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On the convergence of sets and the approximation property for dynamic equations on time scales

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Abstract

The main goal of the paper is to propose a new approach to the problem of approximation of solutions for differential problems. A standard approach is based on discrete approximations. We replace it by a sequence of dynamic equations. In this paper, we investigate the convergence of closed sets being domains of considered problems, i.e. time scales. Then we apply our results for the study of an approximation property of dynamic equations. Our results allow us to characterize a set of solutions for differential problems as a limit of a sequence of dynamic ones.

We point out a kind of convergence of time scales which is applicable and most useful for the study of continuous dependence of solutions for dynamic equations on time scales. It forms an approximation for the differential equations by dynamic equations and allows us to extend the difference approach in numerical algorithms. Finally, we study some Cauchy problems without uniqueness of solutions, which are approximated by simple dynamic problems.

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1 Introduction

When in 1988, Stefan Hilger introduced the calculus of time scales (measure chains) in order to unify continuous and discrete analysis, his supervisor Bernd Aulbach pointed out the three main purposes of this new calculus: unification of separately considered cases, some of their extensions and finally a discretization of continuous problems. The first two goals are widely investigated. We concentrate on the last one. Many results concerning differential problems carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamical systems should help us to understand such a situation and to avoid proving results twice once for differential equations and once again for difference equations. But in some cases the "continuous" differential problem seems to be quite complicated, so the discretized version, i.e. the difference or dynamic equation should be treated as an approximation for an original one. Since it was known at the beginning of time scale calculus, it is a little bit surprising that these kind of results are not intensively studied. The difference equations are usually used as approximations of some differential equations, but in the more general context of dynamic equations it is not well described. This is a basic step for numerical analysis on time scales.

Moreover, some mathematical models are considered for both continuous and discrete versions: in biology ([20] or for population dynamics [29]), economics ([2]) or in the control theory ([3]), for instance. In such cases, it was proposed a unification for both approaches by using dynamical equations. If there is a problem with solving such an equation, then some numerical procedures are used.

If we have a mathematical model based on differential problem, we replace a derivative by some difference scheme, so we approximate it. We propose another approach. We approximate the domain of the considered equation (by a time scale) and we solve exactly the problem on the approximated set. In this paper we describe the method how to construct a sequence of sets convergent to original domain. The construction of approximated problem is based on time scale calculus, so we consider the convergence of sequences of time scales. Instead of different numerical schemes we propose to consider sequences of time scales convergent to the original domain of the problem. In the paper we present a characterization of such a convergence allowing us to extend standard procedures. Recall, that discrete time scales (similar to the usual approach) can be also considered (cf. Section 5.1).

The main idea is clear: by taking a time scale $\mathbb{T}$ which is "close", in some sense, to $\mathbb{R}$ (or another time scale $\mathbb{S}$) we expect that the solution (if exists) of one problem are "close" to solutions for the second one. In fact, we will expect the convergence of sets of solutions when time scales are convergent (continuous dependence of solutions on a time scale). Let us note that the convergence of times scales can replace the previously proposed ([1]) approach: bot problems should be considered on a set of common points for the "target" time scale and an approximated one. The paper is supplemented by an overview of earlier attempts of this type allowing us to carry out the comparative discussion. Let us stress, that the main goal of this paper is to explain the idea of convergence of domains for approximated problems and to compare it with earlier ones. It is the reason for emphasizing on examples and comparative lemmas.

Our main purpose is to define the convergence which is "proper" in the context of dynamic equations. Because such a kind of results is not restricted to dynamic equations, we will compare the proposed algorithm with existing results, we will extend them and *unify* previously introduced types of convergence. Thus we will apply our scheme of proofs for some problems, even for the case which is not considered in difference equations, namely for the problem for which lack of uniqueness of solutions.

There are some earlier results of this type (see [1, 13, 14, 17, 16, 23, 31]). Unfortunately, almost all of them are devoted to study the case of compact time scales and the convergence is considered in the sense of the Hausdorff distance [1, 17, 23, 18]. This is a strong requirement and cannot be extended to the case of arbitrary time scale. There are also another approaches ([13, Definition 22] or when the convergence is taken with respect to the Fell topology [14, 31, 32]). Nevertheless, it is a little bit surprising that they are introduced independently on existing types of convergence in multivalued analysis. We compare the convergence in some topologies defined on families of closed subsets of a space $\mathbb{R}$ and we propose to take the "best" one. In our opinion, the convergence in the sense of Kuratowski as a "proper" choice. Our comparison result for convergence of sets allows us to treat the previous results in a unified manner (at least for the case of time scales). We indicate some new tools allowing us to investigate the convergence of time scales associated with our approximation problem. This paper is complemented by a series of illustrative examples.

We focus on some special time scales, allowing us to solve dynamic problems. In particular, we are interested in covering difference equations (usually Euler scheme with possibly variable time steps $z_{n+1} = z_n + h_n f(z_n)$), but we are not restricted only to this case and we investigate the general case.

2 Time scales and dynamic equations

In this section we briefly recall some basics about time scales and to introduce some notation (see [10] and references therein).

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. By $\mathbb{R}_+$ we will denote the interval $[0,+\infty)$.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum (q-difference) calculus i.e., when $\mathbb{T}=\mathbb{R}, \mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{Z}}=\{q^t : t\in\mathbb{Z}\}\cup\{0\}$, where $q>1$.

Definition 2.1. *The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are defined by $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}$, respectively.*

We put $\inf\emptyset=\sup\mathbb{T}$ (i.e., $\sigma(M)=M$ if $\mathbb{T}$ has a maximum M) and $\sup\emptyset=\inf\mathbb{T}$ (i.e., $\rho(m)=m$ if $\mathbb{T}$ has a minimum m).

The jump operators σ and ρ allow to classify the points in time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$, $\rho(t)<t$, $\rho(t)=t=\sigma(t)$ and $\rho(t)<t<\sigma(t)$, respectively.

The mapping $\mu(t)=\sigma(t)-t$ will be called the graininess of $\mathbb{T}$.

Definition 2.2. *Let $f:\mathbb{T}\to X$ and $t\in\mathbb{T}$. Then define the Δ-derivative $f^{\Delta}(t)$ to be the element of X (if it exists) with the property that for any $\varepsilon>0$ there exists a neighbourhood of t on which*

$$\left\|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right\|\le\varepsilon|\sigma(t)-s|.$$

Remark 2.3. *Concerning the Δ-derivative, it turns out that*

(i) $f^{\Delta}=f'$ *is the usual derivative if* $\mathbb{T}=\mathbb{R}$,

(ii) $f^{\Delta}=\Delta f$ *is the usual forward difference operator if* $\mathbb{T}=\mathbb{Z}$, *and*

(iii) $f^{\Delta}=\Delta_q f$ *is the q-derivative if* $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$, $q>1$.

