Mathematica Aeterna, Vol. 5, 2015, no. 5, 905 - 909

3-Lie algebra Γ_{27} over the prime field Z_2

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Abstract

In this paper, the 8-dimensional 3-Lie algebra Γ_{27} over the prime field Z_2 is constructed by 2-cubic matrix. It is proved that Γ_{27} is a solvable but non-nilpotent 3-Lie algebra. The inner derivation algebra $ad(\Gamma_{27})$ is an 11-dimensional solvable Lie algebra, and the derivation algebra $Der(\Gamma_{27})$ with dimension 18 is solvable but non-nilpotent. And the concrete expression of all derivations are given.

2010 Mathematics Subject Classification: 17B05 17D30
Keywords: N-cubic matrix, 3-Lie algebra, derivation, prime field.

1 Introduction

The notion of n-Lie algebra (or Lie n-algebra, Filippov algebra, Nambu-Poisson algebra and so on) was introduced by Filippov in 1985 [1]. An n-Lie algebra A is a vector space A endowed with a n-ary skew-symmetric multiplication satisfying the n-Jacobi identity:

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n}[x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n].$$

The structure of n-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident $M2$- branes; the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra[2].

In papers [3-7], authors constructed 3-Lie algebras by well known algebras and N-cubic matrices over a field F with $chF \neq 2$, and the structure of 3-Lie algebras is studied.

In this paper, we pay our main attention to 8-dimensional 3-Lie algebras which are constructed by 2-cubic matrix in the prime field $Z_2 = \{0, 1\}$. In the following we suppose that $Z_2 = \{0, 1\}$ is the prime field with characteristic two, for a vector space V and a subset S, the subspace generated by S is denoted by (S).

2 Structure of 3-Lie algebras Γ_{27}

An N-order cubic matrix $A = (a_{ijk})$ over the field Z_2 is an ordered object which the elements with 3 indices, and the element in the position (i, j, k) is $(A)_{ijk} = a_{ijk}$, $1 \leq i, j, k \leq N$ and $a_{ijk} = 0$ or 1. Denote the set of all cubic matrix over Z_2 by Ω_2. Then Ω_2 is an N^3-dimensional vector space with $A + B = (a_{ijk} + b_{ijk}) \in \Omega_2$, $\lambda A = (\lambda a_{ijk}) \in \Omega_2$, for $\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega_2$, $\lambda \in Z_2$, that is, $(A + B)_{ijk} = a_{ijk} + b_{ijk}$, $(\lambda A)_{ijk} = \lambda a_{ijk}$.

Denote $E_{ijk} = (e_{h_1h_2h_3})$, where $e_{h_1h_2h_3} = \delta_{h_1i}\delta_{h_2j}\delta_{h_3k}$, that is when $h_1 = i$, $h_2 = j$, $h_3 = k$, $e_{h_1h_2h_3} = 1$, and elsewhere are zero. Then, $\{E_{ijk} |\ 1 \leq i, j, k \leq n\}$ is a basis of Ω_2.

For all $A = (a_{ijk}), B = (b_{ijk}) \in \Omega_2$, define the multiplication $*_{27}$ in Ω_2 by

$$(A *_{27} B)_{ijk} = \sum_{p,q=1}^{n} a_{qjk} b_{ipk}, 1 \leq i, j, k \leq n. \tag{1}$$

Denote $\langle A \rangle_4 = \sum_{p,q,r=1}^{N} (A)_{pqr} = \sum_{p,q=1}^{N} a_{pqr}$, Then $\langle\ \rangle_4$ is linear function from Ω_2 to Z_2 and satisfies

$$\langle A *_{27} B \rangle_4 = \langle B *_{27} A \rangle_4. \tag{2}$$

Define the multiplication $[, ,]_{27} : \Omega_2 \wedge \Omega_2 \wedge \Omega_2 \to \Omega_2$ as follows:

$$[A, B, C]_{27} = \langle A \rangle_4 (B *_{27} C - C *_{27} B)$$

$$+ \langle B \rangle_4 (C *_{27} A - A *_{27} C) + \langle C \rangle_4 (A *_{27} B - B *_{27} A). \tag{3}$$

Theorem 2.1[4] *The linear space Ω_2 are 3-Lie algebra in the multiplication $[, ,]_{27}$, which is denoted by Γ_{27}, and the multiplication $[, ,]_{27}$ simply denoted by $[, ,]$.*

From above discussion the dimension of Ω_2 is eight and with a basis

$$\{E_{111}, E_{112}, E_{121}, E_{122}, E_{211}, E_{212}, E_{221}, E_{222}\}.$$

And for all $A \in \Omega_2$, $A = \sum_{i,j,k=1}^{2} \lambda_{ijk} E_{ijk}$, $\lambda_{ijk} = 1, 0 \in Z_2$.

