2), we have the following properties:*

(1) *f satisfies the Keller-Osserman condition*

$$\int_1^\infty \frac{dt}{(p'F(t))^{\frac{1}{p}}} < \infty, \quad F(t) = \int_0^t f(s)ds,$$

(2) $\lim_{t\to\infty} \frac{tf(t)}{F(t)} = m+1,$

(3) $\Theta \in NRV_{-\frac{m+1-p}{p}}$, *i.e.,* $\lim_{t\to\infty} \frac{t\Theta'(t)}{\Theta(t)} = -\lim_{t\to\infty} \frac{t}{\Theta(t)(p'F(t))^{\frac{1}{p}}} = -\frac{m+1-p}{p},$

(4) $\lim_{t\to\infty} \frac{(p'F(t))^{\frac{1}{p'}}}{f(t)\Theta(t)} = \frac{p'(m+1-p)}{p(m+1)}.$

Proof. (1) Since $f \in NRV_m$ with $m > p-1$, one can see that the function f can be written as $f(t) = c_0 t^m \widehat{L}(t)$ in $[S_0, \infty)$ for S_0 sufficiently large and some $c_0 > 0$, where $\widehat{L}$ is the normalized slowly varying function at infinity, given in (2.4).

Let $p_1 \in (p-1, m)$. It follows from Proposition 2.8 (2) that

$$\lim_{t\to\infty} t^{m-p_1}\widehat{L}(t) = \infty.$$

Then there exist constants S_1 and S_2 such that $S_2 > S_1 > S_0$ and

$$c_0 t^{m-p_1}\widehat{L}(t) > 1, \quad \forall t \geq S_1, \text{ i.e., } f(t) \geq t^{p_1}, \quad \forall t \geq S_1,$$

$$F(t) \geq \frac{t^{p_1+1}}{2(p_1+1)}, \quad \forall t \geq S_2.$$

Hence, property (1) follows.

By Proposition 2.10, we have

$$F(t) \cong \frac{c_0 t^{m+1}}{m+1}\widehat{L}(t), \quad (p'F(t))^{\frac{1}{p}} \cong \left(\frac{p'c_0 t^{m+1}}{m+1}\widehat{L}(t)\right)^{\frac{1}{p}},$$

$$(p'F(t))^{\frac{1}{p}} \cong \left(\frac{p'c_0 t^{m+1}}{m+1}\widehat{L}(t)\right)^{-\frac{1}{p}}, \quad \Theta(t) \cong \left(\frac{c_0 p'(m+1-p)^p t^{m+1-p}}{p^p(m+1)\widehat{L}(t)}\right)^{-\frac{1}{p}},$$

as $t \to \infty$. Hence, properties (2)-(4) follow.

Lemma 2.14 *Suppose f satisfies (f_1)-(f_3). If q is a constant such that $\frac{m+1-p}{p} < q < m+1$, then we have the following properties:*

(1) $\lim_{t\to\infty} \dfrac{\frac{tf'(t)}{f(t)} - m}{\Theta(t)} = 0,$

(2) $\lim_{t\to\infty} \dfrac{\frac{F(t)}{tf(t)} - \frac{1}{m+1}}{\Theta(t)} = 0,$

(3) $\lim_{t\to\infty} \dfrac{\frac{(p'F(t))^{\frac{1}{p'}}}{f(t)\Theta(t)} - \frac{p'(m+1-p)}{p(m+1)}}{\Theta(t)} = 0,$

(4) $\lim_{t\to\infty} \dfrac{\frac{(f(\xi_0 t))}{\xi_0^{p-1}f(t)} - \xi_0^{m+1-p}}{\Theta(t)} = 0, \quad \xi_0 > 0.$

Proof. We only prove (3) and (4).

$$
\begin{aligned}
(3) \quad & \lim_{t\to\infty} \frac{\frac{(p'F(t))^{\frac{1}{p'}}}{f(t)\Theta(t)} - \frac{p'(m+1-p)}{p(m+1)}}{\Theta(t)} \\
= \;& \lim_{t\to\infty} \frac{\frac{(p'F(t))^{\frac{1}{p'}}}{f(t)\Theta(t)} - \frac{p'(m+1-p)}{p(m+1)}}{\Theta^{2\Theta(t)}(t)} \\
= \;& \lim_{t\to\infty} \frac{p'\frac{(F(t))}{tf(t)} \cdot \frac{tf'(t)}{f(t)} - \frac{p'm}{p(m+1)}}{2\Theta(t)} \\
= \;& \lim_{t\to\infty} \frac{p'\left(\frac{(F(t))}{tf(t)} - \frac{1}{m+1}\right) \cdot \left(\frac{tf'(t)}{f(t)} - m\right) + \frac{p'}{m+1}\left(\frac{tf'(t)}{f(t)} - m\right) + mp'\left(\frac{F(t)}{tf(t)} - \frac{1}{m+1}\right)}{2\Theta(t)} \\
= \;& 0.
\end{aligned}
$$

(4) When $\xi_0 = 1$, the result is obvious. Let $\xi_0 \neq 1$. By (f_2), one can see that

$$\frac{f(\xi_0 t)}{\xi_0^{p-1}f(t)} - \xi_0^{m+1-p} = \xi_0^{m+1-p}\left(\exp\Big(\int_t^{\xi_0 t} \frac{h(s)}{s}ds\Big) - 1\right).$$

By Proposition 2.8, it can be seen that

$$\lim_{t\to\infty} \frac{h(ts)}{s} = 0 \text{ and } \lim_{t\to\infty} \frac{h(ts)}{sh(t)} = s^{-q-1},$$

uniformly for $s \in [1, \xi_0]$ or $[\xi_0, 1]$.