The last property is useful in our consideration:

Lemma 2.4. ([10]) *If the Δ-derivative exists at some point $t\in\mathbb{T}$, then $x(\sigma(t))-x(t)=\mu(t)\cdot x^{\Delta}(t)$.*

3 Convergence of time scales

A very basic question is to discuss and then to define a method for the convergence of time scales, which is useful in our approach. It means, that we need to choose a kind of convergence allowing us to construct some approximated solutions for a given differential problem.

On a family of all closed sets there are many topologies (see [4, 30]). Our approach is based on the fact that for a given time scale $\mathbb{T}$, we *choose* their approximation $\mathbb{T}_1$ in such a way to solve the problem on $\mathbb{T}_1$ and to treat this solution as an approximated solution for the original one. The is one natural requirement - the existence of solutions on $\mathbb{T}$ (but not necessarily the uniqueness of solutions). On the other hand, it means that we construct step-by-step a *sequence* of time scales (rather than the net). Thus the desired "nice" topology of convergence of time scales should be either metrizable or we need to control only sequences of sets. In earlier papers a few topologies were considered. However, we prefer to check sequences of time scales and we will concentrate on the sequential convergence.

Let us present a short historical background. We need to collect all the related results justifying our choice of convergence.

A starting point for a discussion should be the Hausdorff metric (see [4], for instance). By $Cl(X)$ we will denote a family of closed subsets of a metric (X, d). Define a distance function between a point and sets: $dist(a, B) = \inf_{b \in B} d(a, b)$.

For $X, Y \in Cl(X)$ let us define the following quantities:

$$e(X,Y) = \sup_{x \in X} dist(x,Y) = \sup_{x \in X} \inf_{y \in Y} d(x,y) = \inf\{r > 0 \ : \ X \subset B(Y,r)\},$$

and

$$h(X,Y) = \max\{e(X,Y), e(Y,X)\} \ .$$

By convention, $e(\emptyset, Y) = 0$. The quantity $e(X, Y)$ is called the excess (or: non-symmetric distance) between X and Y while $h(X, Y)$ is said to be the Hausdorff distance between X and Y.

Unfortunately, for unbounded closed subsets of X, their Hausdorff distance can be equal to $+\infty$ ($h([-n, n], \mathbb{R}) = +\infty$ for any $n \geq 1$, for instance), so we may exclude this convergence from our consideration (at least for unbounded time scales) (cf. also [14, Section 3]. It is too restrictive type of convergence. Note that for compact time scales it was proposed as a main tool for such a kind of results (see [1, 17, 23]).

Nevertheless, we should be very careful with conclusions, because even equivalent metrics can give us different hyperspaces of closed sets (cf. a version for the truncated metric, non-equivalent to euclidean metric on $\mathbb{R}$ in [31]) and then such a kind of results strictly depends on a considered metric!

Another type of convergence was proposed in [13, Definition 22]. It was observed by the authors that it is not the optimal choice. This definition is based on some ideas taken from the Kuratowski limit of sets, but it is not in a correct form. Note that in this sense we have no convergence for $\mathbb{T}_n = [-n, n]$ (condition (3) is not satisfied) or for $\mathbb{T}_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ (not a closed set cf. the definition of LiA_n below). We will discuss the Kuratowski convergence too.

In [14, 31, 32] the Viétoris topology was investigated, so we are able to exclude this topology from our considerations (see [4]). Note that on a compact space (for bounded time scales, for instance) this topology agree with the Fell topology ([30]) which will be described below.

Finally, the Fell topology should be considered ([14]). This topology is generated by two families of sets. For each subset A of X define

$$A^- = \{B \in 2^X : B \cap A \neq \emptyset\},$$

and

$$A^+ = \{B \in 2^X : B \subset A\}.$$

The *lower Fell topology* on 2^X is generated by subbasis of open sets of the form W^- where W is an open subset of X. The *upper Fell topology* on 2^X is generated by subbasis of open sets of the form C^+ where C has compact complement in X.

Outside the class of locally compact spaces this topology cannot be useful in our investigations (it is not a Hausdorff topology, in general, so we can expect nets convergent to two different limits). Fortunately, all time scales are obviously locally compact (which is not true for metric spaces, cf. [14, Theorem 4.6]).

It is hard to use such a definition for the investigation of convergence of sets. Nevertheless, we have a very interesting characterization:

Proposition 3.1. ([4, Corollary 5.1.7] or [14, Theorem 5.3]) *A sequence of closed sets A_n in $\mathbb{R}$ is convergent to A under the Fell topology if and only if for all compact subsets $K \subset \mathbb{R}$ we have* $\lim_{n\to\infty} e(K \cap A, A_n) = 0$ *and* $\lim_{n\to\infty} e(K \cap A_n, A) = 0$.

In the case of $A = \mathbb{R}$ the second condition is superfluous (as it is always satisfied). This proposition allows us to check the convergence in the Fell topology in the language of excesses between the sets, which will be used later. This means also that the convergence in this topology is relatively easy to be checked (cf. [14, Section 4.2]) and it was suggested as the best choice for the considered problem (cf. also [1, 13, 14, 31, 32, 27]). However, we will show how to simplify this procedure.

Some properties and examples of convergent time scales can be found in [14, 31, 32] or (in an indirect form) in [1].

It seems to be necessary to investigate the convergence in yet one more sense, namely the Kuratowski (or: Kuratowski-Painlevé) convergence. Before describing this kind of convergence let us note that it is strictly related to some upper and lower semi-continuity properties of multifunctions (in the multivalued analysis). Because the set of solutions form a multifunction, it seems to be a very natural approach. Let us mention, for instance, a paper [15] in which the convergence of variational inequalities is treated by this method. In the context of time scales similar idea of convergence of attractors can be found in [24] with the Hausdorff distance of time scales.

Define the so-called upper and lower limits of a sequence $A_n \subset X$ of sets.

