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Positive solution for a class of p-Laplacian fractional q-difference equations involving the integral boundary condition

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Abstract

In this paper, we study the boundary value problem of a class of p-Laplacian fractional q-difference equations with parameter involving the integral boundary condition. The conditions for the existence of at least one positive solution is established together with the estimates of the lower and upper bounds of the solution at any instant of time. Our results are derived based on the method of upper and lower solutions and the Schauder fixed point theorem. As applications, an example is presented to illustrate our main results.

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Keywords: p-Laplacian fractional q-difference equations, integral boundary condition, upper and lower solutions

1 Introduction

The subject of fractional differential equations has been extensively studied in the last two decades. Since integral and derivative operators of fractional order can describe the characteristics exhibited in many complex processes and systems having long-memory in time, many classical integer-order models for complex systems are being substituted by fractional order models. Fractional calculus also provides an excellent tool to describe the hereditary properties of materials and processes, particularly in viscoelasticity, electrochemistry and porous media (see [1-5]). Many successful new applications of fractional calculus in various fields have also been reported recently. Among all the topics, the existence of positive solutions of boundary value problems (BVPs) for fractional differential equations is currently undergoing active investigation; see, for example, [6-12] and the references therein. Many efforts have also been made to develop the theory of discrete fractional calculus in various directions.

The q-difference calculus is an interesting and old subject. It is a necessary part of discrete mathematics. Studies on q-difference equations appeared already at the beginning of the twentieth century in intensive works especially by Jackson [13,14] and Carmichael [15]. The subject has received a considerable interest of many mathematicians and from theoretical and practical aspects. Specifically, q-difference equations have been widely used in mathematical physical problems, dynamical system and quantum models [16], q-analogues of mathematical physical problems including heat and wave equations [17], sampling theory of signal analysis [18, 19]. What is more, the fractional q-difference calculus plays an important role in quantum calculus. As generalizations of integer order q-difference, fractional q-difference can describe physical phenomena much better and more accurately. Recently, an increasing interest in studying the existence of solutions for boundary value problems of fractional q-difference equations, has been observed [20-28].

John R. Graef [25] study the following boundary value problem with fractional q-derivatives

$$\begin{cases} -(D_q^{\nu}u)(t) = f(t, u(t)) = 0 \quad (0 < t < 1, n-1 < \nu \leq n), \\ (D_q^i u)(0) = 0, \quad i = 0, 1, \dots, n-2, \quad (D_q u)(1) = \sum_{j=1}^{m} a_j (D_q u)(t_j) + \lambda. \end{cases}$$

where $m \geq 1$ and $n \geq 2$ are integers, $\lambda \geq 0$ is a parameter, $a_i \geq 0$ and $t_i \in (0,1)$ for $i = 1, \dots, m$, $f(t,u) : [0,1] \times \mathbb{R} \to [0, +\infty)$ is continuous. The uniqueness, existence, and nonexistence of positive solutions are investigated in terms of different ranges of λ.

Miao and Liang [26], by using a fixed-point theorem in partially ordered sets, studied the existence and uniqueness of a positive and nondecreasing solution for the following fractional q-difference boundary value problem involving the p-Laplacian operator:

$$\begin{cases} D_q^{\gamma}(\varphi_p(D_q^{\alpha}u(t))) + f(t, u(t)) = 0 \quad (0 < t < 1, 0 < \gamma < 1, 2 < \alpha < 3), \\ u(0) = (D_q u)(0) = 0, \quad (D_q u)(1) = 0, \quad D_{0+}^{\gamma} u(t) \mid_{t=0} = 0. \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $f(t,u) : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

Bashir Ahmad [27], by applying a fixed point theorem investigated the existence of solutions for the following nonlinear fractional q-difference integral equation (q-variant of the Langevin equation) with two different fractional orders and nonlocal four-point boundary conditions:

$${}^cD_q^{\beta}({}^cD_q^{\gamma} + \lambda)x(t) = pf(t, x(t)) + kI_q^{\xi}g(t, x(t)), \quad 0 \leq t \leq 1, 0 < q < 1,$$

$$\begin{cases} \alpha_1 x(0) - \beta_1 (t^{(1-\gamma)} D_q x(0)) \mid_{t=0} = \sigma_1 x(\eta_1), \\ \alpha_2 x(1) + \beta_2 D_q x(1) = \sigma_2 x(\eta_2), \end{cases}$$

where ${}^cD_q^{\beta}$ and ${}^cD_q^{\gamma}$ denote the fractional q-derivative of the Caputo type, $0 < \beta, \gamma \leq 1$, $I_q^{\xi}(\cdot)$ denotes Riemann-Liouville integral with $0 < \xi < 1, f, g$ are given continuous functions, λ and p, k are real constants and $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}, \eta_i \in (0,1), i = 1, 2$.

