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Fuzzy ideals of pseudo-BCH-algebras

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Abstract

Characterizations of fuzzy ideals of a pseudo-BCH-algebra are established. Conditions for a fuzzy set to be a fuzzy ideal are given. The homomorphic properties of fuzzy ideals are provided. Finally, characterizations of Noetherian pseudo-BCH-algebras and Artinian pseudo-BCH-algebras via fuzzy ideals are obtained.

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1 Introduction

In 1966, Y. Imai and K. Iséki ([10],[11]) introduced BCK- and BCI-algebras. In 1983, Q. P. Hu and X. Li ([9]) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras. J. Neggers and H. S. Kim ([19]) defined d-algebras which are a generalization of BCK-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([7]) introduced pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W. A. Dudek and Y. B. Jun ([1]) introduced pseudo-BCI-algebras as a natural generalization of

BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [5] and [6], respectively. Those algebras were investigated by several authors in [2], [3], [14] and [15]. As a generalization of d-algebras, Y. B. Jun, H. S. Kim and J. Neggers ([13]) introduced pseudo-d-algebras. Recently, A. Walendziak ([21]) defined pseudo-BCH-algebras and considered ideals in such algebras.

Fuzzy ideals of BCK/BCI-algebras were studied in [17] and [18]. See also [12] and [20]. Fuzzy ideals of BCH-algebras were discussed in [8] and [22]. K. J. Lee ([16]) established the fuzzyfication of ideals in pseudo-BCI-algebras. Fuzzy ideals of pseudo-BCK-algebras were investigated in [4].

In this paper we consider the fuzzy ideal theory in pseudo-BCH-algebras. In Section 3 we give characterizations of fuzzy ideals and provide conditions for a fuzzy set to be a fuzzy ideal. Moreover we show that the set of fuzzy ideals of a pseudo-BCH-algebra is a complete lattice. The homomorphic properties of fuzzy ideals of a pseudo-BCH-algebra are provided. Finally, characterizations of Noetherian pseudo-BCH-algebras and Artinian pseudo-BCH-algebras in terms of fuzzy ideals are given in Section 4. For the convenience of the reader, in Section 2 we give the necessary material needed in the sequel, thus making our exposition self-contained.

2 Preliminaries

We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $(x * y) * z = (x * z) * y$;
(BCH-3) $x * y = y * x = 0 \Longrightarrow x = y$.

A BCH-algebra $\mathfrak{X}$ is said to be a *BCI-algebra* if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A *BCK-algebra* is a BCI-algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1. ([1]) A *pseudo-BCI-algebra* is a structure $\mathfrak{X} = (X; \leq, *, \diamond, 0)$, where "$\leq$" is a binary relation on the set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, satisfying the axioms:

(pBCI-1) $(x * y) \diamond (x * z) \leq z * y, \quad (x \diamond y) * (x \diamond z) \leq z \diamond y$;
(pBCI-2) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;
(pBCI-3) $x \leq x$;
(pBCI-4) $x \leq y,\, y \leq x \Longrightarrow x = y$;
(pBCI-5) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo-BCI-algebra $\mathfrak{X}$ is called a *pseudo-BCK-algebra* if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Definition 2.2. ([21]) A *pseudo-BCH-algebra* is an algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;
(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;
(pBCH-3) $x * y = y \diamond x = 0 \Longrightarrow x = y$;
(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Remark 2.3. Observe that if $(X; *, 0)$ is a BCH-algebra, then letting $x \diamond y := x * y$, produces a pseudo-BCH-algebra $(X; *, \diamond, 0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra, then $(X; \diamond, *, 0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [1] we conclude that if $(X; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra.

Example 2.4. Let $(G; \cdot, 1)$ be a group. Define binary operations "$*$" and "$\diamond$" on G by

$$a * b = ab^{-1} \quad \text{and} \quad a \diamond b = b^{-1}a$$

for all $a, b \in G$. It is easy to see that $\mathfrak{G} = (G; *, \diamond, 1)$ is a pseudo-BCH-algebra.

Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Following [21], we define a binary relation $\preceq$ on X by

$$x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0.$$

Proposition 2.5. ([21]) *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then for all $x, y \in X$:*

(a) $x * (x \diamond y) \preceq y$ and $x \diamond (x * y) \preceq y$;
(b) $x * 0 = x \diamond 0 = x$;
(c) $0 * x = 0 \diamond x$;
(d) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$;
(e) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$

Definition 2.6. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. A subset I of X is called an *ideal* of $\mathfrak{X}$ if it satisfies for all $x, y \in X$*:*

(I1) $0 \in I$;
(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\mathrm{Id}(\mathfrak{X})$ the set of all ideals of $\mathfrak{X}$. Obviously, $\{0\}, X \in \mathrm{Id}(\mathfrak{X})$.