Therefore, we can obtain the equalities

$$\lim_{t\to\infty} \int_t^{\xi_0 t} \frac{h(s)}{s}ds = \lim_{t\to\infty} \int_1^{\xi_0} \frac{h(ts)}{s}ds = 0,$$

and

$$\lim_{t\to\infty} \int_1^{\xi_0} \frac{h(ts)}{sh(t)}ds = \lim_{t\to\infty} \int_1^{\xi_0} s^{-q-1}ds = q^{-1}(1 - \xi_0^{-q}),$$

since $e^r - 1 \cong r$ as $r \to 0$, which lead to

$$\frac{f(\xi_0 t)}{\xi_0^{p-1} f(t)} - \xi_0^{m+1-p} = \xi_0^{m+1-p} \lim_{t\to\infty} \frac{\int_1^{\xi_0} \frac{h(ts)}{s} ds}{\Theta(t)}$$

$$= \xi_0^{m+1-p} \lim_{t\to\infty} \frac{h(t)}{\Theta(t)} \cdot \lim_{t\to\infty} \int_1^{\xi_0} \frac{h(ts)}{sh(t)} ds = 0.$$

This completes the proof.

Lemma 2.15 *Under the hypotheses in Theorem 1.1, if ϕ is a solution to the equation*

$$\int_{\phi(t)}^{\infty} (p' F(t))^{-\frac{1}{p}} = t, \quad \forall t > 0,$$

then we have the following properties:

(1) $-\phi'(t) = \Big(p'F(\phi(t))\Big)^{\frac{1}{p}}$, $\phi(t) > 0$, $t > 0$, $\phi(0) := \lim_{t\to 0^+} \phi(t) = \infty$, *and*
$\phi''(t) = (p'-1)\Big(p'F(\phi(t))\Big)^{\frac{2-p}{p}} f(\phi(t))$,

(2) $\displaystyle \lim_{t\to 0^+} \frac{t\phi'(t)}{\phi(t)} = -\frac{p}{m+1-p}$,

(3) $\displaystyle \lim_{t\to 0^+} \frac{\phi'(t)}{t\phi''(t)} = -\frac{m+1-p}{m+1}$,

(4) $\displaystyle \lim_{t\to 0^+} \frac{\phi(t)}{t^2\phi''(t)} = \frac{(m+1-p)^2}{p(m+1)}$,

(5) $\displaystyle \lim_{t\to 0^+} \frac{1}{t}\left(\frac{\phi'(t)}{t\phi''(t)} + \frac{m+1-p}{m+1}\right) = 0$,

(6) $\displaystyle \lim_{t\to 0^+} \frac{1}{t}\left(1 + \frac{\phi'(K(t))}{K(t)\phi''(K(t))} \cdot \frac{K(t)k'(t)}{k^2(t)} - \frac{1}{p-1}\frac{f(\xi_0\phi(K(t)))}{\xi_0^{p-1} f(\phi(K(t)))}\right) = \frac{m+1-p}{m+1} G_k$,
for all $k \in \Lambda$.

Proof. (1) Property (1) easily follows by the definition of ϕ and a direct calculation.

To show properties (2)-(5), set $u = \phi(t)$. By L'Hospital's rule and Lemma 2.14, one can obtain (2)-(5), i.e.,

$$\lim_{t\to 0^+} \frac{t\phi'(t)}{\phi(t)} = -\lim_{t\to 0^+} \frac{t(p'F(\phi(t)))^{\frac{1}{p}}}{\phi(t)}$$

$$
\begin{aligned}
&= -\lim_{u\to\infty}\frac{(p'F(u))^{\frac{1}{p}}\int_u^\infty (p'F(s))^{-\frac{1}{p}}ds}{u} = -\frac{p}{m+1-p},\\
&\lim_{t\to 0^+}\frac{\phi'(t)}{t\phi''(t)} = -\lim_{t\to 0^+}\frac{(p'F(\phi(t)))^{\frac{1}{p}}}{t\phi(t)} = -\lim_{u\to\infty}\frac{(p'F(u))^{\frac{1}{p}}}{(p'-1)f(u)\Theta(u)} = -\frac{m+1-p}{m+1},\\
&\lim_{t\to 0^+}\frac{\phi(t)}{t^2\phi''(t)} = \lim_{t\to 0^+}\frac{\phi(t)}{t\phi'(t)}\cdot\lim_{t\to 0^+}\frac{\phi'(t)}{t\phi''(t)} = \frac{m+1-p}{p}\cdot\frac{m+1-p}{m+1} = \frac{(m+1-p)^2}{p(m+1)},\\
&\lim_{t\to 0^+}\frac{1}{t}\left(\frac{\phi'(t)}{t\phi''(t)}+\frac{m+1-p}{m+1}\right) = -\lim_{u\to\infty}\frac{\frac{(p'F(u))^{\frac{1}{p'}}}{(p'-1)f(u)\Theta(u)}-\frac{m+1-p}{m+1}}{\Theta(u)} = 0.
\end{aligned}
$$