Serkan Araci [28], by using a fixed point theorem proved the existence and uniqueness of a positive and nondecreasing solution for the following fractional q-difference boundary value problem involving the p-Laplacian operator:

$$D_q^\gamma(\varphi_p(D_q^\delta y(t))) + f(t, y(t)) = 0 \quad (0 < t < 1, 0 < \gamma < 1, 3 < \delta < 4),$$

$$\begin{cases} y(0) = (D_q y)(0) = (D_q^2 y)(0) = 0, \\ a_1(D_q y)(1) + a_2(D_q^2 y)(1) = 0, \quad D_{0+}^\gamma u(t) \mid_{t=0} = 0. \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $f(t, y) : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

Motivated essentially by the aforementioned work by Serkan Araci, we investigate the existence of solutions for the following boundary value problem of a class of p-Laplacian fractional q-difference equations with parameter involving the integral boundary condition:

$$\begin{cases} -D_q^\beta(\varphi_p(D_q^\alpha x))(t) = \lambda f(t, x(t)), t \in (0, 1), \\ x(0) = 0, \quad (D_q^\alpha x)(0) = 0, \quad (D_q^\nu x)(1) = \int_0^1 x(s) d_q A(s), \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \nu < 1$, $\alpha - \nu > 1$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $d_q A(s) = D_q A(s) d_q s$, $f(t, x) : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous, $A(s) : [0, 1] \to \mathbb{R}$ is continuous.

By using the method of upper and lower solutions and the Schauder fixed point theorem, so as to determine the interval of eigenvalue for the existence of positive solutions.

2 Preliminary notes

In the following section, we collect some definitions and lemmas about fractional q-integral and fractional q-derivative which are referred to in [20, 29].

let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$(a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

It is easy to see that $[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}$. And note that if $b=0$, then $a^{(\alpha)} = a^\alpha$.

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,\cdots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x\to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x), \quad (D_q^n f)(x) = D_q (D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined on the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0,b].$$

If $a \in [0,b]$, and f is defined on the interval $[0,b]$, its q-integral from a to b is defined by

$$\int_a^b f(x) d_q t = \int_0^b f(x) d_q t - \int_0^a f(x) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined as

$$(I_q^0 f)(x) = f(x), \quad (I_q^n f)(x) = I_q (I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x=0$, then

$$(I_q D_q f)(x) = f(x) - f(0),$$

Basic properties of q-integral operator and q-differential operator can be found in the book [30]. We now point out two formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i).

$${}_tD_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$({}_xD_q \int_0^x f(x,t) d_q t)(x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x).$$

We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

Definition 2.1 [31] *Let $\alpha \geq 0$, and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$, and*

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, \quad x \in [0,1].$$

Definition 2.2 [31] *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$, and*

$$(D_q^\alpha f)(x) = (D_q^p I_q^{p-\alpha} f)(x), \quad \alpha > 0.$$

where p is the smallest integer greater than or equal to α.

Next, we list some properties about q-derivative and q-integral that are already known in the literature.

Lemma 2.3 [31] *Let $\alpha, \beta \geq 0$, $\lambda \in (-1, \infty)$, and f be a function defined on $[0,1]$. Then the following formulas hold:*
(i) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\beta+\alpha} f)(x)$;
(ii) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$;
(iii) $I_q^\alpha [1](x) = \frac{x^\alpha}{\Gamma_q(\alpha+1)}$;
(iv) $I_q^\alpha x^\lambda = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} x^{\alpha+\lambda}$.

Lemma 2.4 [31]*Let $\alpha > 0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha + k - p + 1)} (D_q^k f)(0).$$

Definition 2.5 [12] *A continuous function $\Psi(t)$ is called a lower solution of the boundary value problem (1), if it satisfies*

$$\begin{cases} -D_q^\beta(\varphi_p(D_q^\alpha \Psi))(t) \leq \lambda f(t, \Psi(t)), \quad t \in (0,1), \\ \Psi(0) \geq 0, \quad (D_q^\alpha \Psi)(0) \geq 0, \quad (D_q^\nu \Psi)(1) \geq \int_0^1 \Psi(s) d_q A(s). \end{cases}$$

Definition 2.6 [12] *A continuous function $\Phi(t)$ is called a upper solution of the boundary value problem (1), if it satisfies*

$$\begin{cases} -D_q^\beta(\varphi_p(D_q^\alpha \Phi))(t) \geq \lambda f(t, \Phi(t)), \quad t \in (0,1), \\ \Phi(0) \leq 0, \quad (D_q^\alpha \Phi)(0) \leq 0, \quad (D_q^\nu \Phi)(1) \leq \int_0^1 \Phi(s) d_q A(s). \end{cases}$$

Lemma 2.7 (*Schauder fixed point theorem*) [12] *Let T be a continuous and compact mapping of a Banach space E into itself, such that the set*

$$\{x \in E : x = \sigma T x, \text{ for some } 0 \leq \sigma \leq 1\}$$

is bounded. Then T has a fixed point.

3 Green function and related properties

Denote by $C[0,1]$ the space of all continuous functions on $[0,1]$ with the usual norm $\|u\| = \max_{t\in[0,1]} \mid u(t) \mid$. Indeed, $C[0,1]$ is a Banach space with a partial order, namely for $x, y \in C[0,1], x \leq y \Longleftrightarrow x(t) \leq y(t)$ for $t \in [0,1]$.