Proposition 2.7. ([21]) *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $I \in \mathrm{Id}(\mathfrak{X})$. For any $x, y \in X$, if $y \in I$ and $x \leq y$, then $x \in I$.*

Proposition 2.8. ([21]) *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X satisfying* (I1). *Then I is an ideal of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

(I2') *if $x \diamond y \in I$ and $y \in I$, then $x \in I$.*

Example 2.9. Let $\mathfrak{G}$ be the pseudo-BCH-algebra given in Example 2.4. Let a be an element of G. It is easy to check that $\{a^m : m \in \mathbb{Z}\}$ is an ideal of $\mathfrak{G}$.

Example 2.10. ([21]) Let $X = \{0, a, b, c, d\}$. Define binary operations $*$ and $\diamond$ on X by the following tables:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	b	c	0	d
d	d	d	d	d	0

$\diamond$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	c	a	0	d
d	d	d	d	d	0

Then $\mathfrak{X} = (X; *, \diamond, 0)$ is a pseudo-BCH-algebra. It is easily seen that $\mathrm{Id}(\mathfrak{X}) = \{\{0\}, \{0, a\}, \{0, b\}, \{0, a, b, c\}, X\}$.

Remark 2.11. It is easy to prove that the intersection of an arbitrary number of ideals of a pseudo-BCK-algebra $\mathfrak{X}$ is an ideal of $\mathfrak{X}$. It is also not hard to show that the union of an ascending sequence of ideals of $\mathfrak{X}$ is an ideal of $\mathfrak{X}$.

3 Fuzzy ideals

We now review some fuzzy logic concepts. First, for $\Gamma \subseteq [0; 1]$ we define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a fuzzy set in X is a function $\mu : X \to [0; 1]$.

For any fuzzy sets μ and ν in X, we define

$$\mu \leq \nu \Leftrightarrow \mu(x) \leq \nu(x) \text{ for all } x \in X.$$

A trivial verification shows that this relation is an order relation in the set of fuzzy sets in X.

Let X and Y be any two sets, μ be any fuzzy set in X and $f : X \to Y$ be any function. Set $f^{\leftarrow}(y) = \{x \in X : f(x) = y\}$ for $y \in Y$. The fuzzy set ν in Y defined by

$$\nu(y) = \begin{cases} \bigvee\{\mu(x) : x \in f^{\leftarrow}(y)\} & \text{if } f^{\leftarrow}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Let X and Y be any two sets, $f : X \to Y$ be any function and ν be any fuzzy set in $f(X)$. The fuzzy set μ in X defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in X$$

is called the *preimage* of ν under f and is denoted by $f^{\leftarrow}(\nu)$.

Now, we give the definition of a fuzzy ideal in a pseudo BCH-algebra.

Definition 3.1. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. A fuzzy set μ in X is called a *fuzzy ideal* of $\mathfrak{X}$ if the following conditions are satisfied for all $x, y \in X$:
(F1) $\mu(0) \geq \mu(x)$;
(F2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$.

Denote by $\mathrm{FId}(\mathfrak{X})$ the set of all fuzzy ideals of a pseudo-BCH-algebra $\mathfrak{X}$.

Example 3.2. Let F be a field and $n \in \mathbb{N}$. Let $\mathrm{GL}(n, F)$ be the general linear group of degree n over F and let I_n denote the identity matrix. Consider the pseudo-BCH-algebra $\mathfrak{G} = (G; *, \diamond, I_n)$ given in Example 2.4 for $G = \mathrm{GL}(n, F)$. Define a fuzzy set μ in G by

$$\mu(A) = \begin{cases} \alpha_1 & \text{if } A = I_n, \\ \alpha_2 & \text{if } A = -I_n, \\ \alpha_3 & \text{if } A \in G - \{I_n, -I_n\}, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ and $\alpha_1 > \alpha_2 > \alpha_3$. It is easily checked that μ satisfies (F1) and (F2). Thus $\mu \in \mathrm{FId}(\mathfrak{G})$.

