Mathematica Aeterna, Vol. 5, 2015, no. 5, 911 - 915

q-Lie Algebras and q-3-Lie Algebras

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Abstract

In this paper, the quantum Lie algebras and quantum 3-Lie algebras over a field K with $chK = 0$ are discussed for q generic, where $q \in K, q \neq 0, 1$. A quantum Lie algebra is realized by a Z-graded algebra (Theorem 2.3), and a Lie algebra is realized by a quantum algebra which satisfying the property $q^{-i}x_i(x_jx_k)_q = (x_ix_j)_qx_k$ (Theorem 2.4). From quantum Lie algebras and linear functions, two classes quantum 3-Lie algebras are constructed (Theorem 2.6 and Theorem 2.7).

2010 Mathematics Subject Classification: 17B05 17D30
Keywords: Z-graded space, q-Lie algebra, q-3-Lie algebra.

1 Introduction

Recently one can observe a growing interest in the investigations and explanations of the quantum groups and algebras [1-4]. These structures appeared in the study of integrable models especially during the searching for solutions of the quantum Yang-Baxter equation [3-4]. So in this paper, we construct quantum Lie algebras from quantum algebras which satisfy some conditions, and from quantum Lie algebras, we also can construct general Lie algebras. We also define a class of quantum 3-Lie algebras [5-6], and realized two classes quantum 3-Lie algebras from quantum Lie algebras. In the following, denote K an arbitrary field with $char(K) = 0$, $q \in K, q \neq 0, 1$, and Z be the set of integers. For a positive integer n, set $(n)_q = \frac{1-q^n}{1-q}$.

2 main Result

In this section we study quantum Lie algebras and quantum 3-Lie algebras. For convenience, in the following, for a quantum Lie algebra and a quantum 3-Lie algebra, is simply called a q-Lie algebra and a q-3-Lie algebra for $q \in K$, respectively.

Definitions 2.1. *For a Z-graded vector space $L = \oplus_{i\in Z} L_i$ over a field K equipped with a bilinear q-bracket product $[,]_q$ (where $q \in K, q \neq 0, 1$, dim $L_i < \infty$) satisfying $[L_i, L_j]_q \subseteq L_{i+j}$, and for all $x_i \in L_i$, $\forall i \in Z$, if*

$$[x_i, x_j]_q = -[x_j, x_i]_q, \tag{1}$$

$$(2)_{q^i}[x_i, [x_j, x_k]_q]_q = (2)_{q^j}[x_j, [x_k, x_i]_q]_q + (2)_{q^k}[[x_i, x_j]_q, x_k]_q, \tag{2}$$

are fulfilled under $[,]_q$, then $(L, [,]_q)$ is called a q-Lie algebra, and $[,]_q$ is called the q-Lie product.

Example 2.2. Let K be an arbitrary field with $char(K) \neq 2, 3$, and $q \in K, q \neq 0, 1$ be generic. We define q-differential operator ∂_q over $K[x, x^{-1}]$ by $\partial_q(P) = \frac{P(qx)-P(x)}{qx-x}$, $\forall P \in K[x, x^{-1}]$. Let τ_q denote an algebra automorphism of $K[x, x^{-1}]$ defined by $\tau_q(x) = qx$. The q-differential operator ∂_q is called a τ_q-derivation or skew derivation if for all $P, Q \in K[x, x^{-1}]$, we have

$$\partial_q(PQ) = \partial_q(P)Q + \tau_q(P)\partial_q(Q).$$

Let $Der_q(K[x, x^{-1}])$ denote the set of all τ_q-derivation over $K[x, x^{-1}]$, and let $e_n = x^{n+1}\partial_q$ for all $n \in Z$. If we define a q-bracket product $[,]_q$ on $Der_q(K[x, x^{-1}])$ by

$$[e_i, e_j]_q = [(j+1)_q - (i+1)_q]e_{i+j}, i, j \in Z, \tag{3}$$

then the q-bracket product $[,]_q$ is bilinear over K and satisfies the antisymmetry (1) and the weighted q-Jacobi identity (2). Thus $(Der_q(K[x, x^{-1}]), [,]_q)$ is a q-Lie algebra [1].