Theorem 3.1 *Let $1 < \alpha \leq 2$, $0 < \nu < 1$, $\alpha - \nu > 1$, Suppose that $h(t) : [0,1] \to [0,+\infty)$ is continuous function, $A(s) : [0,1] \to \mathbb{R}$ is continuous, then the boundary value problem*

$$\begin{cases} D_q^\alpha x(t) + h(t) = 0, \\ x(0) = 0, (D_q^\nu x)(1) = \int_0^1 x(s) d_q A(s) \end{cases}$$

has the unique solution

$$x(t) = \int_0^1 J(t,s) h(s) d_q s,$$

where

$$J(t,s) = G(t,qs) + \frac{t^{\alpha-1}}{L} G_A(s), \tag{2}$$

Green function

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} t^{\alpha-1}(1-qs)^{(\alpha-\nu-1)} - (t-qs)^{(\alpha-1)}, & 0 \leq qs \leq t \leq 1, \\ t^{\alpha-1}(1-qs)^{(\alpha-\nu-1)}, & 0 \leq t \leq qs \leq 1. \end{cases}$$

$$G_A(s) = \int_0^1 G(t,qs) d_q A(t), \quad L = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} - \int_0^1 s^{\alpha-1} d_q A(s) \neq 0.$$

Proof In view of item (ii) of Lemma 2.3 and Lemma 2.4, we have

$$x(t) = -I_q^\alpha h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_q s + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{3}$$

Using the boundary condition $x(0) = 0$, we must set $C_2 = 0$. Since

$$D_q^\nu x(t) = -D_q^\nu I_q^\alpha h(t) + C_1 D_q^\nu t^{\alpha-1} = -I_q^{\alpha-\nu} h(t) + C_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} t^{\alpha-\nu-1},$$

we have

$$D_q^\nu x(1) = C_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} - \frac{1}{\Gamma_q(\alpha-\nu)} \int_0^1 (1-qs)^{(\alpha-\nu-1)} h(s) d_q s.$$

From the formula (3), we have

$$\begin{aligned}\int_0^1 x(s)d_qA(s) &= -\frac{1}{\Gamma_q(\alpha)}\int_0^1 [\int_0^s (s-qw)^{(\alpha-1)}h(w)d_qw]d_qA(s) + C_1\int_0^1 s^{\alpha-1}d_qA(s)\\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^1 [\int_{wq}^1 (s-qw)^{(\alpha-1)}d_qA(s)]h(w)d_qw + C_1\int_0^1 s^{\alpha-1}d_qA(s).\end{aligned}$$

Using the boundary condition $D_q^\nu x(1) = \int_0^1 x(s)d_qA(s)$, we have

$$\begin{aligned}C_1[\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} - \int_0^1 s^{\alpha-1}d_qA(s)] &= \frac{1}{\Gamma_q(\alpha-\nu)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs\\ &\quad -\frac{1}{\Gamma_q(\alpha)}\int_0^1 [\int_{wq}^1 (s-qw)^{(\alpha-1)}d_qA(s)]h(w)d_qw.\end{aligned}$$

Therefore, let $L = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} - \int_0^1 s^{\alpha-1}d_qA(s)$, we have

$$C_1 = \frac{1}{L\Gamma_q(\alpha-\nu)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs - \frac{1}{L\Gamma_q(\alpha)}\int_0^1 [\int_{sq}^1 (t-qs)^{(\alpha-1)}d_qA(t)]h(s)d_qs.$$

Thus, we can get

$$\begin{aligned}x(t) &= \frac{t^{\alpha-1}}{L\Gamma_q(\alpha-\nu)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs - \frac{t^{\alpha-1}}{L\Gamma_q(\alpha)}\int_0^1 [\int_{sq}^1 (t-qs)^{(\alpha-1)}d_qA(t)]h(s)d_qs\\ &\quad -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)d_qs\\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs\\ &\quad -\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs + \frac{t^{\alpha-1}}{L}\Big\{\frac{1}{\Gamma_q(\alpha-\nu)}\int_0^1 (1-qs)^{(\alpha-\nu-1)}h(s)d_qs\\ &\quad -\frac{1}{\Gamma_q(\alpha)}\int_0^1 [\int_{sq}^1 (t-qs)^{(\alpha-1)}d_qA(t)]h(s)d_qs\Big\}\\ &= \int_0^1 G(t,qs)h(s)d_qs + \frac{t^{\alpha-1}}{L\Gamma_q(\alpha)}\Big\{\int_0^1 [\int_0^1 t^{\alpha-1}d_qA(t)](1-qs)^{(\alpha-\nu-1)}h(s)d_qs\\ &\quad -\int_0^1 [\int_{sq}^1 (t-qs)^{(\alpha-1)}d_qA(t)]h(s)d_qs\Big\}\\ &= \int_0^1 G(t,qs)h(s)d_qs + \frac{t^{\alpha-1}}{L}\int_0^1 G_A(s)h(s)d_qs\\ &= \int_0^1 J(t,s)h(s)d_qs.\end{aligned}$$

The proof is completed.