Theorem 2.2. *The multiplication of the* 3-*Lie algebra* Γ_{27} *in the basis* $\{E_{111}, E_{112}, E_{121}, E_{122}, E_{211}, E_{212}, E_{221}, E_{222}\}$ *is as follows*

$$\left\{\begin{array}{ll}
[E_{111}, E_{112}, E_{121}] = E_{121} + E_{111}, & [E_{111}, E_{112}, E_{122}] = E_{112} + E_{122},\\
[E_{111}, E_{121}, E_{122}] = E_{122} + E_{112}, & [E_{111}, E_{112}, E_{211}] = E_{111} + E_{211},\\
[E_{112}, E_{121}, E_{211}] = E_{221} + E_{111}, & [E_{112}, E_{122}, E_{211}] = E_{112} + E_{122},\\
[E_{121}, E_{122}, E_{211}] = E_{111} + E_{221}, & [E_{111}, E_{112}, E_{212}] = E_{212} + E_{112},\\
[E_{111}, E_{121}, E_{212}] = E_{111} + E_{121}, & [E_{111}, E_{122}, E_{212}] = E_{222} + E_{112},\\
[E_{121}, E_{122}, E_{212}] = E_{222} + E_{112}, & [E_{111}, E_{112}, E_{221}] = E_{121} + E_{211},\\
[E_{112}, E_{121}, E_{221}] = E_{221} + E_{121}, & [E_{112}, E_{121}, E_{221}] = E_{221} + E_{121},\\
[E_{112}, E_{122}, E_{221}] = E_{112} + E_{122}, & [E_{121}, E_{122}, E_{221}] = E_{121} + E_{221},\\
[E_{111}, E_{112}, E_{222}] = E_{212} + E_{122}, & [E_{111}, E_{121}, E_{222}] = E_{111} + E_{121},\\
[E_{111}, E_{122}, E_{222}] = E_{222} + E_{122}, & [E_{112}, E_{121}, E_{222}] = E_{122} + E_{212},\\
[E_{211}, E_{222}, E_{111}] = E_{111} + E_{211}, & [E_{212}, E_{221}, E_{111}] = E_{121} + E_{211},\\
[E_{212}, E_{222}, E_{111}] = E_{212} + E_{222}, & [E_{221}, E_{222}, E_{111}] = E_{211} + E_{121},\\
[E_{211}, E_{212}, E_{112}] = E_{112} + E_{212}, & [E_{211}, E_{221}, E_{112}] = E_{211} + E_{221},\\
[E_{211}, E_{222}, E_{112}] = E_{122} + E_{212}, & [E_{212}, E_{221}, E_{112}] = E_{212} + E_{112},\\
[E_{211}, E_{222}, E_{121}] = E_{221} + E_{111}, & [E_{212}, E_{221}, E_{121}] = E_{121} + E_{221},\\
[E_{212}, E_{222}, E_{121}] = E_{212} + E_{222}, & [E_{221}, E_{222}, E_{121}] = E_{221} + E_{121},\\
[E_{211}, E_{212}, E_{122}] = E_{112} + E_{222}, & [E_{211}, E_{221}, E_{122}] = E_{211} + E_{221},\\
[E_{211}, E_{222}, E_{122}] = E_{122} + E_{222}, & [E_{212}, E_{221}, E_{122}] = E_{222} + E_{112},\\
[E_{221}, E_{222}, E_{112}] = E_{122} + E_{212}, & [E_{211}, E_{212}, E_{121}] = E_{221} + E_{111},\\
[E_{121}, E_{122}, E_{222}] = E_{222} + E_{122}, & [E_{211}, E_{212}, E_{111}] = E_{211} + E_{111},\\
[E_{111}, E_{122}, E_{221}] = E_{121} + E_{211}, & [E_{112}, E_{121}, E_{212}] = E_{112} + E_{212},\\
[E_{111}, E_{122}, E_{211}] = E_{211} + E_{111}, & [E_{221}, E_{222}, E_{122}] = E_{122} + E_{222},\\
[E_{212}, E_{222}, E_{112}] = E_{112} + E_{122} + E_{222} + E_{212}, & \\
[E_{211}, E_{221}, E_{121}] = E_{111} + E_{221} + E_{121} + E_{211}, & \\
[E_{112}, E_{122}, E_{222}] = E_{222} + E_{212} + E_{112} + E_{122}, & \\
[E_{211}, E_{221}, E_{111}] = E_{111} + E_{121} + E_{221} + E_{211}, & \\
[E_{111}, E_{121}, E_{221}] = E_{221} + E_{211} + E_{111} + E_{121}, & \\
[E_{112}, E_{122}, E_{212}] = E_{222} + E_{112} + E_{212} + E_{122}, & \\
[E_{111}, E_{121}, E_{211}] = E_{221} + E_{211} + E_{111} + E_{121}, & \\
[E_{212}, E_{222}, E_{122}] = E_{112} + E_{122} + E_{222} + E_{212}, &
\end{array}\right. \quad (4)$$

where the zero product of the basis vectors are omitted.