(6) Since $K\in NRV_{C_k^{-1}}$ and $C_k\in(0,1)$, $\lim_{t\to 0^+}\frac{K(t)}{t}=0$. It follows from the choice of ξ_0 given in (1.4), Lemma 2.12 (3), and Lemma 2.14 that

$$1-\frac{m+1-p}{m+1}(1-C_k) = \xi_0^{m+1-p},$$

$$\lim_{t\to 0^+}\frac{1}{t}\left(\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right) = -G_k,$$

$$\lim_{t\to 0^+}\frac{1}{t}\left(\frac{\phi'(K(t))}{K(t)\phi''(K(t))}+\frac{m+1-p}{m+1}\right) = \lim_{t\to 0^+}\frac{K(t)}{t}\cdot\lim_{t\to 0^+}\frac{\left(\frac{\phi'(K(t))}{K(t)\phi''(K(t))}+\frac{m+1-p}{m+1}\right)}{k(t)} = 0,$$

$$\lim_{t\to 0^+}\frac{1}{t}\left(\xi_0^{m+1-p}-\frac{f(\xi_0\phi(K(t)))}{\xi_0^{p-1}f(\phi(K(t)))}\right) = \lim_{t\to 0^+}\frac{K(t)}{t}\cdot\lim_{t\to 0^+}\frac{\xi_0^{m+1-p}-\frac{f(\xi_0\phi(K(t)))}{\xi_0^{p-1}f(\phi(K(t)))}}{K(t)} = 0,$$

$$
\begin{aligned}
&\lim_{t\to 0^+}\frac{1}{t}\left(1+\frac{\phi'(K(t))}{K(t)\phi''(K(t))}\cdot\frac{K(t)k'(t)}{k^2(t)}-\frac{1}{p-1}\frac{f(\xi_0\phi(K(t)))}{\xi_0^{p-1}f(\phi(K(t)))}\right)\\
&= \lim_{t\to 0^+}\left(\frac{\phi'(K(t))}{K(t)\phi''(K(t))}+\frac{m+1-p}{m+1}\right)\cdot\lim_{t\to 0^+}\frac{1}{t}\left(\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right)\\
&\quad +(1-C_k)\lim_{t\to 0^+}\frac{1}{t}\left(\frac{\phi'(K(t))}{K(t)\phi''(K(t))}+\frac{m+1-p}{m+1}\right)\\
&\quad +\frac{m+1-p}{m+1}\lim_{t\to 0^+}\frac{1}{t}\left(\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right)-\lim_{t\to 0^+}\frac{1}{t}\left(\xi_0^{m+1-p}-\frac{f(\xi_0\phi(K(t)))}{\xi_0^{p-1}f(\phi(K(t)))}\right)\\
&= \frac{m+1-p}{m+1}G_k.
\end{aligned}
$$

This completes the proof.

Lemma 2.16 *Suppose that $h_1\in NRV_{-q_1}$ and $k\in\Lambda$. If p is a constant such that $pq_1>(m+1-p)C_k$, then*

$$\lim_{t\to 0}\frac{h_1\left((K(t))^{-\frac{p}{m+1-p}}\right)}{t} = 0.$$

Proof. By Lemma 2.12, we see that $K \in NRV_{C_k^{-1}}$. It follows from Proposition 2.9 that $h_1 \circ K^{-\frac{p}{m+1-p}} \in NRVZ_{\frac{pq_1}{(m+1-p)C_k}}$. Hence, the result follows by Proposition 2.8 (2), since $\frac{pq_1}{(m+1-p)C_k} > 1$.

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The upper and lower solutions method is an important tool in proving the theorem, and so, establishing a comparison principle is very important. Therefore, we first give a comparison principle, shown in [15], for general quasilinear elliptic equations.

Lemma 3.1 (cf. [15])
Suppose that D is a bounded domain in R^N and that $a(x)$ and $\beta(x)$ are continuous functions on D such that $||a||_{L^\infty(D)} < \infty$ and $\beta(x)$ is nonnegative and nontrivial in D. If u_1 and $u_2 \in C^1(D)$ are positive in D and satisfy the following inequalities in the sense of distributions:

$$-\Delta_p u_1 - a(x)u_1^{p-1} - \beta(x)g(u_1) \geq 0 \geq -\Delta_p u_2 - a(x)u_2^{p-1} - \beta(x)g(u_2),$$

$$\overline{\lim}_{d(x,\partial\Omega)\to 0} \left(u_2^{p-1} - u_1^{p-1}\right) \leq 0,$$

where $g \in C^0([0,\infty))$ and $\frac{g(s)}{s^{p-1}}$ is increasing on the interval

$$\Big(\inf_D(u_1,u_2), \sup_D(u_1,u_2)\Big),$$

then we have the inequality $u_1 \geq u_2$ in D.