Theorem 3.2 *Assume* $0 < L < 1$, *and* $G_A(s) \geq 0$ *for* $s \in [0,1]$, *then function* $J(t,s)$ *define by the formula* (2) *satisfies*

(a_1) $J(t,s) \geq 0$, *for all* $s, t \in [0,1]$;

(a_2) *There exist a constant* $C^* = \frac{||G_A(s)||}{L} + \frac{1}{\Gamma_q(\alpha)}$, *such that*

$$\frac{t^{\alpha-1}G_A(s)}{L} \leq J(t,s) \leq C^* t^{\alpha-1} \leq C^*, \quad t,s \in [0,1]. \tag{4}$$

Proof (a_1) The proof is similar to the proof of Lemma 3.0.7 in references [32], so $J(t,s) \geq 0$ is clearly established.

(a_2) Since

$$G(t,qs) \leq \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}(1-qs)^{(\alpha-\nu-1)} \leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}, \quad t,s \in [0,1],$$

according to the formula (2), we can see that

$$\frac{t^{\alpha-1}G_A(s)}{L} \leq J(t,s) \leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{||G_A(s)||t^{\alpha-1}}{L} = C^* t^{\alpha-1} \leq C^*.$$

The proof is completed.

To study the boundary value problem (1), we first consider the associated linear the boundary value problem

$$\begin{cases} -D_q^{\beta}(\varphi_p(D_q^{\alpha}x))(t) = \eta(t), \\ x(0) = (D_q^{\alpha}x)(0) = 0, \quad (D_q^{\nu}x)(1) = \int_0^1 x(s)d_qA(s), \end{cases} \tag{5}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \nu < 1$, $\alpha - \nu > 1$, $\eta(t) : [0,1] \to [0,+\infty)$ is continuous, $A(s) : [0,1] \to \mathbb{R}$ is continuous.

For convenience, let $g > 1$ satisfy the relation $\frac{1}{p} + \frac{1}{g} = 1$, where p is given by the boundary value problem (1), and let $b = [\Gamma_q(\beta)]^{-1}$, then we have the following theorem.

Theorem 3.3 *The associated linear the boundary value problem* (5) *has the unique positive solution*

$$x(t) = \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}\eta(\tau)d_q\tau]^{g-1}d_qs. \tag{6}$$

Proof Let $v = \varphi_p(D_q^{\alpha}x)$. Then the solution of the boundary value problem

$$\begin{cases} -D_q^{\beta}v(t) = \eta(t), \quad t \in (0,1), \\ v(0) = 0 \end{cases}$$

is given by $v(t) = -I_q^{\beta}\eta(t) + C_1 t^{\beta-1}$, $t \in [0,1]$. From the relations $v(0) = 0$ and $0 < \beta \leq 1$, we get $C_1 = 0$, and consequently

$$v(t) = -\frac{1}{\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}\eta(s)d_qs. \tag{7}$$

Therefore the solution of the boundary value problem (5) satisfies

$$\begin{cases} D_q^{\alpha}x(t) = \varphi_p{}^{-1}(-I_q^{\beta}\eta(t)), \\ x(0) = 0, \quad (D_q^{\nu}x)(1) = \int_0^1 x(s)d_qA(s). \end{cases} \tag{8}$$

By theorem 3.1, the solution of the boundary value problem (5) can be written as

$$x(t) = \int_0^1 J(t,s)\varphi_p{}^{-1}(I_q^{\beta}\eta(s))d_qs, \quad t \in [0,1].$$

Since $\eta(s) \geq 0$, $s \in [0,1]$, we have $\varphi_p{}^{-1}(I_q^{\beta}\eta(s)) = (I_q^{\beta}\eta(s))^{g-1}$, which implies that the solution of the boundary value problem (5) is

$$x(t) = \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}\eta(\tau)d_q\tau]^{g-1}d_qs, \quad t \in [0,1].$$

The proof is completed.

Remark 3.4 *Assume $0 < L < 1$ and $G_A(s) \geq 0$ for $s \in [0,1]$, and $x \in C([0,1],\mathbb{R})$ satisfies $x(0) = 0$, $(D_q^{\nu}x)(1) = \int_0^1 x(s)d_qA(s)$, and $-D_q^{\alpha}x(t) \geq 0$ for any $t \in [0,1]$. Then $x(t) \geq 0$, $t \in [0,1]$.*

4 Main results

To establish the existence of a solution to the boundary value problem (1), we need to make the following assumptions.

(H_0) $A(s) : [0,1] \to \mathbb{R}$ is a continuous function satisfying $G_A(s) \geq 0$ for $s \in [0,1]$; and $0 < L < 1$.