Proof The result follows from the direct complication according to the definition of $*_{27}$ and Eqs.(1), (2) and (3).

Theorem 2.3 *The* 3-*Lie algebra* Γ_{27} *is a non-nilpotent indecomposable* 3-*Lie algebra with a basis* $e_1 = E_{111}, e_2 = E_{112} + E_{111}, e_3 = E_{111} + E_{121}, e_4 = E_{112} + E_{122}$, $e_5 = E_{211} + E_{111}$,$e_6 = E_{212} + E_{112}, e_7 = E_{211} + E_{221} + E_{111} + E_{121}$, $e_8 = E_{212} + E_{222} + E_{112} + E_{122}$. *And the multiplication in it is as follows:*

$$\left\{\begin{array}{l}
[e_1, e_2, e_3] = e_3, [e_1, e_2, e_4] = e_4, [e_1, e_2, e_5] = e_5,\\
[e_1, e_3, e_5] = e_7, [e_1, e_2, e_6] = e_6, [e_1, e_4, e_6] = e_8.
\end{array}\right. \quad (5)$$

Proof It is clear that $\{e_1, \cdots, e_8\}$ is linearly independent, so it is a basis of Ω_2. By the definition of $*_{27}$, we obtain Eq.(5). Since Γ_{27} can not be written as the direct sum of two proper ideals, Γ_{27} is indecomposable.

From $\Gamma_{27}^1 = [\Gamma_{27}, \Gamma_{27}, \Gamma_{27}] = (e_3, e_4, e_5, e_6, e_7, e_8)$, $\Gamma_{27}^2 = [\Gamma_{27}^1, \Gamma_{27}, \Gamma_{27}] = (e_3, e_4, e_5, e_6, e_7, e_8)$, then for all positive integer $s > 1$, we have $\Gamma_{27}^s = \Gamma_{27}^1 \neq 0$. Therefore, Γ_{27} is non-nilpotent.

Theorem 2.4 *The subalgebra $H = (e_1, e_2, e_7, e_8)$ is a Cartan subalgebra of the 3-Lie algebra Γ_{27}. And the decomposition of Γ_{27} associate to H is* $\Gamma_{27} = H \dot{+} \Gamma_\alpha \dot{+} \Gamma_{-\alpha}$, *where $\alpha : (H \otimes H) \to Z_2$, $\alpha(e_1, e_2) = 1$, and others are zero, $\Gamma_\alpha = (e_3, e_6)$, $\Gamma_{-\alpha} = (e_4, e_5)$.*

Proof From Theorem 2.3, $H = (e_1, e_2, e_7, e_8)$ is a Cartan subalgebra of Γ_{27}. Denote $\alpha : H \otimes H \to Z_2$, $\alpha(e_1, e_2) = 1$, $\alpha(e_1, e_7) = \alpha(e_1, e_8) = \alpha(e_2, e_7) = \alpha(e_2, e_8) = \alpha(e_7, e_8) = 0$, we have $ad(e_1, e_2)(e_3) = e_3$, $ad(e_1, e_2)(e_4) = e_4$, $ad(e_1, e_2)(e_5) = e_5$, $ad(e_1, e_2)(e_6) = e_6$, $ad^2(e_1, e_7)e_i = ad^2(e_1, e_8)(e_i) = ad^2(e_2, e_7)(e_i) = 0$, $ad^2(e_2, e_8)(e_i) = ad^2(e_7, e_8)(e_i) = 0$, $i = 3, 4, 5, 6$. We obtain the result.

Now we study the inner derivation algebra $ad(\Gamma_{27})$. For $e_i, e_j \in \Omega_2$, denote $ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l$, where $a_{kl}^{ij} = -a_{kl}^{ji} = 0$ or $1 \in Z_2$. Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl} are the matrix units.