Example 3.3. Let I be an ideal of a pseudo-BCH-algebra $\mathfrak{X}$ and let $\alpha, \beta \in [0; 1]$ with $\alpha \geq \beta$. Define $\mu_I^{\alpha,\beta}$ as follows:

$$\mu_I^{\alpha,\beta}(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise.} \end{cases}$$

We denote $\mu_I^{\alpha,\beta} = \mu$. Since $0 \in I$, $\mu(0) = \alpha \geq \mu(x)$ for all $x \in X$. To prove (F2), let $x, y \in X$. If $x \in I$, then $\mu(x) = \alpha \geq \mu(x * y) \wedge \mu(y)$. Suppose now that $x \notin I$. By the definition of an ideal, $x * y \notin I$ or $y \notin I$. Therefore, $\mu(x * y) \wedge \mu(y) = \beta = \mu(x)$. Thus μ is a fuzzy ideal of $\mathfrak{X}$.
In particular, the characteristic function χ_I of I:

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy ideal of $\mathfrak{X}$.

Proposition 3.4. *Let μ be a fuzzy ideal of a pseudo-BCH-algebra $\mathfrak{X}$. Then, for any $x, y \in X$, if $x \preceq y$, then $\mu(x) \geq \mu(y)$.*

Proof. If $x \preceq y$, then $x * y = 0$. Hence, by (F2), we have $\mu(x) \geq \mu(x*y) \wedge \mu(y) = \mu(0) \wedge \mu(y) = \mu(y)$. ■

Proposition 3.5. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. A fuzzy set μ in X is a fuzzy ideal of $\mathfrak{X}$ if and only if μ satisfies* (F1) *and*

(F2') $\mu(x) \geq \mu(x \diamond y) \wedge \mu(y)$.

Proof. It suffices to prove that if (F2) is satisfied, then (F2') is also satisfied. The proof of the converse of this implication is analogous. From Proposition 2.5 (a) we know that $x * (x \diamond y) \preceq y$. Thus, by Proposition 3.4, $\mu(y) \leq \mu(x * (x \diamond y))$. Hence

$$\mu(x \diamond y) \wedge \mu(y) \leq \mu(x \diamond y) \wedge \mu(x * (x \diamond y)). \tag{1}$$

By (F2),

$$\mu(x \diamond y) \wedge \mu(x * (x \diamond y)) \leq \mu(x). \tag{2}$$

Applying (1) and (2) we obtain $\mu(x \diamond y) \wedge \mu(y) \leq \mu(x)$, so (F2') holds. ■

Proposition 3.6. *Let μ be a fuzzy ideal of a pseudo-BCH-algebra $\mathfrak{X}$. Then*

$$\mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x)) \geq \mu(x)$$

for all $x \in X$.

Proof. Let μ be a fuzzy ideal of $\mathfrak{X}$ and let $x \in X$. By Proposition 2.5 (c), $\mu(0 * (0 \diamond x)) = \mu(0 \diamond (0 * x))$. Applying (F2), (pBCH-2), (pBCH-1) and (F1) we have

$$\begin{aligned} \mu(0 \diamond (0 * x)) &\geq \mu((0 \diamond (0 * x)) * x) \wedge \mu(x) = \\ &= \mu((0 * x) \diamond (0 * x)) \wedge \mu(x) = \mu(0) \wedge \mu(x) = \mu(x) \end{aligned}$$

and the proof is complete. ■

Proposition 3.7. *Let μ be a fuzzy ideal of a pseudo-BCH-algebra $\mathfrak{X}$. Let $\widetilde{\mu}$ be the fuzzy set defined by*

$$\widetilde{\mu}(x) = \mu(0 * (0 \diamond x))$$

for any $x \in X$. Then $\widetilde{\mu}$ is a fuzzy ideal of $\mathfrak{X}$ and $\widetilde{\mu} \geq \mu$.

Proof. By Proposition 3.6, $\widetilde{\mu}(x) = \mu(0 * (0 \diamond x)) \geq \mu(x)$ for any $x \in X$ and hence $\widetilde{\mu} \geq \mu$.

Now we show that $\widetilde{\mu}$ is a fuzzy ideal of $\mathfrak{X}$. Let $x, y \in X$. Since μ is a fuzzy ideal, we obtain $\widetilde{\mu}(0) = \mu(0 * (0 \diamond 0)) = \mu(0) \geq \mu(x)$. Thus (F1) holds.