(H_1) $f(t,x) : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous and is non-increasing in $x > 0$; and for any constant $r \in (0,1)$, there exists a constant $\epsilon > 0$, such that, for any $(t,x) \in [0,1] \times [0,+\infty)$ have

$$f(t,rx) \leq r^{-\epsilon}f(t,x). \tag{9}$$

Remark 4.1 *For constant $r > 1$, we have $\frac{1}{r} \in (0,1)$, and $y = rx \in [0,\infty)$, given $x \in [0,+\infty)$, and thus from (H_1), we have*

$$f(t,\frac{y}{r}) \leq (\frac{1}{r})^{-\epsilon}f(t,y),$$

using $y = rx$, we have

$$f(t,rx) \geq r^{-\epsilon}f(t,x). \tag{10}$$

For notational convenience, we write $u(t) = f(t, t^{\alpha-1})$, for any $t \in [0,1]$. The main result of this paper is the assertion in Theorem 4.2.

Theorem 4.2 *Suppose* (H_0) *and* (H_1) *hold. Then there exists a constant* $\lambda^* > 0$, *such that the boundary value problem (1) has at least one positive solution* $w(t)$ *for any* $\lambda \in (\lambda^*, \infty)$, *and moreover there exist two constants* $l \in (0,1)$ *and* $L_1 \in (1, \infty)$ *such that* $lt^{\alpha-1} \leq w(t) \leq L_1 t^{\alpha-1}$.

Proof Let $E = C[0,1]$, and define a subset P of E as follows:
$P = \{x \in E$: there exists a constant $l \in (0,1)$, such that $lt^{\alpha-1} \leq x(t) \leq l^{-1}t^{\alpha-1}, t \in [0,1]\}$.
Clearly, P is a nonempty set since $t^{\alpha-1} \in P$. Now define the operator T_λ in E:

$$T_\lambda x(t) = \lambda \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)} f(\tau, x(\tau)) d_q\tau]^{g-1} d_q s, \quad t \in [0,1]. \tag{11}$$

We assert that T_λ is well defined and $T_\lambda(P) \subset P$. In fact, for any $x \in P$, there exists a positive number $l_x \in (0,1)$ such that $l_x t^{\alpha-1} \leq x(t) \leq l_x^{-1} t^{\alpha-1}$, $t \in [0,1]$. And thus, by Theorem 3.2 and (H$_1$), one gets

$$\begin{aligned}
T_\lambda x(t) &= \lambda b^{g-1} \int_0^1 J(t,s)[\int_0^s (s-q\tau)^{(\beta-1)} f(\tau, x(\tau)) d_q\tau]^{g-1} d_q s \\
&\leq \lambda b^{g-1} \int_0^1 J(t,s)[\int_0^s (s-q\tau)^{(\beta-1)} f(\tau, l_x\tau^{\alpha-1}) d_q\tau]^{g-1} d_q s \\
&\leq \lambda b^{g-1} l_x^{-\epsilon(g-1)} C^* t^{\alpha-1} \int_0^1 [\int_0^s (s-q\tau)^{(\beta-1)} u(\tau) d_q\tau]^{g-1} d_q s \\
&\leq \lambda b^{g-1} l_x^{-\epsilon(g-1)} C^* t^{\alpha-1} ||u||^{g-1} \int_0^1 [\int_0^s (s-q\tau)^{(\beta-1)} d_q\tau]^{g-1} d_q s \\
&= \lambda l_x^{-\epsilon(g-1)} C^* t^{\alpha-1} ||u||^{g-1} \int_0^1 \frac{s^{\beta(g-1)}}{[\Gamma_q(\beta+1)]^{g-1}} d_q s \\
&= \frac{\lambda l_x^{-\epsilon(g-1)} C^* t^{\alpha-1} ||u||^{g-1}}{[\Gamma_q(\beta+1)]^{g-1}} \frac{1}{[\beta(g-1)+1]_q} < +\infty.
\end{aligned} \tag{12}$$

On the other hand, from formulas (4) and (10), we have

$$\begin{aligned}
T_\lambda x(t) &= \lambda b^{g-1} \int_0^1 J(t,s)[\int_0^s (s-q\tau)^{(\beta-1)} f(\tau, x(\tau)) d_q\tau]^{g-1} d_q s \\
&\geq \lambda b^{g-1} \int_0^1 J(t,s)[\int_0^s (s-q\tau)^{(\beta-1)} f(\tau, l_x^{-1}\tau^{\alpha-1}) d_q\tau]^{g-1} d_q s \\
&\geq \frac{\lambda b^{g-1} t^{\alpha-1} l_x^{\epsilon(g-1)}}{L} \int_0^1 G_A(s)[\int_0^s (s-q\tau)^{(\beta-1)} u(\tau) d_q\tau]^{g-1} d_q s.
\end{aligned} \tag{13}$$