Fix $\epsilon > 0$, and for all $\delta > 0$ we define a set $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since $\partial\Omega$ is C^4-smooth, we can choose $\delta_1 \in (0,\delta_0)$ such that $d \in C^4(\Omega_{\delta_1})$ and

$$|\nabla d(x)| = 1, \quad \triangle d(x) = -(N-1)H(\overline{x})d(x) + o(1), \quad \forall x \in \Omega_{\delta_1}. \tag{3.1}$$

Let $a_0 \in \left(0, \min\{1, \frac{m^2+m-p}{p}\}\right)$ and let

$$w_\pm = \xi_0\phi(K(d(x)))\Big(1 + (C_1 \pm \varepsilon)d(x) + C_2H(\overline{x})d(x)\Big), \quad x \in \Omega_{\delta_1}.$$

By the Lagrange mean value theorem, one can see that there exist constants $\lambda_\pm \in (0,1)$ such that

$$f(w_\pm(x)) = f(\xi_0\phi(K(d(x)))) + \xi_0\phi(K(d(x)))f'(\phi_\pm(d(x)))\Big((C_1 \pm \epsilon)d(x) + C_2H(\overline{x})d(x)\Big),$$

for all $x \in \Omega_{\delta_1}$, where

$$\phi_\pm(d(x)) = \xi_0\phi(K(d(x)))\Big(1 + \lambda_\pm(C_1 \pm \varepsilon)d(x) + C_2H(\overline{x})d(x)\Big).$$

Since $f \in NRV_m$, we get the limits

$$\lim_{d(x)\to 0} \frac{f(\xi_0\phi(K(d(x))))}{f(\phi_\pm(d(x)))} = \lim_{d(x)\to 0} \frac{f'(\xi_0\phi(K(d(x))))}{f'(\phi_\pm(d(x)))} = 1,$$

by Proposition 2.4. Set $r = d(x) = |x - \overline{x}|$ and set

$$M_1 = \frac{1}{r}\left(1 + \frac{\phi'(K(r))}{K(r)\phi''(K(r))} \cdot \frac{K(r)k'(r)}{k^2(r)} - \frac{1}{p-1}\frac{f(\xi_0\phi(K(r)))}{\xi_0^{p-1}f(\phi(K(r)))}\right),$$

$$M_{2\pm} = (C_1 \pm \varepsilon)\left[1 + \frac{\phi'(K(r))}{K(r)\phi''(K(r))}\left(\frac{K(r)k'(r)}{k^2(r)} + \frac{2K(r)}{rk(r)}\right)\right.$$
$$\left. -\frac{1}{p-1}\frac{f'(\phi_\pm(K(r)))}{f'(\phi(K(r)))}\frac{\phi(K(r))f'(\phi(K(r)))}{\xi_0^{p-2}f(\phi(K(r)))}\right]$$
$$-\frac{1}{p-1}(B_0 \mp a_0\epsilon)\frac{f(\xi_0\phi(K(r)))}{\xi_0^{p-1}f(\phi(K(r)))},$$

$$M_3 = H(\overline{x})\left[C_2\left(1 + \frac{\phi'(K(r))}{K(r)\phi''(K(r))}\Big(\frac{K(r)k'(r)}{k^2(r)} + \frac{2K(r)}{rk(r)}\Big)\right.\right.$$
$$\left. -\frac{1}{p-1}\frac{f'(\phi_\pm(K(r)))}{f'(\phi(K(r)))}\frac{\phi(K(r))f'(\phi(K(r)))}{\xi_0^{p-2}f(\phi(K(r)))}\right)$$
$$\left. -((N-1)+o(1))\frac{\phi'(K(r))}{K(r)\phi''(K(r))}\frac{K(r)}{rk(r)}\right],$$

$$M_{4\pm} = r\frac{\phi'(K(r))}{K(r)\phi''(K(r))}\frac{K(r)}{rk(r)}\Big(C_1 \pm \varepsilon + C_2H(\overline{x})\Big)\Delta d(x)$$
$$+\frac{\phi(K(r))}{K^2(r)\phi''(K(r))}\frac{K(r)}{rk(r)}\frac{K(r)}{k(r)}\Big(C_1 \pm \varepsilon + C_2H(\overline{x})\Big)\Delta d(x)$$
$$-(B_0 \mp a_0\epsilon)r\frac{f'(\phi_\pm(K(r)))}{f'(\phi(K(r)))}\frac{\phi(K(r))f'(\phi(K(r)))}{\xi_0^{p-2}f(\phi(K(r)))}\Big(C_1 \pm \varepsilon + C_2H(\overline{x})\Big),$$

$$M_{5\pm} = \Big|\nabla d(x)\Big(1 + (C_1 \pm \epsilon)r + C_2H(\overline{x})r\Big)$$
$$+\frac{\phi(K(r))}{K(r)\phi'(K(r))}\frac{K(r)}{k(r)}\Big((C_1 \pm \varepsilon)\nabla d(x) + C_2H(\overline{x})\nabla d(x)\Big)\Big|.$$