Applying Proposition 2.5 we get

$$\begin{aligned}\widetilde{\mu}(x*y) &= \mu(0*(0\diamond(x*y))) = \\ &= \mu(0\diamond(0*(x*y))) = \\ &= \mu(0\diamond((0\diamond x)\diamond(0*y))) = \\ &= \mu((0*(0\diamond x))*(0\diamond(0*y))).\end{aligned}$$

Then

$$\begin{aligned}\widetilde{\mu}(x*y)\wedge\widetilde{\mu}(y) &= \mu((0*(0\diamond x))*(0\diamond(0*y)))\wedge\mu(0\diamond(0*y)) \leq \\ &\leq \mu(0*(0\diamond x)) = \widetilde{\mu}(x),\end{aligned}$$

since μ is a fuzzy ideal. Hence $\widetilde{\mu}$ satisfies (F2). Thus $\widetilde{\mu}$ is a fuzzy ideal of $\mathfrak{X}$. ■

Proposition 3.8. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. A fuzzy set μ in X is a fuzzy ideal of $\mathfrak{X}$ if and only if it satisfies* (F1) and

(F3) *for all $x, y, z \in X$, if $(x*y)*z = 0$, then $\mu(x) \geq \mu(y)\wedge\mu(z)$.*

Proof. Let $\mu \in \mathrm{FId}(\mathfrak{X})$ and $x, y, z \in X$. Suppose that $(x*y)*z = 0$. By (F2), $\mu(x*y) \geq \mu((x*y)*z)\wedge\mu(z) = \mu(0)\wedge\mu(z) = \mu(z)$ and $\mu(x) \geq \mu(x*y)\wedge\mu(y)$. Therefore, $\mu(x) \geq \mu(y)\wedge\mu(z)$.

Conversely, let μ satisfy (F3). Applying (pBCH-1) we have $(x*y)*z = 0$, where $z = x*y$. From (F3) it follows that $\mu(x) \geq \mu(y)\wedge\mu(z) = \mu(y)\wedge\mu(x*y)$. Thus μ satisfies (F2) and hence μ is a fuzzy ideal of $\mathfrak{X}$. ■

Proposition 3.9. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. A fuzzy set μ in X is a fuzzy ideal of $\mathfrak{X}$ if and only if it satisfies* (F1) and
(F3') *for all $x, y, z \in X$, if $(x\diamond y)\diamond z = 0$, then $\mu(x) \geq \mu(y)\wedge\mu(z)$.*

Proof. Similar to the proof of Proposition 3.8. ■

Theorem 3.10. *Let μ be a fuzzy set of a pseudo-BCH-algebra $\mathfrak{X}$. Then μ is a fuzzy ideal if and only if its nonempty level subset*

$$U(\mu;\alpha) := \{x \in X : \mu(x) \geq \alpha\}$$

is an ideal of $\mathfrak{X}$ for all $\alpha \in [0,1]$.

Proof. Assume that $\mu \in \mathrm{FId}(\mathfrak{X})$ and let $\alpha \in [0,1]$ be such that $U(\mu;\alpha) \neq \emptyset$. Then $\mu(x_0) \geq \alpha$ for some $x_0 \in X$. Since $\mu(0) \geq \mu(x_0)$, we have $0 \in U(\mu;\alpha)$. Let $x, y \in X$ be such that $x*y, y \in U(\mu;\alpha)$. Then $\mu(x*y) \geq \alpha$ and $\mu(y) \geq \alpha$. It follows from (F2) that

$$\mu(x) \geq \mu(x*y)\wedge\mu(y) \geq \alpha,$$

so that $x \in U(\mu;\alpha)$. Therefore $U(\mu;\alpha)$ is an ideal of $\mathfrak{X}$.

Conversely, suppose that for each $\alpha \in [0,1]$, $U(\mu;\alpha) = \emptyset$ or $U(\mu;\alpha)$ is an ideal of $\mathfrak{X}$. If (F1) is not valid, then there exists $x_0 \in X$ such that $\mu(0) < \mu(x_0) := \beta$. Then $U(\mu;\beta) \neq \emptyset$ and by assumption, $U(\mu;\beta)$ is an ideal of $\mathfrak{X}$. Hence $0 \in U(\mu;\beta)$ and consequently, $\mu(0) \geq \beta$. This is a contradiction and (F1) is valid. Now assume that (F2) does not hold. Then there are $a, b \in X$ such that $\mu(a) < \mu(a * b) \wedge \mu(b)$. Taking

$$\beta = \tfrac{1}{2}(\mu(a) + \mu(a * b) \wedge \mu(b)),$$

we get $\mu(a) < \beta < \mu(a * b) \wedge \mu(b) \leq \mu(a * b)$ and $\beta < \mu(b)$. Therefore $a * b, b \in U(\mu;\beta)$ but $a \notin U(\mu;\beta)$. This is impossible, and μ is a fuzzy ideal of $\mathfrak{X}$. ■