Definition 3.2. ([6, 26] or [28, Definition 1.9]) *Let (A_n) be a sequence of closed sets in a metric space X. Then the Kuratowski limit inferior LiA_n is the set of points $y \in X$ each neighbourhood of which meets all but finitely many sets A_n. By the Kuratowski limit superior LsA_n we will mean a set of all points $y \in X$ each neighbourhood of which meets infinitely many sets A_n.*

If $LiA_n = LsA_n = A$ we simply write $LimA_n = A$ and we will say that (A_n) is convergent to A in the sense of Kuratowski. The Kuratowski convergence is not topological (Mrówka's Theorem, see [28, Proposition 2.2]), but in an interesting case of $X = \mathbb{R}$ we have a topology associated with this kind of convergence (since this is a Hausdorff and locally compact space, cf. [5] and [28, Corollary 2.5]). It is useful to consider sequences of time scales and we do not investigate the nets. This kind of convergence is known ([13, 14, 31]), but we need to clarify the problem. We can use also the following characterizations of the Kuratowski limits:

$$LsA_n = \{x \in X : \liminf_{n\to\infty} dist(x, A_n) = 0\},$$

$$LiA_n = \{x \in X : \lim_{n\to\infty} dist(x, A_n) = 0\}$$

and

$$LsA_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} A_k} \quad and \quad \overline{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k} \subset LiA_n.$$

First, the Fell topology on a family of closed subsets of a space X is metrizable iff X is locally compact and second countable [4, Theorem 5.1.5]). As $X = \mathbb{R}$ satisfies both conditions we are able to restrict our attention to sequences of time scales. This is the reason for which just sequences are used in [14, Section 5.3]. For a general version (with nets) of Theorem 5.3 in [14], see Corollary 5.1.7 in [4].

Now, it should be clear that both the convergence in the Fell topology, as well as the Kuratowski convergence seem to be adequate for our problem. In our opinion, the last one appear to be more natural and easier (cf. our example below) and we recommend its use. In the case of an arbitrary topological space X denote by τ_F the Fell topology and by τ_K the Kuratowski topology. Indeed, we have the following:

Lemma 3.3. (cf. [30, Theorem 2.6]) *For an arbitrary topological space X we have: $\tau_F \subset \tau_K$.*

However, in the interesting case $X = \mathbb{R}$, we get the following result (cf. also [30, Theorem 1.1]):

Lemma 3.4. ([4, Theorem 5.2.10] or [28, Proposition 2.4]) *For a locally compact space X, a sequence of closed sets is convergent in τ_F if and only if it is convergent in τ_K and the limits coincides.*

We need to compare an approach from [14, Theorems 5.2, 5.3 and 5.5] with our. Let us present a special case of the above lemma. It allows us to indicate the role of compact sets in the mentioned results.

Proposition 3.5. ([4, Proposition 5.2.5]) *Consider a sequence of closed sets (A_n) in a locally compact metric space X. The following are equivalent:*

- [K1] $A \subset LiA_n$ *and for each compact subset K of X, we have* $Ls(K \cap A_n) \subset A$,
- [K2] A_n *is convergent in the Fell topology to A.*

Note that for compact spaces S the Fell topology is equivalent with the Hausdorff topology ([4, Corollary 5.1.11]). Thus for bounded time scales the Hausdorff distance is still in use. Earlier results on continuous dependence were proved for time scales "close" to each other with respect to the Hausdorff topology ([1, 17, 23]), but thanks to this remark, they can be interpreted as particular cases of our treatment.

The readers more interested on some topologies sequentially equivalent to the Kuratowski convergence are referred to the interesting papers [8, 9]. Very interesting results of Arzela-Ascoli type for sequences of sets convergent in the sense of Kuratowski can be found in [6, 7]. For a survey about different topologies on families of closed sets we refer to [4] or [28].

We need to solve one more problem: if there exists sequence of time scales convergent in the sense of Kuratowski to a fixed one? We have very interesting result:

Proposition 3.6. ([4, Theorem 5.2.12]) *Let X be second countable Hausdorff space and let A_n be a sequence in $Cl(X)$. Then (A_n) has a subsequence which is convergent in the sense of Kuratowski.*

Note that clearly $\mathbb{R}$ satisfies both assumptions from the above result. We need to construct such a sequence of time scales $(\mathbb{T}_n)$ in such a way to easily solve the considered problem on each $\mathbb{T}_n$. Such solutions will be treated as approximated solutions for the original one - as will be described later.

Let us conclude this section by recalling, that a very interesting characterization for the Kuratowski convergence (and other types of convergence of sets) in terms of distance functions can be found in [6, Section 4] or [7].

3.1 Difference approximations

In this part we will compare different types of convergence of time scales by considering some of them being of particular interest. The most important one is $\mathbb{T}_n = h_n \cdot \mathbb{Z}_+$ (the so-called Eulerian time scales cf. [19]). If $h_n \to 1$, then $\mathbb{T}_n \to \mathbb{Z}_+$ ([14, Lemma 4.3]) in the Fell topology. We will study the case $h_n \to 0$, which seems to be more interesting. In particular, we investigate the difference approach for differential problems.

It is intuitively expected that if $h_n \to 0$, then this sequence should be convergent to $\mathbb{R}_+$ (cf. [13, Section 6]. We will check this property for some different topologies and then we will compare the convergence of the obtained sequences for considered topologies. This idea was exploited in [13, Example 14].

Now, we will check the convergence in details. This will explain our main convergence theorem and our motivations.

Lemma 3.7. *Put* $\mathbb{T}_n = \frac{1}{10^n} \cdot \mathbb{Z}_+$ *and* $\mathbb{T} = \mathbb{R}_+ = [0, +\infty)$.
[H] $(\mathbb{T}_n)$ *is not convergent to* $\mathbb{T}$ *in the Hausdorff metric.*
[DHOV] $(\mathbb{T}_n)$ *is not convergent to* $\mathbb{T}$ *in the sense of [13, Definition 22].*
[F] $(\mathbb{T}_n)$ *is convergent to* $\mathbb{T}$ *in the Fell topology.*
[K] $(\mathbb{T}_n)$ *is convergent to* $\mathbb{T}$ *in the sense of the Kuratowski convergence.*

Proof: We use the previously considered properties of convergent sequences of sets..

- [H] The Hausdorff distance: $h(\mathbb{T}_n, \mathbb{T}) = +\infty$, so it is not a convergent sequence in the Hausdorff metric.

- [DHOV] (In the sense of [13, Definition 22]) It is clear that $\mathbb{T}_n$ form an increasing sequence of sets (note that for the case $h_n = (\frac{1}{\pi^n}$ this is not true!). But the condition (3) [13, Definition 22] is not satisfied: $\bigcup_{n=1}^{\infty} \mathbb{T}_n$ is not equal $\mathbb{T}$. There is no convergence of $\mathbb{T}_n$ in this sense.