With Lemma 2.15 and the choices of ξ_0, C_1, and C_2 given in Theorem 1.1, the following Lemma can be obtained:

Lemma 3.2 *Under the hypotheses in Theorem 1.1, we have the limits*

(1) $\lim_{r\to 0} M_1(r) = \frac{m+1-p}{m+1}G_k,$

(2) $\lim_{r\to 0} M_{2\pm}(r) = -\left(C_1\frac{(m+1-p)(m+1)C_k+p}{m+1} + B_0\frac{p+(m+1-p)C_k}{m+1}\right.$
$\left.\mp\epsilon\Big(\frac{(m+1-p)(m+1)C_k+p}{m+1} + a_0\frac{p+(m+1-p)C_k}{m+1}\Big)\right),$

(3) $\lim_{r\to 0} M_3(r) = \lim_{r\to 0} M_{4\pm}(r) = 0$,

(4) $\lim_{r\to 0} \Big(M_1(r) + M_{2\pm}(r) + M_3(r) + M_{4\pm}(r)\Big)$
$= \mp\epsilon\Big(\frac{(m+1-p)(m+1)C_k+p}{m+1} + a_0\frac{p+(m+1-p)C_k}{m+1}\Big)$,

(5) $\lim_{r\to 0} M_{5\pm}(r) = 1$.

- *Proof of Theorem 1.1.*

By Lemma 3.2, (b_1) and (b_2), and $K(x) \in C[0, \delta_0)$ with $K(0) = 0$, one can see that there are constants $\delta_{1\epsilon}$ and $\delta_{2\epsilon} \in \Big(0, \min\{1, \frac{\delta_1}{2}\}\Big)$, which are corresponding to $\varepsilon > 0$ sufficiently small, such that

(i) $0 \le K(t) \le 2\delta_{1\epsilon}, \quad t \in (0, 2\delta_{2\epsilon})$,

(ii) $k^p(d(x))\Big(1 + (B_0 - a_0\epsilon)d(x)\Big) \le b(x) \le k^p(d(x))\Big(1 + (B_0 + a_0\epsilon)d(x)\Big)$, $x \in \Omega_{2\delta_{1\epsilon}}$,

(iii) $\lim_{r\to 0} M_{5+}(r) \le 1, \quad x \in \Omega_{2\delta_{1\epsilon}}$,

(iv) $\lim_{r\to 0} M_{5-}(r) \ge 1, \quad x \in \Omega_{2\delta_{1\epsilon}}$,

(v) $\lim_{r\to 0} \Big(M_1(r) + M_{2+}(r) + M_3(r) + M_{4+}(r)\Big) \le 0, \quad \forall (x, r) \in \Omega_{2\delta_{1\epsilon}} \times (0, 2\delta_{2\epsilon})$,

(vi) $\lim_{r\to 0} \Big(M_1(r) + M_{2-}(r) + M_3(r) + M_4(r)\Big) \ge 0, \quad \forall (x, r) \in \Omega_{2\delta_{1\epsilon}} \times (0, 2\delta_{2\epsilon})$.

Let

$$d_1(x) = d(x) - \rho, \;\; d_2(x) = d(x) + \rho, \;\; \rho \in (0, 2\delta_{1\epsilon}), \tag{3.2}$$

$$D_\rho^- = \Omega_{2\delta_{1\epsilon}}/\overline{\Omega}_\rho, \;\; D_\rho^+ = \Omega_{2\delta_{1\epsilon}-\rho}, \tag{3.3}$$

and let

$$\overline{u}_\epsilon(x) = \xi_0\phi(K(d_1(x)))\Big(1 + (C_1 + \varepsilon)d_1(x) + C_2H(\overline{x})d_1(x)\Big), \quad x \in D_\rho^-. \tag{3.4}$$

We then have

$$\begin{aligned} f(\overline{u}_\epsilon(x)) \;\; &= \;\; f(\xi_0\phi(K(d_1(x)))) \\ &+ \;\; \xi_0\phi(K(d_1(x)))f'(\phi_+(d_1(x)))\Big((C_1 + \epsilon)d_1(x) + C_2H(\overline{x})d_1(x)\Big), \; x \in D_\rho^-, \end{aligned}$$

where $\lambda_+ \in (0, 1)$ and

$$\phi_+(d_1(x)) = \xi_0\phi(K(d_1(x)))\Big(1 + \lambda_+(C_1 + \varepsilon)d_1(x) + C_2H(\overline{x})d_1(x)\Big).$$