Example 3.11. Consider the pseudo-BCH-algebra $\mathfrak{X} = (X; *, \diamond, 0)$ given in Example 2.10. Let μ be a fuzzy set in X such that $\mu(0) = \alpha_1, \mu(a) = \alpha_2, \mu(b) = \mu(c) = \alpha_3$, and $\mu(d) = \alpha_4$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$ and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. Observe that μ is a fuzzy ideal of $\mathfrak{X}$. It is easy to check that for all $\alpha \in [0;1]$ we have

$$U(\mu;\alpha) = \begin{cases} \emptyset & \text{if } \alpha > \alpha_1, \\ \{0\} & \text{if } \alpha_2 < \alpha \leq \alpha_1, \\ \{0,a\} & \text{if } \alpha_3 < \alpha \leq \alpha_2, \\ \{0,a,b,c\} & \text{if } \alpha_4 < \alpha \leq \alpha_3, \\ X & \text{if } \alpha \leq \alpha_4. \end{cases}$$

Since $\{0\}, \{0,a\}, \{0,a,b,c\}$ and X are ideals of $\mathfrak{X}$, from Theorem 3.10 we conclude that μ is a fuzzy ideal of $\mathfrak{X}$.

Corollary 3.12. *If μ is a fuzzy ideal of a pseudo-BCH-algebra $\mathfrak{X}$, then the set*

$$X_b := \{x \in X : \mu(x) \geq \mu(b)\}$$

is an ideal of $\mathfrak{X}$ for every $b \in X$.

By Corollary 3.12, we have the following.

Corollary 3.13. *If μ is a fuzzy ideal of a pseudo-BCK algebra $\mathfrak{X}$, then the set*

$$X_\mu := \{x \in X : \mu(x) = \mu(0)\}$$

is an ideal of $\mathfrak{X}$.

The following example shows that the converse of Corollary 3.13 does not hold.

Example 3.14. Let $\mathfrak{X}$ be a pseudo-BCK-algebra. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} 0.5 & \text{if } x = 0, \\ 0.6 & \text{if } x \neq 0. \end{cases}$$

Then $X_\mu = \{0\}$ and it is an ideal of $\mathfrak{X}$ but $\mu \notin \mathrm{FId}(\mathfrak{X})$, because μ does not satisfy (F1).

Let T be a nonempty set of indexes. Let $\mu_t \in \mathrm{FId}(\mathfrak{X})$ for $t \in T$. The meet $\bigwedge_{t\in T} \mu_t$ of fuzzy ideals μ_t of $\mathfrak{X}$ is defined as follows:

$$\left(\bigwedge_{t\in T} \mu_t\right)(x) = \bigwedge\{\mu_t(x) : t \in T\}.$$

Proposition 3.15. *Let* $\mu_t \in \mathrm{FId}(\mathfrak{X})$ *for* $t \in T$. *Then* $\bigwedge_{t\in T} \mu_t \in \mathrm{FId}(\mathfrak{X})$.

Proof. Let $\mu = \bigwedge_{t\in T} \mu_t$. Then, by (F1),

$$\mu(0) = \bigwedge\{\mu_t(0) : t \in T\} \geq \bigwedge\{\mu_t(x) : t \in T\} = \mu(x)$$

for all $x \in X$. Let $x, y \in X$. Since $\mu_t \in \mathrm{FId}(\mathfrak{X})$, we have $\mu_t(x) \geq \mu_t(x * y) \wedge \mu_t(y)$. Hence

$$\begin{aligned} \bigwedge\{\mu_t(x) : t \in T\} &\geq \bigwedge\{\mu_t(x * y) \wedge \mu_t(y) : t \in T\} \\ &= \bigwedge\{\mu_t(x * y) : t \in T\} \wedge \bigwedge\{\mu_t(y) : t \in T\}. \end{aligned}$$

Consequently, $\mu(x) \geq \mu(x * y) \wedge \mu(y)$ and therefore $\mu \in \mathrm{FId}(\mathfrak{X})$. ■