Choose

$$\tilde{l}_x = \min\left\{\frac{1}{2}, \frac{\lambda b^{g-1} l_x^{\epsilon(g-1)}}{L}\int_0^1 G_A(s)[\int_0^s (s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs,\right.$$

$$\left.\frac{[\Gamma_q(\beta+1)]^{g-1}[\beta(g-1)+1]_q}{\lambda l_x^{-\epsilon(g-1)}C^*||u||^{g-1}}.\right\} \tag{14}$$

then it follows from formulas (12)-(14) that

$$\tilde{l}_x t^{\alpha-1} \leq (T_\lambda x)(t) \leq \tilde{l}_x^{-1} t^{\alpha-1},$$

which implies that T_λ is well defined and $T_\lambda(P) \subset P$. Furthermore, comparing formulas (6) and (11), $T_\lambda x(t)$ must also satisfy the boundary value problem (5) with $\eta(t)$ replaced by $\lambda f(t, x(t))$, namely

$$\begin{cases} -D_q^\beta(\varphi_p(D_q^\alpha(T_\lambda x)))(t) = \lambda f(t, x(t)), \\ (T_\lambda x)(0) = 0, (D_q^\alpha(T_\lambda x))(0) = 0, (D_q^\nu(T_\lambda x))(1) = \int_0^1 (T_\lambda x)(s)d_qA(s). \end{cases} \tag{15}$$

Next we shall devote to finding the upper and lower solutions of the the boundary value problem (1). Firstly, let

$$e(t) = \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs, \quad t \in [0,1].$$

By Theorem 3.2, we have

$$e(t) \geq \frac{t^{\alpha-1}}{L}\int_0^1 G_A(s)[\int_0^s b(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs, \quad t \in [0,1],$$

and consequently there exists a constant $\lambda_1 \geq 1$ such that

$$\lambda_1 e(t) \geq t^{\alpha-1}, \quad t \in [0,1]. \tag{16}$$

On the other hand, by (H$_1$), we know that the operator T_λ is decreasing, and thus for any $\lambda > \lambda_1$, from (12), we have

$$\begin{aligned} & \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau, \lambda e(\tau))d_q\tau]^{g-1}d_qs \\ \leq & \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau, \lambda_1 e(\tau))d_q\tau]^{g-1}d_qs \\ \leq & \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau, \tau^{\alpha-1})d_q\tau]^{g-1}d_qs \\ = & \int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs < +\infty, \end{aligned}$$

and

$$\begin{aligned}
e(t) &\leq C^*b^{g-1}\int_0^1[\int_0^s(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs\\
&\leq C^*||u||^{g-1}b^{g-1}\int_0^1[\int_0^s(s-q\tau)^{(\beta-1)}d_q\tau]^{g-1}d_qs\\
&= C^*||u||^{g-1}\int_0^1[\frac{s^\beta}{\Gamma_q(\beta+1)}]^{g-1}d_qs\\
&= \frac{C^*||u||^{g-1}}{[\Gamma_q(\beta+1)]^{g-1}}\int_0^1 s^{\beta(g-1)}d_qs\\
&= \frac{C^*||u||^{g-1}}{[\Gamma_q(\beta+1)]^{g-1}}\frac{1}{[\beta(g-1)+1]_q} < +\infty.
\end{aligned}$$

Now let $\rho = \frac{C^*||u||^{g-1}}{[\Gamma_q(\beta+1)]^{g-1}[\beta(g-1)+1]_q]} + 1$, and take

$$\lambda^* = max\left\{\lambda_1, \{\frac{\rho^{-\epsilon(g-1)}}{L}\int_0^1 G_A(s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,1)d_q\tau]^{g-1}d_qs\}^{\frac{1}{\epsilon(g-1)-1}}\right\}.$$

Then by Theorem 3.2 and (10), we obtain

$$\begin{aligned}
+\infty &> \lambda^*\int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,\lambda^*e(\tau))d_q\tau]^{g-1}d_qs\\
&\geq (\lambda^*)^{1-\epsilon(g-1)}\frac{t^{\alpha-1}}{L}\int_0^1 G_A(s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,e(\tau))d_q\tau]^{g-1}d_qs\\
&\geq (\lambda^*)^{1-\epsilon(g-1)}\frac{t^{\alpha-1}}{L}\int_0^1 G_A(s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,\rho)d_q\tau]^{g-1}d_qs\\
&\geq (\lambda^*)^{1-\epsilon(g-1)}\frac{t^{\alpha-1}}{L}\rho^{-\epsilon(g-1)}\int_0^1 G_A(s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,1)d_q\tau]^{g-1}d_qs
\end{aligned}$$

$$\geq t^{\alpha-1},\quad t\in[0,1]. \tag{17}$$

Let

$$\phi(t) = \lambda^*e(t), \psi(t) = \lambda^*\int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}f(\tau,\lambda^*e(\tau))d_q\tau]^{g-1}d_qs,$$

then

$$\phi(t) = T_{\lambda^*}(t^{\alpha-1}),\quad \psi(t) = T_{\lambda^*}(\phi(t)). \tag{18}$$

It follows from formulas (16) and (17) that for any $t\in[0,1]$,

$$\psi(t)\geq t^{\alpha-1}, \phi(t) = \lambda^*\int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs \geq \lambda_1e(t)\geq t^{\alpha-1}. \tag{19}$$

Moreover, by formulas (15) and (18), we know

$$\begin{cases} \phi(0)=0, \quad (D_q^\alpha\phi)(0)=0, \quad (D_q^\nu\phi)(1)=\int_0^1\phi(s)d_qA(s), \\ \psi(0)=0, \quad (D_q^\alpha\psi)(0)=0, \quad (D_q^\nu\psi)(1)=\int_0^1\psi(s)d_qA(s). \end{cases} \tag{20}$$