- [F] The Fell topology: it is clear that we can use Proposition 3.1 instead of the direct use of the definition, which simplify the proof. This will be also one of our goals of the paper - to indicate equivalent conditions for the Kuratowski (or: Fell) convergence. This will made convergence easy to be checked.

  Take an arbitrary compact set $K \subset \mathbb{R}_+$. Then $K \cap \mathbb{T} = K \cap \mathbb{R}_+ = K$ and $e(K \cap \mathbb{T}, \mathbb{T}_n) = \frac{1}{10^n}$. Finally, $\lim_{n\to\infty} e(K \cap \mathbb{T}, \mathbb{T}_n) = 0$. We need to check the second condition. For arbitrary sets A, B, we have $e(A, B) = 0 \iff A \subset \overline{B}$ and $K \cap \mathbb{T}_n \subset \mathbb{T}$, we get $e(K \cap \mathbb{T}_n, \mathbb{T}) = 0$ for any n. Note that $K \cap \mathbb{T}_n$ is a finite set. By Proposition 3.1 this sequence is convergent with respect to the Fell topology to $\mathbb{R}_+$.

  To show some advantages of the result by Esty and Hilger [14] and our Theorem 4.1 let us present now a direct proof derived directly from the definition of the Fell topology. Such results was presented in [14, Lemmas 4.1-4.5], for instance.

Let $V = \mathbb{R}_+ \in V^+$. Then $\mathbb{R}_+ \subset V$ and $\mathbb{R}_+ = V$. Thus for arbitrary $n \in \mathbb{N}$, we $\mathbb{T}_n \subset \mathbb{R}_+ \subset V$. Whence all time scales $\mathbb{T}_n$ belong to V^+ and we have then this sequence is convergent in the upper Fell topology to $\mathbb{R}_+$.

Consider the second case. For open sets $U_1, U_2, ..., U_k$ (k - finite) let $\mathbb{R}_+ \in U_1^- \cap U_2^- \cap ... ... \cap U_k^-$. For each $i = 1, 2, ..., k$ choose an arbitrary point $u_i \in U_i$. As U_i are open we can separate these points by choosing numbers $\delta_i > 0$ such that the balls centered at u_i with radius δ_i are subsets of U_i. Put $\delta = \min_{i=1,2,...,k} \delta_i$. As $\frac{1}{10^n}$ is convergent to zero, then there exists a number $N \in \mathbb{N}$ such that for all $n > N$ we have $\frac{1}{10^N} < \delta$.

Then for all $n > N$ and arbitrary $i = 1, 2, ..., k$ $u_i + \frac{1}{10^n} \leq u_i + \delta \in U_i$. Finally for such indices n, we have $\mathbb{T}_n \in U_1^- \cap U_2^- \cap ... \cap U_k^-$ and $(\mathbb{T}_n)$ is convergent to $\mathbb{R}_+$ in the lower Fell topology. We are done.

- [K1] Let us now check the convergence in the Kuratowski topology. Take an arbitrary $x \in \mathbb{R}_+$. Denote by $ent(x)$ an integer part of x. Then $ent(x) \leq x \leq ent(x) + 1$, whence

$$\frac{ent(10^n x)}{10^n} \leq x \leq \frac{ent(10^n x) + 1}{10^n}$$

  for any n. Because both $ent(10^n x)$ and $ent(10^n x)+1$ are integer numbers, we get $\frac{ent(10^n x)}{10^n}, \frac{ent(10^n x)+1}{10^n} \in \mathbb{T}_n$. It is clear that they are both converging to x (as decimal approximations of x). Thus $x \in Lim\mathbb{T}_n$ and finally $\mathbb{T} = Lim\mathbb{T}_n$.

- [K2] It is not a problem to investigate the Kuratowski convergence by using the properties of this limit. In our case we have $\mathbb{T}_n \subset \mathbb{T}_{n+1} \subset \mathbb{R}_+$. To simplify the proof, we use some known properties of increasing sequences of sets with respect to the Kuratowski limit and then $Lim\mathbb{T}_n = \overline{\bigcup_{n=1}^{\infty} \mathbb{T}_n}$. Since the closure is taken with respect to the metric topology, it is also a sequential closure and we need only to prove that the last set is equal to be $\mathbb{R}_+$. Indeed, take an arbitrary nonnegative number $x \in \mathbb{R}_+$. Note that the sequence constructed above is convergent to x. Thus sequential closure (as well as the topological one) of $\bigcup_{n=1}^{\infty} \mathbb{T}_n$ is equal to $\mathbb{R}_+$.

□

3.2 Some convergent sequences of time scales

Let us start with the lemma, which will clarify the motivation of our result. It will be directly used in our Example 5.1, clarifying our main result.

Lemma 3.8. *For an arbitrary sequence* (h_n) *of positive numbers convergent to* 0 *a sequence of time scales of the form* $\mathbb{T}_n = h_n \cdot \mathbb{Z}_+$ *converge to* $\mathbb{R}_+$ *under the Fell (or: Kuratowski) topology.*

Proof: It suffice to note that for an arbitrary sequence $h_n \to 0$ as $n \to \infty$ and for sufficiently big indices n, there exists a number $N \in \mathbb{N}$ with $\frac{1}{10^N} \leq h_n \leq \frac{1}{10^{N-1}}$. This means that we can repeat the proof from the above subsection. □

Example 3.9. It is a good place to stress on the role of the Kuratowski convergence for our type of problems.