Let ν be a fuzzy set in X. A fuzzy ideal μ of $\mathfrak{X}$ is said to be *generated by* ν if $\nu \leq \mu$ and for any fuzzy ideal ρ of $\mathfrak{X}$, $\nu \leq \rho$ implies $\mu \leq \rho$. The fuzzy ideal generated by ν will be denoted by $(\nu]$. The fuzzy ideal $(\nu]$ we can define equivalently as follows:

$$(\nu] = \bigwedge\{\rho \in \mathrm{FId}(\mathfrak{X}) : \rho \geq \nu\}.$$

For $\mu, \nu \in \mathrm{FId}(\mathfrak{X})$ let $\mu \vee \nu$ denote the join of μ and ν, that is, $\mu \vee \nu = (\rho]$, where ρ is the fuzzy set in X defined by $\rho(x) = \mu(x) \vee \nu(x)$ for all $x \in X$.

From Proposition 3.15 we obtain the following theorem.

Theorem 3.16. *Let* $\mathfrak{X}$ *be a pseudo-BCH-algebra. Then* $(\mathrm{FId}(\mathfrak{X}); \wedge, \vee)$ is a complete lattice.

The following two theorems give the homomorphic properties of fuzzy ideals.

Theorem 3.17. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras and let $f : X \to Y$ be a homomorphism and $\nu \in \mathrm{FId}(\mathfrak{Y})$. Then $f^{\leftarrow}(\nu) \in \mathrm{FId}(\mathfrak{X})$.*

Proof. Let $x \in X$. Since $f(x) \in Y$ and $\nu \in \mathrm{FId}(\mathfrak{Y})$, we have $\nu(0) \geq \nu(f(x)) = (f^{\leftarrow}(\nu))(x)$, but $\nu(0) = \nu(f(0)) = (f^{\leftarrow}(\nu))(0)$. Thus we get $(f^{\leftarrow}(\nu))(0) \geq (f^{\leftarrow}(\nu))(x)$ for any $x \in X$, that is, $f^{\leftarrow}(\nu)$ satisfies (F1).

Now let $x, y \in X$. Since $\nu \in \mathrm{FId}(\mathfrak{Y})$, we obtain

$$\nu(f(x)) \geq \nu(f(x) * f(y)) \wedge \nu(f(y)) = \nu(f(x * y)) \wedge \nu(f(y))$$

and hence $(f^{\leftarrow}(\nu))(x) \geq (f^{\leftarrow}(\nu))(x * y) \wedge (f^{\leftarrow}(\nu))(y)$. Consequently, $f^{\leftarrow}(\nu) \in \mathrm{FId}(\mathfrak{X})$. ■

Lemma 3.18. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras and let $f : X \to Y$ be a homomorphism and $\mu \in \mathrm{FId}(\mathfrak{X})$. Then, if μ is constant on $\ker f = f^{\leftarrow}(0)$, then $f^{\leftarrow}(f(\mu)) = \mu$.*

Proof. Let $x \in X$ and $f(x) = y$. Hence

$$(f^{\leftarrow}(f(\mu)))(x) = (f(\mu))(f(x)) = (f(\mu))(y) = \bigvee\{\mu(a) : a \in f^{\leftarrow}(y)\}.$$

For all $a \in f^{\leftarrow}(y)$, we have $f(x) = f(a)$. Hence $f(a * x) = 0$, i.e., $a * x \in \ker f$. Thus $\mu(a * x) = \mu(0)$. Therefore, $\mu(a) \geq \mu(a * x) \wedge \mu(x) = \mu(0) \wedge \mu(x) = \mu(x)$. Similarly, $\mu(x) \geq \mu(a)$. Hence $\mu(x) = \mu(a)$. Thus

$$(f^{\leftarrow}(f(\mu)))(x) = \bigvee\{\mu(a) : a \in f^{\leftarrow}(y)\} = \mu(x),$$

that is, $f^{\leftarrow}(f(\mu)) = \mu$. ■

Theorem 3.19. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras and let $f : X \to Y$ be a surjective homomorphism and $\mu \in \mathrm{FId}(\mathfrak{X})$ such that $A_\mu \supseteq \ker f$. Then $f(\mu) \in \mathrm{FId}(\mathfrak{Y})$.*