By Lemma 3.2 and a direct calculation, one can obtain that for all $x \in D_\rho^-$

$$\begin{aligned}
& \triangle_p \overline{u}_\varepsilon(x) - k^p(d_1(x))\Big(1+(B_0-\varepsilon)d_1(x)\Big)f(\overline{u}_\epsilon(x)) \\
= \ & (p-1)\xi_0^{p-1}\Big|\phi'(K(d_1(x)))\Big|^{p-2}\Big|k^{p-2}(d_1(x))\Big|\Big|\nabla d(x)\Big(1+(C_1+\varepsilon)d_1(x)+C_2H(\overline{x})d_1(x)\Big) \\
& + \frac{\phi(K(d_1(x)))}{K(d_1(x))\phi''(K(d_1(x)))}\cdot\frac{K(d_1(x))}{k(d_1(x))}\Big((C_1+\varepsilon)\nabla d(x)+C_2H(\overline{x})\nabla d(x)\Big)\Big|^{p-2}\cdot \\
& \Big[\phi''(K(d_1(x)))k^2(d_1(x))\Big(1+(C_1+\varepsilon)d_1(x)+C_2H(\overline{x})d_1(x)\Big) \\
& +\phi'(K(d_1(x)))k'(d_1(x))\Big(1+(C_1+\varepsilon)d_1(x)+C_2H(\overline{x})d_1(x)\Big) \\
& +\phi'(K(d_1(x)))k(d_1(x))\Delta d(x)\Big(1+(C_1+\varepsilon)d_1(x)+C_2H(\overline{x})d_1(x)\Big) \\
& +2\phi'(K(d_1(x)))k(d_1(x))\nabla d(x)\Big((C_1+\varepsilon)\nabla d(x)+C_2H(\overline{x})\nabla d(x)\Big) \\
& +\phi(K(d_1(x)))\Big((C_1+\varepsilon)\triangle d(x)+C_2H(\overline{x})\triangle d(x)\Big)\Big] \\
& -k^p(d_1(x))\Big(1+(B_0-\varepsilon)d_1(x)\Big)\Big[f(\xi_0\phi(K(d_1(x)))) \\
& +\xi_0\phi(K(d_1(x)))f'(\phi_+(d_1(x)))\Big((C_1+\epsilon)d_1(x)+C_2H(\overline{x})d_1(x)\Big)\Big] \\
= \ & (p-1)\xi_0^{p-1}\Big|\phi'(K(d_1(x)))\Big|^{p-2}\phi''(K(d_1(x)))k^p(d_1(x))d_1(x)M_{5+}(x)\cdot \\
& \Big(M_1(r)+M_{2+}(r)+M_3(x)+M_{4+}(x)\Big) \\
\leq \ & 0,
\end{aligned}$$

where $r=d_1(x)$, which implies that $\overline{u}_\varepsilon$ is an upper solution of equation (1.1) in D_ρ^-.

By a similar argument, it can be shown that

$$\underline{u}_\epsilon(x)=\xi_0\phi(K(d_2(x)))\Big(1+(C_1-\varepsilon)d_2(x)+C_2H(\overline{x})d_2(x)\Big),\quad x\in D_\rho^+ \tag{3.5}$$

is a lower solution of (1.1) in D_ρ^+.

Let u be an arbitrary solution of problem (1.1) and let $C_1(\delta_\varepsilon)=\max_{d(x)\geq\delta_\epsilon}u(x)$. We then have the inequality

$$u\leq C_1(\delta_{1\varepsilon})+\overline{u}_\epsilon \text{ on } \partial D_\rho^-.$$

Since ϕ_1 is decreasing, we get the inequalities

$$\underline{u}_\epsilon\leq\xi_0\phi(K(2\delta_\varepsilon)):=C_2(\delta_\varepsilon),\quad \text{whenever}\quad d(x)=2\delta_\varepsilon-\rho,$$

and

$$\underline{u}_\epsilon\leq u+C_2(\delta_\varepsilon) \text{ on } \partial D_\rho^+.$$

It follows from (f_1) and Lemma 3.1 that

$$u\leq C_1(\delta_\varepsilon)+\overline{u}_\epsilon,\ \ x\in D_\rho^- \ \text{ and } \ \underline{u}_\epsilon\leq u+C_2(\delta_\varepsilon),\quad x\in D_\rho^+.$$

Hence, letting $\rho \to 0$, we have that for all $x \in D_\rho^- \cap D_\rho^+$,

$$1 + (C_1 - \epsilon)d(x) + C_2 H(\overline{x})d(x) - \frac{C_2(\delta_\epsilon)}{\xi_0 \phi(K(d(x)))} \leq \frac{u(x)}{\xi_0 \phi(K(d(x)))},$$

and

$$\frac{u(x)}{\xi_0 \phi(K(d(x)))} \leq 1 + (C_1 + \epsilon)d(x) + C_2 H(\overline{x})d(x) + \frac{C_1(\delta_\epsilon)}{\xi_0 \phi(K(d(x)))}.$$

Consequently, we obtain the inequalities

$$C_1 - \epsilon + C_2 H(\overline{x}) \leq \liminf_{d(x)\to 0} (d(x))^{-1}\Big(\frac{u(x)}{\xi_0 \phi(K(d(x)))} - 1\Big),$$

and

$$\limsup_{d(x)\to 0} (d(x))^{-1}\Big(\frac{u(x)}{\xi_0 \phi(K(d(x)))} - 1\Big) \leq C_1 + \epsilon + C_2 H(\overline{x}).$$

Letting $\epsilon \to 0$ we have the limit

$$\lim_{d(x)\to 0} (d(x))^{-1}\Big(\frac{u(x)}{\xi_0 \phi(K(d(x)))} - 1\Big) = C_1 + C_2 H(\overline{x}).$$

This completes the proof.