- [S1] (cf. [14, Lemma 4.1]) $\mathbb{T}_n = [-n, n]$ is Kuratowski convergent to $\mathbb{R}$. Indeed, this is an increasing sequence of sets and then $Lim\mathbb{T}_n = \overline{\bigcup_{k=1}^{\infty} \mathbb{T}_k} = \mathbb{R}$.
- [S2] (cf. [14, Lemma 4.3]) $\mathbb{T}_n = (1-\frac{1}{n})\cdot\mathbb{Z}$ It is not an increasing sequence, but $\bigcap_{k=n}^{\infty} \mathbb{T}_k = \mathbb{Z}$ so is $\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} \mathbb{T}_k = \bigcup_{n=1}^{\infty} \mathbb{Z} = \mathbb{Z}$. Moreover, for any n, $\bigcup_{k=n}^{\infty} \mathbb{T}_k = \mathbb{T}_n$ and then $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} \mathbb{T}_k = \bigcap_{n=1}^{\infty} \mathbb{T}_n = \mathbb{Z}$. The Kuratowski limit exists and $Lim\mathbb{T}_n = \mathbb{Z}$.
- [S3] ([14, Lemma 4.4] without the proof) Let (h_n) be an arbitrary increasing sequence of positive numbers convergent to 1 (in case of $h_n \geq 1$ we have nothing to do). Consider $\mathbb{T}_n = \bigcup_{z\in\mathbb{Z}_+}[z, z+h_n]$. Then it is an increasing sequence of sets and $Lim\mathbb{T}_n = \overline{\bigcup_{k=1}^{\infty} \mathbb{T}_k} = \mathbb{R}_+$.
- [S4] Finally, maybe the simplest, but an illustrative example. Let $\mathbb{T}_n = \{1, \frac{1}{2}, ..., \frac{1}{n}\}$. For any $n \in \mathbb{N}$, $\overline{\bigcup_{k=n}^{\infty} \mathbb{T}_n} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n}^{\infty} \mathbb{T}_n} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Similarly $\bigcap_{k=n}^{\infty} \mathbb{T}_n = \mathbb{T}_n$ and $\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} \mathbb{T}_n = \{\frac{1}{n} : n \in \mathbb{N}\}$ and then the sequence is Kuratowski convergent to $\{0\}\cup\{\frac{1}{n} : n \in \mathbb{N}\}$. The proof based on limits and accumulation points of $\mathbb{T}_n$ is also quite obvious.

Note that the examples considered in [31, Section 3], i.e. for $\mathbb{T}_n = [0, n]$ and $\mathbb{T}_n = \mathbb{Z} + \frac{1}{n}$ are obvious and the sequences are convergent to $\mathbb{R}_+$ and $\mathbb{Z}$, respectively.

We have also one more, very natural, example of a sequence of time scales justifying our choice of a type of convergence.

Example 3.10. Let us recall a construction of a Cantor set $\mathbb{K}$. We will show that this set is, in fact, the Kuratowski limit of time scales C_n used in a classical construction. Recall that $\mathbb{K}$ is closed, compact, perfect and uncountable. The *Cantor set* $\mathbb{K}$ is defined as

$$\mathbb{K} := \bigcap_{k=1}^{\infty} C_k = C_0 \setminus \bigcup_{n=1}^{\infty} R_n, \tag{1}$$

where $R_1 := (\frac{1}{3}, \frac{2}{3})$. We define C_1 as the two remaining pieces

$$C_1 := C_0 \setminus R_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]. \tag{2}$$

Now repeat the process on each remaining segment, removing the open set

$$R_2 := \left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right) \tag{3}$$

to form the four-piece set

$$C_2 := C_1 \setminus R_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right]. \tag{4}$$

Continue the process, forming $C_3, C_4, \ldots$ Note that C_k has 2^k pieces.

Because the sequence (C_n) is obviously decreasing, by the properties of the Kuratowski limit we have $Lim C_k = \bigcap_{k=1}^{\infty} C_k =: \mathbb{K}$.

Some time scales should be even *defined* as Kuratowski limits of (very natural) time scales ("fat Cantor sets", for instance). The direct proof of convergence of C_k with respect to the Fell topology is left to the reader. As a consequence we obtain the following result:

Lemma 3.11. *A Cantor set $\mathbb{K}$ is the Kuratowski limit of a sequence of time scales C_n taken from the classical construction.*

4 Convergence conditions

As a conclusion of our considerations, let us present a theorem which will be useful for studying the convergence of time scales. This idea is suggested by Esty and Hilger [14] in the case of $\mathbb{T} = \mathbb{R}$. We stress on applicability of such kind of results, we collect some related results and we present a general version. This seems to be also a unification tool for many problems considered in the context of differential equations or inclusions.

We are in a position to formulate some equivalent conditions allowing us to investigate the convergence of time scales. Now let us summarize all our previous results.

Theorem 4.1. *Let $\mathbb{T}_n, \mathbb{T}$ be a sequence of time scales. The following are equivalent:*

(F) *$\mathbb{T}_n$ is convergent to $\mathbb{T}$ with respect to the Fell topology,*

(E) *For any compact set K of $\mathbb{R}$, we have $\lim_{n\to\infty} e(K \cap \mathbb{T}_n, \mathbb{T}) = 0$ and $\lim_{n\to\infty} e(K \cap \mathbb{T}, \mathbb{T}_n) = 0$,*

(K) $\mathbb{T}_n$ *is convergent to* $\mathbb{T}$ *in the sense of Kuratowski,*

(D) *For any compact set, K we have* $\{x : \liminf_{n\to\infty} dist(x, K\cap\mathbb{T}_n) = 0\} \subset \mathbb{T}$ *and* $\mathbb{T} \subset \{x : \lim_{n\to\infty} dist(x, \mathbb{T}_n) = 0\}$.

Proof: $(F) \iff (E)$ is proved in [14, Theorem 4.6].

The equivalence of (F) and (K) is established in Lemma 3.4.

$(K) \iff (D)$ follows from the definition of the Kuratowski convergence and our Proposition 3.5. □

If $\mathbb{T} = \mathbb{R}$ then obviously some of the above conditions can be simplified (they are trivially satisfied, cf. [14, section 5.2]). For compact times scales the Hausdorff distance can be used to express such a convergence, but it is covered in (E) (see also [1], [17], [23]).

Let us explain the role of compact sets K in our condition (D) by recalling the following lemma (adapted from [6, Lemma 3.4]):

Lemma 4.2. *Assume, that a sequence* (f_n) *of distance functions* $f_n(x) = dist(x, \mathbb{T}_n)$ *is pointwise convergent to a finite-valued function* f. *Then* $\mathbb{T}_n$ *is convergent in the sense of Kuratowski. If a function* f *is continuous then the limit* $\mathbb{T} = \{x : f(x) = 0\}$.

Example 4.3. Let us stress on a role of compactness of sets in the above conditions. Let $\mathbb{T}_n = \{0, n\}$. It is clear that the sets of all limit points and accumulation points are both the same, namely $\{0\}$, so it is a Kuratowski convergent sequence to $\mathbb{T} = \{0\}$. However, $e(\mathbb{T}_n, \mathbb{T}) = dist(n, \{0\}) = n$ is not convergent to 0. But for an arbitrary compact set K we have $K \subset [-N, N]$ and then $K \cap \mathbb{T}_n = \{0\}$ for $n > N$. Then $e(K \cap \mathbb{T}_n), \mathbb{T}) = dist(0, \{0\}) = 0$ for such n and then condition (E) is satisfied.