Proof. Since μ is a fuzzy ideal of $\mathfrak{X}$ and $0 \in f^{\leftarrow}(0)$, we have

$$(f(\mu))(0) = \bigvee\{\mu(a) : a \in f^{\leftarrow}(0)\} = \mu(0) \geq \mu(x)$$

for any $x \in X$. Hence

$$(f(\mu))(0) \geq \bigvee\{\mu(x) : x \in f^{\leftarrow}(y)\} = (f(\mu))(y)$$

for any $y \in Y$. Thus $f(\mu)$ satisfies (F1). Suppose that

$$(f(\mu))(y_1) < (f(\mu))(y_1 * y_2) \wedge (f(\mu))(y_2)$$

for some $y_1, y_2 \in Y$. Since f is surjective, there are $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence

$$(f(\mu))(f(x_1)) < (f(\mu))(f(x_1 * x_2)) \wedge (f(\mu))(f(x_2)).$$

Therefore

$$(f^{\leftarrow}(f(\mu)))(x_1) < (f^{\leftarrow}(f(\mu)))(x_1 * x_2) \wedge (f^{\leftarrow}(f(\mu)))(x_2).$$

Since $A_\mu \supseteq \ker f$, μ is constant on $\ker f$. Then, by Lemma 3.18, we get

$$\mu(x_1) < \mu(x_1 * x_2) \wedge \mu(x_2),$$

which is a contradiction with the fact that μ is a fuzzy ideal. Thus, we obtain $f(\mu) \in \mathrm{FId}(\mathfrak{Y})$. ■

4 Fuzzy characterizations of Noetherian and Artinian pseudo-BCH-algebras

In this section we characterize Noetherian pseudo-BCH-algebras and Artinian pseudo-BCH-algebras using some fuzzy concepts, in particular, fuzzy ideals.

A pseudo-BCH-algebra $\mathfrak{X}$ is called *Noetherian* if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of $\mathfrak{X}$ there exists $k \in N$ such that $I_n = I_k$ for all $n \geq k$. A pseudo-BCH-algebra $\mathfrak{X}$ is called *Artinian* if for every descending sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of $\mathfrak{X}$ there exists $k \in N$ such that $I_n = I_k$ for all $n \geq k$.

We first prove

Lemma 4.1. *Let $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ be a strictly ascending sequence of ideals in a pseudo-BCH-algebra $\mathfrak{X}$ and (t_n) be a strictly decreasing sequence in $(0;1)$. Let μ be the fuzzy set in X defined by*

$$\mu(x) = \begin{cases} 0 & if\ x \notin I_n \text{ for each } n \in \mathbb{N}, \\ t_n & if\ x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases}$$

where $I_0 = \emptyset$. Then μ is a fuzzy ideal of $\mathfrak{X}$.

Proof. Let $I = \bigcup_{n \in \mathbb{N}} I_n$. By Remark 2.11, I is an ideal of $\mathfrak{X}$. Obviously, $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$, that is, (F1) holds. Now we show that μ satisfies (F2). Let $x, y \in X$. We have two cases.

Case 1: $x \notin I$.

Then $x * y \notin I$ or $y \notin I$. Therefore $\mu(x * y) \wedge \mu(y) = 0 = \mu(x)$.

Case 2: $x \in I_n - I_{n-1}$ for some $n \in N$.
Then $x * y \notin I_{n-1}$ or $y \notin I_{n-1}$. Hence $\mu(x * y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x * y) \wedge \mu(y) \leq t_n = \mu(x)$.
Thus (F2) is also satisfied and consequently, μ is a fuzzy ideal of $\mathfrak{X}$. ∎

Theorem 4.2. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. The following statements are equivalent:*
(a) *$\mathfrak{X}$ is Noetherian,*
(b) *for each fuzzy ideal μ of $\mathfrak{X}$, $\mathrm{Im}(\mu) := \{\mu(x) : x \in X\}$ is a well-ordered set.*

Proof. (a) $\Longrightarrow$ (b): Assume that $\mathfrak{X}$ is Noetherian and μ is a fuzzy ideal of $\mathfrak{X}$ such that $\mathrm{Im}(\mu)$ is not a well-ordered subset of $[0;1]$. Then there exists a strictly decreasing sequence $(\mu(x_n))$, where $x_n \in X$. Let $t_n = \mu(x_n)$ and $U_n = U(\mu; t_n) = \{x \in X : \mu(x) \geq t_n\}$. Then, by Theorem 3.10, U_n is an ideal of $\mathfrak{X}$ for every $n \in \mathbb{N}$. So $U_1 \subset U_2 \subset \ldots$ is a strictly ascending sequence of ideals of $\mathfrak{X}$. This is a contradiction with the assumption that $\mathfrak{X}$ is Noetherian. Therefore $\mathrm{Im}(\mu)$ is a well-ordered set for each fuzzy ideal μ of $\mathfrak{X}$.