Theorem 2.5 *the inner derivation algebra $ad(\Gamma_{27})$ is solvable but indecomposable Lie algebra with dimension 11, and $X_1 = E_{33} + E_{44} + E_{55} + E_{66}$, $X_2 = E_{23} + E_{57}$, $X_3 = E_{24} + E_{68}$, $X_4 = E_{25} + E_{37}$, $X_5 = E_{26} + E_{48}$, $X_6 = E_{13}$, $X_7 = E_{14}$, $X_8 = E_{15}$, $X_9 = E_{16}$, $X_{10} = E_{17}$, $X_{11} = E_{18}$, is a basis. And the multiplication in it is*

$$\begin{cases} [X_1, X_2] = X_2, \ [X_1, X_3] = X_3, \ [X_1, X_4] = X_4, \\ [X_1, X_5] = X_5, \ [X_1, X_6] = X_6, \ [X_1, X_7] = X_7, \\ [X_1, X_8] = X_8, \ [X_1, X_9] = X_9, \ [X_2, X_8] = X_{10}, \\ [X_3, X_9] = X_{11}, \ [X_4, X_6] = X_{10}, \ [X_5, X_7] = X_{11}. \end{cases} \tag{6}$$

Proof By a direct computation according to Eq.(5) we have that $ad(e_1, e_2) = E_{33} + E_{44} + E_{55} + E_{66}$, $ad(e_1, e_3) = E_{23} + E_{57}$, $ad(e_1, e_4) = E_{24} + E_{68}$, $ad(e_1, e_5) = E_{25} + E_{37}$, $ad(e_1, e_6) = E_{26} + E_{48}$, $ad(e_2, e_3) = E_{13}$, $ad(e_2, e_4) = E_{14}$, $ad(e_2, e_5) = E_{15}$, $ad(e_2, e_6) = E_{16}$, $ad(e_3, e_5) = E_{17}$, $ad(e_4, e_6) = E_{18}$. Then $\{X_1, \cdots, X_{11}\}$ is a basis of $ad(\Gamma_{27})$. From $[ad(e_i, e_j), ad(e_k, e_l)] = ad([e_i, e_j, e_k], e_l) + ad(e_k, [e_i, e_j e_l])$, we obtain Eq.(6). And $ad^1(\Gamma_{27}) = [ad(\Gamma_{27}), ad(\Gamma_{27})] = (X_{10}, X_{11})$, $[X_{10}, X_{11}] = 0$ then $ad(\Gamma_{27})$ is solvable. Since $ad(X_1)$ is non-nilpotent, and $ad(\Gamma_{27})$ can not be written as the direct sum of two proper ideals, $ad(\Gamma_{27})$ is indecomposable non-nilpotent.

Now,we discuss the derivation algebra $Der\Gamma_{27}$.

Theorem 2.6 *The derivation algebra* $Der(\Gamma_{27})$ *with a basis* $\{X_1, \cdots, X_{18}\}$, *where* $X_{12} = E_{11} + E_{22} + E_{77} + E_{88}$, $X_{13} = E_{33} + E_{77}$, $X_{14} = E_{44} + E_{88}$,$X_{15} = E_{55} + E_{77}$,$X_{16} = E_{28}$, $X_{17} = E_{12}$, *and* $X_{18} = E_{27}$, X_i, $1 \leq i \leq 11$ *are in Theorem 2.5. The multiplication in it is*

$$\begin{cases} [X_1, X_2] = X_2, [X_1, X_3] = X_3, [X_1, X_4] = X_4, [X_1, X_5] = X_5, \\ [X_1, X_6] = X_6, [X_1, X_7] = X_7, [X_1, X_8] = X_8, [X_1, X_9] = X_9, \\ [X_2, X_8] = X_{10}, [X_3, X_9] = X_{11}, [X_4, X_6] = X_{10}, [X_5, X_7] = X_{11}, \\ [X_2, X_{12}] = X_2, [X_3, X_{12}] = X_3, [X_4, X_{12}] = X_4, [X_5, X_{12}] = X_5, \\ [X_6, X_{12}] = X_6, [X_7, X_{12}] = X_7, [X_8, X_{12}] = X_8, [X_9, X_{12}] = X_9, \\ [X_2, X_{13}] = X_2, [X_6, X_{13}] = X_6, [X_{10}, X_{13}] = X_{10}, [X_{18}, X_{13}] = X_{18}, \\ [X_3, X_{14}] = X_3, [X_7, X_{14}] = X_7, [X_{11}, X_{14}] = X_{11}, [X_{16}, X_{14}] = X_{16}, \\ [X_4, X_{15}] = X_4, [X_8, X_{15}] = X_8, [X_{10}, X_{15}] = X_{10}, [X_{18}, X_{15}] = X_{18}, \\ [X_2, X_{17}] = X_6, [X_3, X_{17}] = X_7, [X_4, X_{17}] = X_8, [X_5, X_{17}] = X_9, \\ [X_{18}, X_{17}] = X_{10}, [X_{16}, X_{17}] = X_{11}. \end{cases}$$

And $Der(\Gamma_{27}) = ad(\Gamma_{27})\dot{+}(X_{12}, \cdots, X_{18})$ is solvable but non-nilpotent.

Proof The result follows from a direct computation according to Theorem 2.4.

Acknowledgements

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

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Received: October, 2015