It is clear that if possible, we can use some parameters specific to this kind of sets for the investigations of convergence of time scales (cf. the condition (M)). Especially, the convergence of graininess or forward jump functions seem to be useful. As will be indicated in the next example we should be very careful. We present some conditions related to properties of time scales, as well as of points in time scales. Following Esty and Hilger [14] let us introduce the following parameter:

Definition 4.4. ([14]) *Let* $a, b \in \mathbb{R}$. *If* $\inf \mathbb{T} \le a \le b \le \sup \mathbb{T}$, *then we define* $\overline{\mu}_{a,b}(\mathbb{T}) = \max\{\mu(t) : [t, \sigma(t)] \cap [a, b] \neq \emptyset\}$ *and* $+\infty$ *if the last set is empty.*

Are the *properties* of points important? We need to adapt the notion of almost uniform convergence to the case of time scales. By considering a compact set $K \subset \mathbb{R}$ we have $T_K = \mathbb{T} \cap K$ is closed, so is a time scale. We denote the graininess on such a time scale by μ^K.

Definition 4.5. *Let $\mathbb{T}_n, \mathbb{T}$ be a sequence of time scales equipped with the graininess functions μ_n and μ, respectively. We will say that a sequence (μ_n) is almost uniform convergent to μ^K if for each compact subset $K \subset \mathbb{R}$ a sequence of truncated graininess functions μ_n^K is uniformly convergent to μ^K i.e., $\sup_{t\in\mathbb{T}_n\cap K} \mu_n^K$ is convergent.*

This definition allows to reject from considerations some "pathological" points.

Example 4.6. Let $\mathbb{T}_n = \{-n, 0, 2\}$. Put $a = -\frac{1}{2} \geq \inf \mathbb{T}_n$ and $b = 1 \leq \sup \mathbb{T}_n$. Then the set of all points $t \in \mathbb{T}_n$ (say S_n) satisfying the condition $[t, \sigma(t)] \cap [a, b] \neq \emptyset$ is just $\{-n, 0\}$. This means that $\sup_{t\in S_n} \mu(t) = n$ (cf. [14, Section 5.3]).

It is easy to check that $\mathbb{T}_N$ converges to $\mathbb{T} = \{0, 2\}$ in the Kuratowski (or: Fell) topology. But $\sup_{t\in\mathbb{T}} \mu(t) = 2$. An estimation of $\lim_{n\to\infty} \sup_{t\in S_n} \mu(t)$ by $\sup_{t\in\mathbb{T}} \mu(t)$ is not feasible. The problem lies in a fact of divergent sequence $(-n)$.

Now, let us check the almost uniform convergence. For any fixed compact set K, there exists $N \in \mathbb{N}$ such that $\mathbb{T}_n \cap K = \{0, 2\}$. Thus $\mu_n^K(0) = 2 = \mu(0)$ and $\mu_n^K(2) = 0 = \mu(2)$ and then for an arbitrary compact K, (μ_n^K) is uniformly convergent to μ.

Example 4.7. In [14, p. 1015] a sequence of time scales $\mathbb{T}_n = (-\infty, -\frac{1}{n}] \cup \{n\}$ is indicated as (Fell) convergent. It is true, but this is a good example to show the difference between original graininess μ_n on $\mathbb{T}_n$ and a truncated μ_n^K for compact set K. It is clear that $\sup_{t\in\mathbb{T}_n} \mu_n(t) = n + \frac{1}{n} \to \infty$, but this sequence is almost uniformly convergent to μ. Note that $\mu_n(-\frac{1}{n}) = n + \frac{1}{n}$, but $\mu_n^K(t) = 0$ for each $t \in \mathbb{T}_n$ (because $\sigma(-\frac{1}{n}) = -\frac{1}{n}$ on $\mathbb{T}_n \cap K$)!

We are in a position to discuss some sufficient conditions for convergence expressed in terms of the above definitions (cf. [14]) (it is possible for points in $t \in \mathbb{T}_n \cap \mathbb{T}$).

Theorem 4.8. *For $t \in \mathbb{T}_n \cap \mathbb{T}$ consider the following conditions*

(M) $\lim_{n\to\infty} \mu_n(t) = \mu(t)$ *and the convergence is almost uniform on* $\mathbb{T}$,

(M2) *For all $a < t < b$ we have* $\lim_{n\to\infty} \overline{\mu}_{a,b}(\mathbb{T}_n) = \overline{\mu}_{a,b}(\mathbb{T})$,

(M3) $\lim_{n\to\infty} \mu_n(t) = \mu(t)$,

(S) *for an arbitrary $t \neq \sup \mathbb{T}$ we have* $\lim_{n\to\infty} \sigma_n(t) = \sigma(t)$.

Then (M2) $\Rightarrow$ (M) $\Rightarrow$ (S) and (M2) $\Rightarrow$ (M) $\Rightarrow$ (M3). If for all n and all compact subsets $K \subset \mathbb{R}$ we have $\mathbb{T}_n \subset \mathbb{T}$ and $\sup_{t\in K\cap\mathbb{T}_n} \mu_n^K(t) \to 0$, then (M) $\Rightarrow$ (E) (or any equivalent condition).

Proof: For any compact set K, there exists an interval $[a,b]$ such that $K \subset [a,b]$ and then

$$0 \le \sup_{t\in K\cap \mathbb{T}_n} \mu_n^K(t) \le \sup_{t\in[a,b]} \mu_n(t) \le \overline{\mu}_{a,b}(\mathbb{T}_n).$$

By applying (M2) we get (M). Since singletons are compact, obviously $(M) \Rightarrow (M3)$.

Because $\mu_n(t) = \sigma_n(t) - t$ and by the condition (M), $\mu_n(t) \to \mu(t) = \sigma(t) - t$ as $n \to \infty$, then $\sigma_n(t) \to \sigma(t)$ (a Hausdorff space), so we get (S).

It remains to prove, that under an additional assumption we have the implication (M) $\Rightarrow$ (E). Thus assume that for all n and all compact subsets $K \subset \mathbb{R}$, we have $\mathbb{T}_n \subset \mathbb{T}$ and $\sup_{t\in K\cap\mathbb{T}_n} \mu_n^K(t) \to 0$.

Observe that $K \cap \mathbb{T}_n \subset \mathbb{T}_n \subset \mathbb{T}$. Then $e(K \cap \mathbb{T}_n, \mathbb{T}) = 0$.

To prove the second condition in (E), fix an arbitrary $x \in K$. Consider the set L of all points t from $\mathbb{T}_n$ such that $t \le x$ and R as the set of all points t from $\mathbb{T}_n$ such that $x \le t$. Either L has its maximum or R has its minimum. Note that we can omit the points x for which such sets are empty (there are no relations with the function μ_n^K).