(b) $\Longrightarrow$ (a): Assume that (b) is true. Suppose that $\mathfrak{X}$ is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ of ideals of $\mathfrak{X}$. Let μ be a fuzzy set in X such that

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\ \frac{1}{n} & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases}$$

where $I_0 = \emptyset$. By Lemma 4.1, $\mu \in \mathrm{FId}(\mathfrak{X})$, but $\mathrm{Im}(\mu)$ is not a well-ordered set, which is imposible. Therefore $\mathfrak{X}$ is Noetherian. ∎

Corollary 4.3. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. If for every fuzzy ideal μ of $\mathfrak{X}$, $\mathrm{Im}(\mu)$ is a finite set, then $\mathfrak{X}$ is Noetherian.*

Theorem 4.4. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0\}$, where (t_n) is a strictly decreasing sequence in $(0;1)$. Then the following conditions are equivalent:*
(a) *$\mathfrak{X}$ is Noetherian,*
(b) *for each fuzzy ideal μ of $\mathfrak{X}$, if $\mathrm{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.*

Proof. (a) $\Longrightarrow$ (b): Assume that $\mathfrak{X}$ is Noetherian. Let μ be a fuzzy ideal of $\mathfrak{X}$ such that $\mathrm{Im}(\mu) \subseteq T$. From Theorem 4.2 we know that $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0;1]$. Then, since $1 > t_1 > t_2 > \cdots > t_n > \cdots > 0$ and $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots\} \cup \{0\}$, there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.

(b) $\Longrightarrow$ (a): Assume that (b) is true. Suppose that $\mathfrak{X}$ is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \ldots$ of ideals of $\mathfrak{X}$.

Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\ t_n & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases}$$

where $I_0 = \emptyset$. By Lemma 4.1, μ is a fuzzy ideal of $\mathfrak{X}$. This is a contradiction with our assumption. Thus $\mathfrak{X}$ is Noetherian. ■

Theorem 4.5. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0, 1\}$, where (t_n) is a strictly increasing sequence in $(0;1)$. Then the following conditions are equivalent:*
(a) *$\mathfrak{X}$ is Artinian,*
(b) *for each fuzzy ideal μ of $\mathfrak{X}$, if $\mathrm{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0, 1\}$.*

Proof. (a) $\Longrightarrow$ (b): Suppose that $t_{i_1} < t_{i_2} < \cdots < t_{i_m} < \cdots$ is a strictly increasing sequence of elements of $\mathrm{Im}(\mu)$. Let $U_m = U(\mu; t_{i_m})$ for $m \in \mathbb{N}$. It is immediately seen that $U_1 \supset U_2 \supset \cdots \supset U_m \supset \cdots$ is a strictly descending sequence of ideals of $\mathfrak{X}$. This contradicts the assumption that $\mathfrak{X}$ is Artinian.

(b) $\Longrightarrow$ (a): Assume that (b) is true. Suppose that $\mathfrak{X}$ is not Artinian. Then there exists a strictly descending sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of ideals of $\mathfrak{X}$. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I_1, \\ t_n & \text{if } x \in I_n - I_{n+1} \text{ for } n = 1, 2, \ldots, \\ 1 & \text{if } x \in \bigcap\{I_n : n \in \mathbb{N}\}. \end{cases}$$

Obviously, $\mu(0) = 1 \geq \mu(x)$ for all $x \in X$, that is, (F1) holds. Now we show that μ satisfies (F2). Let $x, y \in X$. We have three cases.

Case 1: $x \notin I_1$.
Then $x * y \notin I_1$ or $y \notin I_1$. Therefore $\mu(x * y) \wedge \mu(y) = 0 = \mu(x)$.

Case 2: $x \in I_n - I_{n+1}$ for some $n \in \mathbb{N}$.
Then $x * y \notin I_{n+1}$ or $y \notin I_{n+1}$. Hence $\mu(x * y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x * y) \wedge \mu(y) \leq t_n = \mu(x)$.

Case 3: $x \in \bigcap\{I_n : n \in \mathbb{N}\}$.
Obvious.

Thus μ is a fuzzy ideal of $\mathfrak{X}$. This contradicts our assumption. Thus $\mathfrak{X}$ is Artinian. ■

Corollary 4.6. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. If for every fuzzy ideal μ of $\mathfrak{X}$, $\mathrm{Im}(\mu)$ is a finite set, then $\mathfrak{X}$ is Artinian.*

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