Theorem 2.3. *For a Z-graded vector space $L = \oplus_{i\in Z} L_i$ over a field K equipped with a bilinear multiplication satisfying $L_iL_j \subset L_{i+j}$, and*

$$(2)_{q^{-i}}x_i(x_jx_k) = (2)_{q^k}(x_ix_j)x_k. \tag{4}$$

Then for all $x_i \in L_i$, and $x_j \in L_i$, $\forall i, j \in Z$, define the q-bracket product

$$[x_i, x_j]_q = q^{i+1}x_ix_j - q^{j+1}x_jx_i, \tag{5}$$

$(L, [,]_q)$ *is a q-Lie algebra, where $q \in K, q \neq 0, 1$, $dimL_i < \infty$.*

Proof The bilinearity of the q-bracket product $[,]_q$ is obvious over K, since

$$[x_j, x_i]_q = q^{j+1}x_jx_i - q^{i+1}x_ix_j = -[x_i, x_j]_q,$$

we only need to prove the identity (2). Now for all $x_i \in L_i$, $x_j \in L_i$, and $x_k \in L_k$, $\forall i, j, k \in Z$,

$$\begin{aligned}
&(2)_{q^i}[x_i, [x_j, x_k]_q]_q + (2)_{q^j}[x_j, [x_k, x_i]_q]_q + (2)_{q^k}[x_k, [x_i, x_j]_q]_q \\
&= (2)_{q^i}[x_i, q^{j+1}x_jx_k - q^{k+1}x_kx_j]_q + (2)_{q^j}[x_j, q^{k+1}x_kx_i - q^{i+1}x_ix_k]_q \\
&+(2)_{q^k}[x_k, q^{i+1}x_ix_j - q^{j+1}x_jx_i]_q \\
&= (1+q^i)\cdot q^{j+1}[x_i, x_jx_k]_q - (1+q^i)\cdot q^{k+1}[x_i, x_kx_j]_q + (1+q^j)\cdot q^{k+1}[x_j, x_kx_i]_q \\
&-(1+q^j)\cdot q^{i+1}[x_j, x_ix_k]_q + (1+q^k)\cdot q^{i+1}[x_k, x_ix_j]_q - (1+q^k)\cdot q^{j+1}[x_k, x_jx_i]_q \\
&= q^{j+1}[x_i, x_jx_k]_q + q^{i+j+1}[x_i, x_jx_k]_q - q^{k+1}[x_i, x_kx_j]_q - q^{i+k+1}[x_i, x_kx_j]_q \\
&+q^{k+1}[x_j, x_kx_i]_q + q^{j+k+1}[x_j, x_kx_i]_q - q^{i+1}[x_j, x_ix_k]_q - q^{i+j+1}[x_j, x_ix_k]_q \\
&+q^{i+1}[x_k, x_ix_j]_q + q^{i+k+1}[x_k, x_ix_j]_q - q^{j+1}[x_k, x_jx_i]_q - q^{j+k+1}[x_k, x_jx_i]_q = 0.
\end{aligned}$$

Therefore, $(L, [,]_q)$ is a q-Lie algebra.

Theorem 2.4. *If a Z-graded algebra $L = \oplus_{i \in Z} L_i$ over a field K satisfies $L_iL_j \subset L_{i+j}$, and*

$$q^{-i}x_i(x_jx_k)_q = (x_ix_j)_qx_k. \tag{6}$$

Then $(L, [,])$ is a Lie algebra, where for $\forall x_i \epsilon L_i, x_j \epsilon L_j$, the product $[,]$ is defined by

$$[x_i, x_j] = q^{i+1}(x_ix_j)_q - q^{j+1}(x_jx_i)_q, \tag{7}$$

where $q \in K, q \neq 0, 1$, $dim\ L_i < \infty$.