4 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. To show the result of this theorem, we first introduce a lemma with proof below:

Let $a_0 \in \Big(0, \min\{1, \frac{m^2+m-p}{p}\}\Big)$ and let

$$w_\pm(x) = \xi_1 (K(d(x)))^{-\frac{p}{m+1-p}} \Big(1 + (C_3 \pm \epsilon)d(x) + C_4 H(\overline{x})d(x)\Big),$$

where $x \in \Omega_{\delta_1}$. Then, we get

$$f(w_\pm(x)) = c_0 \xi_1^m (K(d(x)))^{-\frac{mp}{m+1-p}} (1 + g_1(w_\pm(x)))\Big(1 + m(C_3 \pm \epsilon)d(x) + mC_4 H(\overline{x})d(x)\Big),$$

where $x \in \Omega_{\delta_1}$.

Set $r = d(x) = |x - \overline{x}|$ and set

$$\begin{aligned} M_1(r) \quad = \quad & \frac{1}{r}\Bigg(\frac{p(m+1)}{(m+1-p)^2} - \frac{p}{m+1-p}\frac{K(r)k'(r)}{k^2(r)} \\ & - \frac{c_0 \xi_1^{m+1-p}}{p-1}\Big(\frac{p}{m+1-p}\Big)^{-(p-2)}(1 + g(w_\pm(x)))\Bigg), \end{aligned}$$

$$\begin{aligned} M_{2\pm}(r) &= (C_3 \pm \varepsilon)\left(\frac{p(m+1)}{(m+1-p)^2} - \frac{p}{m+1-p}\frac{K(r)k'(r)}{k^2(r)}\right. \\ &\left. -\frac{2p}{m+1-p}\frac{K(r)}{rk(r)} - \frac{c_0 m}{p-1}\Big(\frac{p}{m+1-p}\Big)^{-(p-2)}\xi_1^{m+1-p}\right) \\ &- \frac{c_0}{p-1}\left(\frac{p}{m+1-p}\right)^{-(p-2)}\xi_1^{m+1-p}(B_0 \mp a_0\epsilon), \end{aligned}$$

$$\begin{aligned} M_3(r) &= H(\overline{x})C_4\left(\frac{p(m+1)}{(m+1-p)^2} - \frac{p}{m+1-p}\frac{K(r)k'(r)}{k^2(r)}\right. \\ &\left. -\frac{2p}{m+1-p}\frac{K(r)}{rk(r)} - \frac{c_0 m}{p-1}\Big(\frac{p}{m+1-p}\Big)^{-(p-2)}\xi_1^{m+1-p}\right) \\ &- \left(\frac{p}{m+1-p}\right)\frac{K(r)}{rk(r)}\triangle d(x), \end{aligned}$$

$$\begin{aligned} M_{4\pm}(r) &= -\frac{1}{p-1}\cdot\frac{p}{m+1-p}\frac{K(r)}{k(r)}\big(C_3 \pm \varepsilon + C_4 H(\overline{x})\big) \\ &+ r\left(\frac{K(r)}{rk(r)}\right)^2\big(C_3 + \varepsilon + C_4 H(\overline{x})\big)\triangle d(x) \\ &- \frac{c_0}{p-1}\left(\frac{p}{m+1-p}\right)^{-(p-2)}\xi_1^{m+1-p}\Big((B_0 \mp a_0\varepsilon)g(w_\pm)(B_0 \mp a_0\varepsilon)r \\ &+ g(w_\pm(x))(B_0 \mp a_0\varepsilon)rg(w_\pm(x))(m(C_3 \pm \varepsilon) + mC_4 H(\overline{x}))\Big), \end{aligned}$$

$$\begin{aligned} M_{5\pm}(r) &= -\frac{p}{m+1-p} - \triangledown d(x)\big(1 + C_3 \pm \varepsilon + C_4 H(\overline{x})\big) \\ &+ \frac{K(d_2(x))}{k(d_2(x))}\big(C_3 \pm \varepsilon + C_4 H(\overline{x})\big)\triangledown d(x). \end{aligned}$$

Combining Lemmas 2.12 and 2.16 with the choices of ξ_1, C_3, and C_4 given in Theorem 1.2, the following lemma can be obtained:

Lemma 4.1 *Under the hypotheses given in Theorem 1.2, we have the properties*

(1) $\lim_{r\to 0} M_1 = \frac{p}{m+1-p}G_k,$

(2) $\lim_{r\to 0} M_{2\pm} = -p(C_1 \pm \varepsilon)\left(\frac{(m+1-p)(m+1)C_K - (m-1)}{(m+1-p)^2}\right)$
$- p(B_0 \mp a_0\varepsilon)\frac{(m+1-p)C_K + p}{(m+1-p)^2},$

(3) $\lim_{r\to 0} M_3 = \lim_{r\to 0} M_{4\pm} = 0,$

(4) $\lim_{r\to 0} M_{5\pm} = -\frac{p}{m+1-p},$

(5) $\lim_{r\to 0} \Big(M_1+M_{2\pm}+M_3+M_{4\pm}\Big) = \mp p\varepsilon \frac{(m+1-p)(m+1-a_0)C_k - p(m+1-a_0)}{(m+1-p)^2}.$