Proceeding as in formulas (12)-(14), we get that $\phi(t)$, $\psi(t)\in P$. By formulas (18) and (19), for $t\in(0,1)$, we have

$$t^{\alpha-1}\leq\psi(t)=T_{\lambda^*}(\phi(t))\leq\lambda^*\int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}u(\tau)d_q\tau]^{g-1}d_qs=\phi(t). \tag{21}$$

Thus, taking account of f being non-increasing in $x>0$, and by formulas (15), (18), (19) and (21), we have

$$\begin{aligned} &D_q^\beta(\varphi_p(D_q^\alpha\psi))(t)+\lambda^*f(t,\psi(t))=D_q^\beta(\varphi_p(D_q^\alpha(T_{\lambda^*}(\phi))))(t)+\lambda^*f(t,\psi(t)) \\ &\geq D_q^\beta(\varphi_p(D_q^\alpha(T_{\lambda^*}(\phi))))(t)+\lambda^*f(t,\phi(t))=-\lambda^*f(t,\phi(t))+\lambda^*f(t,\phi(t))=0, \end{aligned} \tag{22}$$

and

$$\begin{aligned} &D_q^\beta(\varphi_p(D_q^\alpha\phi))(t)+\lambda^*f(t,\phi(t))=D_q^\beta(\varphi_p(D_q^\alpha(T_{\lambda^*}(t^{\alpha-1}))))(t)+\lambda^*f(t,\phi(t)) \\ &=-\lambda^*f(t,t^{\alpha-1})+\lambda^*f(t,\phi(t))\leq-\lambda^*f(t,t^{\alpha-1})+\lambda^*f(t,t^{\alpha-1})=0. \end{aligned} \tag{23}$$

It follows from formulas (20)-(23) that $\phi(t)$, $\psi(t)$ are upper and lower solutions of the boundary value problem (1), and $\psi(t)$, $\phi(t)\in P$.

Now let us define a function

$$F(t,x)=\begin{cases} f(t,\psi(t)), x<\psi(t); \\ f(t,x(t)), \psi(t)\leq x\leq\phi(t); \\ f(t,\phi(t)), x>\phi(t). \end{cases} \tag{24}$$

It then follows from (H_1) and the formula (24) that $F(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous.

We now show that the boundary value problem

$$\begin{cases} -D_q^\beta(\varphi_p(D_q^\alpha x))(t)=\lambda^*F(t,x(t)), t\in(0,1), \\ x(0)=0, (D_q^\alpha x)(0)=0, (D_q^\nu x)(1)=\int_0^1 x(s)d_qA(s). \end{cases} \tag{25}$$

has a positive solution.

Define the operator B_{λ^*} by

$$B_{\lambda^*}x(t)=\lambda^*\int_0^1 J(t,s)[\int_0^s b(s-q\tau)^{(\beta-1)}F(\tau,x(\tau))d_q\tau]^{g-1}d_qs, \quad t\in[0,1]. \tag{26}$$

Then $B_{\lambda^*}: C[0,1] \to C[0,1]$, and a fixed point of the operator B_{λ^*} is a solution of the boundary value problem (25).

On the other hand, from the definition of F and the fact that the function $f(t,x)$ is nonincreasing in x, we obtain $f(t,\phi(t)) \le F(t,x(t)) \le f(t,\psi(t))$ provided that $\psi(t) \le x \le \phi(t)$, $F(t,x(t)) = f(t,\psi(t))$ provided that $x < \psi(t)$, and $F(t,x(t)) = f(t,\phi(t))$ provided that $x > \phi(t)$. So we have

$$f(t,\phi(t)) \le F(t,x(t)) \le f(t,\psi(t)), \quad \forall x \in E. \tag{27}$$

Furthermore, by formulas (19) and (27), we have

$$f(t,\phi(t)) \le F(t,x(t)) \le f(t,t^{\alpha-1}) = u(t), \quad \forall x \in E. \tag{28}$$

It follows from formulas (4) and (28) that for any $x \in E$,

$$\begin{aligned} B_{\lambda^*}x(t) &= \lambda^* \int_0^1 J(t,s)\Big[\int_0^s b(s-q\tau)^{(\beta-1)} F(\tau,x(\tau))d_q\tau\Big]^{g-1} d_qs \\ &\le \lambda^* C^* \int_0^1 \Big[\int_0^s b(s-q\tau)^{(\beta-1)} u(\tau)d_q\tau\Big]^{g-1} d_qs \\ &\le \frac{\lambda^* C^* ||u||^{g-1}}{[\Gamma_q(\beta+1)]^{g-1}} \frac{1}{[\beta(g-1)+1]_q} < +\infty. \end{aligned} \tag{29}$$

namely, the operator B_{λ^*} is uniformly bounded.

On the other hand, let $\Omega \subset E$ be bounded. As the function $J(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, $B_{\lambda^*}(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we have $B_{\lambda^*}: E \to E$ is completely continuous. Moreover, the formula (29) implies that Lemma 2.7 holds, thus, by using the Schauder fixed point theorem, B_{λ^*} has at least one fixed point w such that $w = B_{\lambda^*}w$.