Proof The bilinearity of the product $[,]$ is obvious over K. The anti-symmetry (1) is clear according to identity (7). Now we consider the Jacobi identity of Lie algebras. For all $x_i \in L_i$, $x_j \in L_j$ and $x_k \in L_k$, $\forall i, j, k \in L$, from

$$[x_k, [x_i, x_j]] = [x_k, q^{i+1}(x_ix_j)_q - q^{j+1}(x_jx_i)_q]$$

$$= q^{i+1}[q^{k+1}x_k(x_ix_j)_q - q^{i+j+1}(x_ix_j)_qx_k] - q^{j+1}[q^{k+1}x_k(x_jx_i)_q - q^{i+j+1}(x_jx_i)_qx_k].$$

And the cyclic permutation of (i, j, k), we have $[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0$. It follows the result.

In the following, we construct quantum 3-Lie algebras from quantum Lie algebras. First we give the following definition.

Definitions 2.5 *For a Z-graded vector space $L=\oplus_{i\in Z}L_i$ over a field K equipped with a 3-ary linear q-3-bracket product $[,,]_q$ satisfying $[L_i,L_j,L_k]_q\subseteq L_{i+j+k}$. If for all $x_i\in L_i, x_j\in L_j, x_k\in L_k$, we have*

$$[x_1,x_2,x_3]_q=sgn(\sigma)[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}],\forall x_1,x_2,x_3\in L \tag{8}$$

and the weighted q-Jacobi identity

$$(2)_{q^{i+j}}[x_i,x_j,[x_k,x_s,x_t]_q]_q=(2)_{q^{s+t}}[[x_i,x_j,x_k]_q,x_s,x_t]_q$$
$$+(2)_{q^{k+t}}[x_k,[x_i,x_j,x_s]_q,x_t]_q+(2)_{q^{k+s}}[x_k,x_s,[x_i,x_j,x_k]_q]_q, \tag{9}$$

$(L,[,,]_q)$ is called a q-3-Lie algebra, where $q\in K, q\neq 0,1$, $dim\ L_i<\infty$.

Theorem 2.6 *Let $(L,[,]_q)$ be a q-Lie algebra over a field K, and $x_0\notin L$. Define the q-3-bracket on vector space $A=L\dot{+}Fx_0$ by*

$$\begin{cases}[x_i,x_j,x_0]_q=[x_i,x_j]_q,\\ [x_i,x_j,x_k]_q=0,\end{cases} \tag{10}$$

for all $x_i\in L_i, x_j\in L_j$ and $x_k\in L_k$. Then $(A,[,,]_q)$ is a q-3-Lie algebra.

Proof It is clear that the q-3-bracket is skew-symmetric, so we need to consider the weighted q-Jacobi identity on $(2)_{q^j}[x_0,x_j,[x_s,x_t,x_0]_q]_q$. From

$$(2)_{q^t}[[x_0,x_j,x_s]_q,x_t,x_0]_q+(2)_{q^s}[x_s,[x_0,x_j,x_t]_q,x_0]_q$$
$$=(2)_{q^t}[[x_j,x_s]_q,x_t]_q+(2)_{q^s}[x_s,[x_j,x_t]_q]_q=(2)_{q^j}[x_j,[x_s,x_t]_q]_q.$$

The result follows.

Theorem 2.7 *Let $(L,[,]_q)$ be a q-Lie algebra over a field K, $f:L\to K$ be a linear function satisfying $f([x_i,x_j]_q)=0$, for all $x_i\in L_i$ and $x_j\in L_j$. Define q-3-bracket product $[,,]_q$ on L by*

$$[x_i,x_j,x_k]_q=f(x_i)q^{j+k}[x_j,x_k]_q+f(x_j)q^{k+i}[x_k,x_i]_q+f(x_k)q^{i+j}[x_i,x_j]_q, \tag{11}$$

then $(L,[,,])$ is a q-3-Lie algebra.