Assume that L is nonempty and has a maximal point (other cases can be studied by a similar manner). Put τ as its supremum. Thus $\sigma_n(\tau) \in \mathbb{T}_n$ is bigger than x. We have $\tau \le x \le \sigma_n(\tau)$. Denote by $a = \sigma_n(\tau) - x$ and by $b = x - \tau$. As $\min(a,b) \le \frac{a+b}{2}$ we get

$$\min(a,b) \le \frac{(\sigma(\tau) - x) + (x - \tau)}{2} = \frac{\sigma_n(\tau) - \tau}{2} = \frac{\mu_n(\tau)}{2}.$$

If $\tau = \sup \mathbb{T}_n \cap K$, then $x = \tau$ and then $\mu_n(\tau) = 0$. Otherwise we have an immediate estimation $\mu_n(\tau) \le \sup_{t\in K\cap\mathbb{T}_n} \mu_n^K(\tau) =: \alpha_n^K$. Then at least one of the points τ and $\sigma_n(\tau)$ lies in the ball $B(x, \alpha_n^K/2)$ and then $\mathbb{T}_n \cap B(x, \alpha_n^K/2) \ne \emptyset$. But x is arbitrarily chosen, so $e(K, \mathbb{T}_n) \le \alpha_n^K/2$.

Since $e(K \cap \mathbb{T}, \mathbb{T}_n) \le e(K, \mathbb{T}_n)$ we get

$$0 \le e(K \cap \mathbb{T}, \mathbb{T}_n) \le \frac{\alpha_n^K}{2}.$$

By (M), $\alpha_n^K \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} e(K \cap \mathbb{T}, \mathbb{T}_n) = 0$. The condition (E) is proved. Due to Theorem 4.1 we can take an arbitrary equivalent condition. The case when R has its minimum can be studied by similar manner, so let us omit the details. □

Note that the additional condition mentioned above is satisfied for $\mathbb{T} = \mathbb{R}$ or, for increasing approximations for time scales of the type $[a,b]$ or $[a,\infty)$, for instance. It is well-known that for compact time scales we have some equivalences (but let us recall Example 4.6 for a general lack of equivalence).

Corollary 4.9. ([14, Theorem 5.2]) *If* $\mathbb{T} = \mathbb{R}$ *in Theorem 4.1, then each of the equivalent conditions implies that* $\lim_{k\to\infty} \inf \mathbb{T}_k = -\infty$ *and* $\lim_{k\to\infty} \sup \mathbb{T}_k = +\infty$.

Remark 4.10. Note that such a set of conditions is very useful when we study different sequences of time scales. As an example may serve the Lemma 3.11, where the direct use of the Kuratowski convergence allows one to check the convergence. However, a calculation of graininess for both the Cantor set, as well as particular steps in the construction require more efforts.

Remark 4.11. Let us note that a suggested choice for the convergence of time scales with respect to the Fell topology seems to be proper. However, it is worthwhile to study such problems by the well-known methods from multivalued analysis. It is much simpler to study this convergence via the Kuratowski convergence. Moreover, we obtain a detailed answer to [31, Conjecture 3.3]: positive in the case of time scales, nonetheless negative for an arbitrary space X.

We still have an open problem: at least for some time scales $\mathbb{T}$ an upper ($LsA_n = A$) or lower ($LiA_n = A$) Kuratowski convergence seems to be satisfactory, when we study approximated solutions on $\mathbb{T}$. How to characterize such time scales?

Since our main goal of the paper is to characterize all "proper" kind of convergence of sequences of time scales and to give as detailed theory as possible we restrict ourselves to relatively simple examples of applications. We are motivated by some studies on dynamic approximations of differential problems (see the next Section). For a more detailed theory about continuous dependence of solutions of dynamic problems on their domains (time scales) we refer the reader to [12].

5 An example

It seems to be very hard to imagine a general "convergence" result, i.e. $\mathbb{T}_n \to \mathbb{T}$ in the Kuratowski (or any different) sense should implies convergence of appropriate sets of solutions for an *arbitrary* dynamical equation. But even particular cases seems to be interesting and realizing one of the main goals of dynamic equations, as well as for difference equations. Recall that the goal of this paper is to check what happens when one considers a dynamical equation on a time scale near to the original domain of the problem.

Let us consider now a case of the Cauchy problem without uniqueness of solutions. We are able to show how to apply our approach in such a case.

It was suggested many times that this seems to be most interesting case when we deal with such a property of dynamic equations. Note that it is an initial requirement for difference treatment of differential problems. Surprisingly, the proposed approach was never investigated. We study the dynamic problem with sufficiently regular right-hand side, i.e. with function f satisfying all conditions for the Peano theorem on time scales (cf. [10, 11], [22] and a correction in [21]) - is rd-continuous and continuous with respect to the second argument. The existence of approximated solutions is therefore guaranteed. However lack of uniqueness of solutions restricts the use of classical numerical methods.

To be familiar with earlier investigation we start with classical differential problem. However, we will also present more general result.

Example 5.1. A) Consider the following problem on $\mathbb{R}_+$:

$$x'(t) = 2\sqrt{x} \quad , \quad x(0) = 0. \tag{5}$$

It is well known that it has a trivial solution $x(t) \equiv 0$ and a family of solutions indexed by $C \geq 0$ of the form: $x(t) = 0$ for $t \leq C$ and $x(t) = (t - C)^2$ for $t > C$.

B) Now, consider the time scale $\mathbb{T} = h\mathbb{Z}_+$ with some (fixed) $h > 0$. This is, in fact, classical discretization for differential problems, but in the considered case there is lack of uniqueness of solutions, so usually it was not investigated as a difference equation. This dynamical equation over the time scale $\mathbb{T} = h\mathbb{Z}_+$ is in fact the Euler (forward) numerical scheme with constant time step h applied to the differential equation ("the Eulerian time scale").

We prefer the dynamic approach, so let us consider two cases. Fix an initial condition $x(0) = 0$. In this example, we will assume that the points t are taken only from $\mathbb{T}$.

$$x^{\Delta}(t) = 2\sqrt{x(t)} \tag{6}$$

and

$$x^{\Delta}(t) = 2\sqrt{x(\sigma((t))}. \tag{7}$$

Both problems are considered in many papers. Let us recall that an interesting discussion about the two forms, in particular about their differences, can be found in [25]. We will indicate one more difference, which suggest the approach via equation (7).