Now we prove

$$\psi(t) \le w(t) \le \phi(t), \quad t \in [0,1].$$

Since w is a fixed point of B_{λ^*}, by formulas (20) and (25), we have

$$\begin{cases} w(0) = 0, \quad (D_q^\alpha w)(0) = 0, \quad (D_q^\nu w)(1) = \int_0^1 w(s)d_qA(s), \\ \phi(0) = 0, \quad (D_q^\alpha \phi)(0) = 0, \quad (D_q^\nu \phi)(1) = \int_0^1 \phi(s)d_qA(s). \end{cases} \tag{30}$$

From formulas (15), (18), (28) and noticing that w is a fixed point of B_{λ^*}, we also have

$$D_q^\beta(\varphi_p(D_q^\alpha\phi))(t) - D_q^\beta(\varphi_p(D_q^\alpha w))(t) = -\lambda^* f(t,t^{\alpha-1}) + \lambda^* F(t,w(t)) \le 0, \forall t \in [0,1].$$

Let $z(t) = \varphi_p(D_q^\alpha\phi)(t) - \varphi_p(D_q^\alpha w)(t)$. Then

$$\begin{aligned} &(D_q^\beta z)(t) = D_q^\beta(\varphi_p(D_q^\alpha\phi))(t) - D_q^\beta(\varphi_p(D_q^\alpha w))(t) \le 0, \\ &z(0) = \varphi_p(D_q^\alpha\phi)(0) - \varphi_p(D_q^\alpha w)(0) = 0. \end{aligned}$$

It follows from the formula (7) that $z(t) \leq 0$, that is

$$\varphi_p(D_q^\alpha \phi)(t) \leq \varphi_p(D_q^\alpha w)(t).$$

Noticing that φ_p is monotone increasing, we have

$$(D_q^\alpha \phi)(t) \leq (D_q^\alpha w)(t), \quad i.e., \quad (D_q^\alpha(\phi - w))(t) \leq 0.$$

It follows from Remark 3.4 and the formula (30) that $\phi(t) - w(t) \geq 0$. Thus we have $\phi(t) \geq w(t)$, $t \in [0,1]$. By the same way, we also have $w(t) \geq \psi(t)$, $t \in [0,1]$. So

$$\psi(t) \leq w(t) \leq \phi(t), \quad t \in [0,1]. \tag{31}$$

Consequently, $F(t, w(t)) = f(t, w(t))$. Hence $w(t)$ is a positive solution of the boundary value problem (1).

Finally, by the formula (31) and ϕ, $\psi \in P$, we have

$$l_\psi t^{\alpha-1} \leq \psi(t) \leq w(t) \leq \phi(t) \leq l_\phi t^{\alpha-1}.$$

The proof is completed.

5 An example

In this section, we present a simple example to explain the main result.

Example 5.1 *Consider the following fractional q-differential equations with parameter involving the integral boundary condition*

$$\begin{cases} -D_{0.5}^{0.5}(\varphi_p(D_{0.5}^{1.8}x))(t) = \lambda(x+t)^{-\frac{1}{6}}, \quad t \in (0,1), \\ x(0) = 0, (D_{0.5}^{1.8}x)(0) = 0, (D_{0.5}^{0.3}x)(1) = \int_0^1 x(s)d_qs^2. \end{cases} \tag{32}$$

where $q = 0.5$, $\alpha = 1.8$, $\beta = 0.5$, $\nu = 0.3$, $A(s) = s^2$, $f(t,x) = (x+t)^{-\frac{1}{6}}$.

It is obvious that a direct calculation shows that $G_A(s) = \int_0^1 G(t,qs)d_qA(t) \geq 0$, and

$$\begin{aligned} L = & \tfrac{\Gamma_{0.5}(1.8)}{\Gamma_{0.5}(1.5)} - \int_0^1 s^{0.8}d_qs^2 \approx 1.0401 - [2]_q \int_0^1 s^{1.8}d_qs \\ = & \quad 1.0401 - \tfrac{[2]_q}{[2.8]_q} \approx 1.084 - 0.8757 = 0.1649. \end{aligned}$$

Clearly, $0 < L < 1$. So (H_0) holds.

On the other hand, let $\epsilon = \frac{1}{6}$, then for all positive numbers $r < 1$, we have

$$f(t, rx) = (rx+t)^{-\frac{1}{6}} = r^{-\frac{1}{6}}(x + \frac{t}{r})^{-\frac{1}{6}} \leq r^{-\frac{1}{6}}(x+t)^{-\frac{1}{6}} = r^{-\epsilon}f(t,x).$$

So (H_1) holds. By Theorem 4.2 there exists a constant $\lambda^* > 0$, such that for any $\lambda \in (\lambda^*, \infty)$, the boundary value problem (32) has at least one positive solution $w(t)$, and there exist two constants $l \in (0, 1)$ and $L_1 \in (1, \infty)$, such that $lt^{\alpha-1} \leq w(t) \leq L_1 t^{\alpha-1}$.

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