Proof From Eq.(11), the q-3-bracket is skew-symmetric. So we only need to prove the q-Jacobi identity (9). Since

$$q^{i+j}[x_i,x_j,[x_s,x_t,x_r]_q]_q$$
$$=q^{i+j}f(x_i)q^{j+s+t+r}(f(x_s)q^{t+r}[x_j,[x_t,x_r]_q]_q+f(x_t)q^{r+s}[x_j,[x_r,x_s]_q]_q$$
$$+f(x_r)q^{s+t}[x_j,[x_s,x_t]_q]_q)+q^{i+j}f(x_j)q^{i+s+t+r}(f(x_s)q^{t+r}[[x_t,x_r]_q,x_i]_q$$
$$+f(x_t)q^{r+s}[[x_r,x_s]_q,x_i]_q+f(x_r)q^{s+t}[[x_s,x_t]_q,x_i]_q).$$

$$q^{t+r}[[x_i,x_j,x_s]_q,x_t,x_r]_q+q^{r+s}[x_s,[x_i,x_j,x_t]_q,x_r]_q+q^{s+t}[x_s,x_r,[x_i,x_j,x_r]_q,]_q$$
$$=q^{t+r}f(x_t)q^{i+j+r+s}[x_r,[x_i,x_j,x_s]_q]_q+q^{t+r}f(x_r)q^{i+j+s+t}[[x_i,x_j,x_s]_q,x_t]_q$$
$$+q^{s+r}f(x_s)q^{i+j+t+r}[[x_i,x_j,x_t]_q,x_r]_q+q^{s+r}f(x_r)q^{i+j+s+t}[x_s,[x_i,x_j,x_t]_q]_q$$
$$+q^{s+t}f(x_s)q^{i+j+t+r}[x_t,[x_i,x_j,x_r]_q]_q+q^{s+t}f(x_t)q^{i+j+s+r}[[x_i,x_j,x_r]_q,x_s]_q$$
$$=q^{t+r}f(x_t)q^{i+j+r+s}(f(x_i)q^{j+s}[x_r,[x_j,x_s]_q]_q+f(x_j)q^{i+s}[x_r,[x_s,x_i]_q]_q$$
$$+f(x_s)q^{i+j}[x_r,[x_i,x_j]_q]_q)+q^{t+r}f(x_r)q^{i+j+s+t}(f(x_i)q^{j+s}[[x_j,x_s]_q,x_t]_q$$

$+f(x_j)q^{i+s}[[x_s, x_i]_q, x_t]_q + f(x_s)q^{i+j}[[x_i, x_j]_q, x_t]_q)$
$+q^{s+r}f(x_s)q^{i+j+t+r}(f(x_i)q^{j+t}[[x_j, x_t]_q, x_r]_q + f(x_j)q^{t+i}[[x_t, x_i]_q, x_r]_q$
$+f(x_t)q^{i+j}[[x_i, x_j]_q, x_r]_q)+q^{s+r}f(x_r)q^{i+j+s+t}(f(x_i)q^{j+t}[x_s, [x_j, x_t]_q]_q$
$+f(x_j)q^{t+i}[x_s, [x_t, x_i]_q]_q + f(x_t)q^{i+j}[x_s, [x_i, x_j]_q]_q)$
$+q^{s+t}f(x_s)q^{i+j+t+r}(f(x_i)q^{j+r}[x_t, [x_j, x_r]_q]_q + f(x_j)q^{r+i}[x_t, [x_r, x_i]_q]_q$
$+f(x_r)q^{i+j}[x_t, [x_i, x_j]_q]_q) + q^{s+t}f(x_t)q^{i+j+s+r}(f(x_i)q^{j+r}[[x_j, x_r]_q, x_s]_q$
$+f(x_j)q^{r+i}[[x_r, x_i]_q, x_s]_q + f(x_r)q^{i+j}[[x_i, x_j]_q, x_s]_q)$.

Therefore, $(L, [, ,]_q)$ is a q-3-Lie algebra.

Acknowledgements

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

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Received: October, 2015