- *Proof of Theorem 1.2.*

Let

$$\overline{u}_\epsilon(x) = \xi_1(K(d(x)))^{-\frac{p}{m+1-p}}\Big(1+(C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big),\ x\in D_\rho^-.$$

By a direct calculation, we have that for all $x \in D_\rho^-$,

$$\begin{aligned}
&\Delta_p\overline{u}_\epsilon - k^p(d_2(x))\Big(1+(B_0-\epsilon)d_2(x)\Big)f(\overline{u}_\epsilon(x))\\
= \;& (p-1)\xi_1^{p-1}\Big(K(d_2(x))\Big)^{-\frac{(m+1)(p-2)}{m+1-p}}k^{p-2}(d_2(x))\Bigg|-\frac{p}{m+1-p}\nabla d(x)\cdot\\
&\Big(1+(C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)+\frac{K(d_2(x))}{k(d_2(x))}\Big((C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)\Bigg|^{p-2}\\
&-\Bigg[\frac{p(m+1)}{(m+1-p)^2}\Big(K(d_2(x))\Big)^{-\frac{m+1}{m+1-p}-1}k^2(d_2(x))\Big(1+(C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)\\
&-\frac{p}{m+1-p}\Big(K(d_2(x))\Big)^{-\frac{m+1}{m+1-p}}k'(d_2(x))\Big(1+(C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)\\
&-\frac{p}{m+1-p}\Big(K(d_2(x))\Big)^{-\frac{m+1}{m+1-p}}k(d_2(x))\triangle d(x)\Big(1+(C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)\\
&-\frac{2p}{m+1-p}\Big(K(d_2(x))\Big)^{-\frac{m+1}{m+1-p}}k(d_2(x))\nabla d(x)\Big((C_3+\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big)\\
&-\Big(K(d_2(x))\Big)^{-\frac{p}{m+1-p}}\triangle d(x)\Big((C_3+\epsilon)+C_4H(\overline{x})\Big)\Bigg]\\
&-k^p(d_2(x))\Big(1+(B_0-\epsilon)d_2(x)\Big)c_0\xi_1^m\Big(K(d(x))\Big)^{-\frac{mp}{m+1-p}}\Big(1+g_1(w_+(x))\Big)\cdot\\
&\Big(1+m(C_3+\epsilon)d(x)+mC_4H(\overline{x})d(x)\Big)\\
= \;& (p-1)\left(\frac{p}{m+1-p}\right)^{p-2}\xi_1^{p-1}\Big(K(d_2(x))\Big)^{-\frac{mp}{m+1-p}}k^p(d_2(x))d_2(x)\cdot\\
&M_{5+}\Big(M_1(r)+M_{2+}(x)+M_3(r)+M_{4+}(x)\Big)\\
\le \;& 0,
\end{aligned}$$

which implies that $\overline{u}_\varepsilon$ is a lower solution to equation (1.1) in D_ρ^-.

Similarly, it can be shown that

$$\overline{u}_\epsilon = \xi_1\Big(K(d(x))\Big)^{-\frac{p}{m+1-p}}\Big(1+(C_3-\epsilon)d_2(x)+C_4H(\overline{x})d_2(x)\Big),\ x\in D_\rho^+, \quad (3.5)$$

is a lower solution of (1.1) in D_ρ^+.

Letting $\rho\to 0$, we can obtain the inequalities

$$1+(C_3-\epsilon)d(x)+C_4H(\overline{x})d(x)-\frac{C_4(\delta_\epsilon)}{\xi_1(K(d(x)))^{-\frac{p}{m+1-p}}}\le\frac{u(x)}{\xi_1(K(d(x)))^{-\frac{p}{m+1-p}}},$$

and

$$\frac{u(x)}{\xi_1(K(d(x)))^{-\frac{p}{m+1-p}}}\le 1+(C_3+\epsilon)d(x)+C_4H(\overline{x})d(x)+\frac{C_4(\delta_\epsilon)}{\xi_0(K(d(x)))^{-\frac{p}{m+1-p}}},$$

for all $x\in D_\rho^-\cap D_\rho^+$.

Consequently, we can get the inequalities

$$C_3-\epsilon+C_4H(\overline{x})\le\liminf_{d(x)\to 0}(d(x))^{-1}\left(\frac{u(x)}{\xi_0(K(d(x)))^{-\frac{p}{m+1-p}}}-1\right),$$

and

$$\limsup_{d(x)\to 0}(d(x))^{-1}\left(\frac{u(x)}{\xi_0(K(d(x)))^{-\frac{p}{m+1-p}}}-1\right)\le C_3+\epsilon+C_4H(\overline{x}).$$

Letting $\epsilon\to 0$, we have the limit

$$\lim_{d(x)\to 0}(d(x))^{-1}\left(\frac{u(x)}{\xi_0(K(d(x)))^{-\frac{p}{m+1-p}}}-1\right)=C_3+C_4H(\overline{x}).$$

This completes the proof.

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