[Case I.] By the definition of the Δ-derivative we get

$$\frac{x(\sigma(0)) - x(0)}{\mu(t)} = 2\sqrt{x(0)}.$$

This implies that $x(\sigma(0)) = 0$ and then by repeating this procedure we get only the trivial solution $x(t) \equiv 0$.

[Case II.] We have

$$\frac{x(\sigma(0))) - x(0)}{\mu(t)} = 2\sqrt{x(\sigma((0)))}$$

and then

$$x(\sigma(0)) = 2h\sqrt{x(\sigma(0))}.$$

It means that we have two possible solutions: either we get $x(\sigma(0)) = 0$ or $x(\sigma(0)) = (2h)^2$. If we choose the first one, in the next step we will only have the same choice. Thus if we get $x(\sigma(t)) = 0$ in every step, we get trivial solution.

If not, i.e. for a fixed t_0 in some step we choose $x(t_0) = (2h)^2$, we have a positive value of x, which implies that for t bigger than t_0 we have increasing function and there is no "return" to the previous choice. But this means that the choice of numerical scheme ("Euler forward or backward scheme") does not matter.

For $t > t_0$ we will have

$$\frac{x(\sigma(t)) - x(t)}{\mu(t)} = 2\sqrt{x(\sigma((t)))}$$

and then

$$x(\sigma(t)) = 2h\sqrt{x(\sigma((t)))} + x(t).$$

In particular,

$$x(\sigma(t_0)) = 2h\sqrt{x(\sigma((t_0)))} + (2h)^2.$$

We get

$$x(\sigma(t_0)) = [h \cdot (1 + \sqrt{5})]^2.$$

The remaining values for $t > t_0$ can be obtained recursively:

$$x(\sigma(t)) = h + \sqrt{h^2 + x(t)}.$$

As t_0 is of the form $t_0 = k \cdot h$ for some $k \in \mathbb{Z}_+$, we get the direct formula

$$x_{k+1} = h + \sqrt{h^2 + x_k}. \tag{8}$$

For $t < t_0$ we have $y(t) = 0 = x(t)$. In a point $t_0 = k \cdot h$ we have, $y((\sigma(t_0)) = 4h^2 = x(t_0)$. Then in the next points of our time scale we have the values of y: $9h^2$, $16h^2$, ... The approximated values of x can be easily calculated: $(1+\sqrt{5})^2 \cdot h^2$, $(1+\sqrt{1 + (1 + \sqrt{5})^2})^2 \cdot h^2$, ... Note that the values are independent on the choice of t_0.

Letting h be arbitrarily small (by taking a sequence is convergent to 0, for instance), we see that in every point t we have a good estimation for an exact solution y of (5): $|x(t) - y(t)| \leq const. \cdot h^2$.

Let us note that x is arbitrary, so the uniqueness is not required for (5).

As a consequence we get the following result. Let us stress on a specific kind of dependence of solutions on time scales allowing us to move domains $\mathbb{T}_n$.

Theorem 5.2. *Let y be an arbitrary solution of (5) on $\mathbb{R}_+$. Take an arbitrary number $\varepsilon > 0$. Then for any sequence of time scales $(\mathbb{T}_n)_n = (h_n\mathbb{Z}_+)_n$ there exists a limit x of solutions (7) on $\mathbb{T}_n$ with $|x(t) - y(t)| < \varepsilon$ for $t \in \mathbb{T}_n$. Conversely, for every $\varepsilon > 0$ and every solution x_n of (7) on $\mathbb{T}_n$ there exists a solution y for (5) such that $|x_n(t_n) - y(t)| < \varepsilon$ for $t \in \mathbb{T}$ for some sequence (t_n) with $t_n \in \mathbb{T}_n$, $t_n \to t \in \mathbb{T}$.*

Proof: Let us sketch the proof. For any fixed $C \geq 0$, we take a solution $y(t)$ of (5) as in part "A)". For $n \in \mathbb{N}$ there exists a point $t_0^n \in \mathbb{T}_n$ "close" to C. Assume that we constructed solutions of (7) on $\mathbb{T}_n$ by the above method with such a fixed $t_0^n < C$ and $\sigma(t_0^n) > C$ and call the appropriate solutions $x_n^{t_0^n}$ and $x_n^{\sigma(t_0^n)}$, respectively. Then by the above consideration we have

$$x_n^{t_0^n}(t) \leq y(t) \leq x_n^{\sigma(t_0^n)}(t)$$

for $t \in \mathbb{T}$ and $t > t_0^n$ (note that $x_{t_0^n}(t) = y(t) = x_n^{\sigma(t_0^n)} = 0$ for t less than t_0^n). We have $y(t) = (t - C)^2$ and $x_n^{(t_0^n)}(t) = x_n^{\sigma(t_0^n)}(t + h)$.

If we put $h_n \to 0$ in the direct formula (8) (with h_n instead of h), then $x_{k+1}^2(t) \approx x_k(t)$ and for sufficiently small h we have a desired estimation. Both $x_n^{t_0^n}$ and $x_n^{\sigma(t_0^n)}$ can be used in our thesis with $t_n = t_0^n$ or $t_n = \sigma(t_0^n)$, respectively.

Our previous considerations allow us to say that an arbitrary solution x_n of (7) is , in fact, an approximation for a solution of (5). $\square$

6 Conclusion and future directions

Besides previously mentioned problems in the paper let us add one more remark. It seems to be clear that by taking a differential (or: dynamic) problem on $\mathbb{T}$ we are able to construct a sequence of time scales $\mathbb{T}_n$ convergent to $\mathbb{T}$ in the sense of Kuratowski. Since we are not forced to do it for an arbitrary sequence, we can suppose that our sequence should satisfy some additional conditions. It facilitates us to check the convergence and allows us to solve the approximated problem. Then on a intersection of time scales, we can compare solutions or simply treat the obtained solution on $\mathbb{T}_n$ as a approximation for that in $\mathbb{T}$ (cf. [1]). Let us stress that even a direct form of the Kuratowski convergence seems to be optimal when study dependence of solutions on time scales and approximation problems. However, it it worthwhile to study some equivalent conditions for convergence of time scales specified for dynamic problems (cf. Theorem 4.1). Note that we need to choose also a proper form of the approximated problem, i.e. either (5) or (7). We are not restricted to the case of problems with uniqueness of solutions for the original problem. This should be an advantage of the theory of dynamic equations. But for "spares" time scales, the intersection of $\mathbb{T}$ and $\mathbb{T}_n$ can be an empty set and we need a convergence of solutions as in Theorem 5.2.

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