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Some estimates on the Simpson's type inequalities through s-convex and quasi-convex stochastic processes

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Abstract

We obtain several inequalities of Simpson's type by giving explicit error bounds in the Simpson's rules, by means of Peano type kernels and results from the modern theory of inequalities. A midpoint type inequalities are given. The approach presented here using s-convex and quasi-convex stochastic processes in terms of up to second derivatives are obtained for first time.

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1 Introduction

In numerical analysis, the Simpson's quadrature is a numerical integration method used to obtain the approximate value of definite integrals. This quadrature is a combination of the Simpson's rule and inequality. To show this, we define the Simpson's rule first. For $f : [a, b] \to \mathbf{R}$ a continuous function, the Simpson's rule approximates the value of the integral of f in (a, b) as follow:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \quad (1)$$

One of the most known numerical integration results is the Simpson's inequality. Suppose that $f : [a, b] \to \mathbf{R}$ is a four times continuously differentiable mapping on (a,b) and having the fourth derivative bounded on (a,b), that is, $||f^{(4)}||_\infty := \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$, then the following inequality:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{1}{2880}||f^{(4)}||_\infty (b-a)^5, \tag{2}$$

holds and it is well known in the literature as Simpson's inequality.

Now, if we assume that $\pi : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we have the Simpson's quadrature formula:

$$\int_a^b f(x)dx = P_S(f,\pi) + R_S(f,\pi), \tag{3}$$

where $P_S(f,\pi)$ is the Simpson's rule:

$$P_S(f,\pi) := \frac{1}{6}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i, \tag{4}$$

and the remainder term $R_S(f,\pi)$ satisfies the estimation:

$$|R_S(f,\pi)| := \frac{1}{2880}||f^{(4)}||_\infty \sum_{i=0}^{n-1} h_i^5, \tag{5}$$

where $h_i := x_{i+1} - x_i$ for $i = 0, \dots n-1$.

The Simpson's inequality and quadrature formula are one of the most used quadrature formula in practical applications [1]. Many authors have established error estimations for the Simpson's inequality, for refinements, counterparts, generalizations and new Simpson's type inequality. In the years 1998 and 1999, Dragomir in [6], [7], [8], worked in estimations of remainder for Simpson's quadrature formula for mapping of bounded variation, Lipschitzians and differentiable mappings whose derivatives belong to L_p spaces and applications in theory of special means. However, in 2000, Dragomir *et. al* [9], established some very recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth. In the same year, Dragomir *et. al* in [10] gave new trapezoid inequality as well as Simpson and Ostrowski type inequalities for monotonic functions. They provided their applications in probability theory, numerical analysis and for special means [10]. In 2000, Pečarić *et. al* in [19] generalized the results obtained by Dragomir in [7] using functions whose n^{th} derivatives, $n \in \{2,3,4\}$, belong to L_p spaces. Later, in 2004 Ujević in [24], presented two sharp inequalities, the first is a sharp Simpson's inequality and the second is a sharp inequality of Ostrowski type. The mentioned inequalities given error bounds for some known quadrature rules together with applications in numerical analysis were given as well. Then in the same year Ujević in [25] established two sharp inequalities if Simpson type whose first derivatives are absolutely continuous and second derivatives belong to $L_2(a, b)$ and gave some applications in numerical integration. Also, they obtained a sharp inequality with also gives an error bound for Simpson's quadrature rule for functions whose second derivatives are absolutely continuous and third derivatives belong to $L_2(a, b)$. The sharpness is demonstrated by showing an equality for a particular function of such type getting therefore an error bound for Simpson's quadrature rule. Subsequently, in 2005, Ujević in [26] presented a generalization of the modified Simpson's rule and established various error bounds for this generalization and, in the year 2007 in [27] derived a new error bounds for the well-known Simpson's quadrature rule. Using these bounds the Simpson's rule can then be applied to functions whose first, second and third derivatives are unbounded below or above, furthermore, these error bounds can be much better than some recently obtained bounds. In 2008, Alomari *et. al* in [1] introduced, in terms of the first derivative, some inequalities of Simpson's type based on s-convexity and gave best Midpoint type inequalities as well as obtained error estimates for special means and some numerical quadrature rules. Recently, in 2010, Sarikaya *et. al* in [20], obtained for differentiable convex mappings which were connected with Simpson's inequality and, in the same year, [21] introduced some new inequalities of Simpson's type based on s-convexity and some applications to special means of real numbers. However, in the same year, Alomari *et. al* in [2] introduced some inequalities of Simpson's type for quasi-convex functions and restrict the conditions on f to get best error estimates for the midpoint rule than the original. Also, Alomari *et. al* in [3] establish some inequalities of Simpson's type for quasi-convex functions in terms of third derivatives and gave some applications to Simpson's numerical quadrature rule.

In 1974, Nagy [16] applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation. Soon after, in 1980, K. Nikodem in [17], established some properties of convex stochastic processes and, in [18], introduced properties of quasi-convex stochastic processes. Later, D. Kotrys in 2011 presented in [14] an inequality of Hermite-Hadamard type for Jensen-convex stochastic processes and N. Merentes *et al.*, proved in [4] a generalization for h-convex stochastic processes. In particular, with the function h equals to the identity, a Hermite-Hadamard inequality type for convex stochastic processes were obtained in [4]. Nevertheless, in 2014 Set *et. al* in [22], presented the s-convex stochastic processes in the second sense and some well-known results concerning s-convex functions are extended to s-convex stochastic processes in the second sense. Also, they investigated a relation between s-convex stochastic processes in the second sense and convex stochastic processes. Then, in 2015 Set *et. al* [15], obtained a similar result to the previous one but for s-convex functions. The previous result extended the concept of s-convex functions, which was introduced and improved by Hudzik, Maligranda [12] and Dragomir, Fitzpatrik [5] to s-convex stochastic processes and obtained some results similar to the ones in s-convex functions. Recently, in 2015, Merentes *et. al* [11] presented some estimates of the left and right-hand side of the Hermite-Hadamard inequality for convex stochastic processes with convex or quasi-convex first second derivatives in absolute value establishing for the first time an estimate of error for this kind of inequalities in stochastic processes.

The aim of this paper is establish a counterpart of the results for s-convex and quasi-convex functions of Alomari *et. al* in [1] - [3], to convex stochastic processes with s-convex and quasi-convex derivatives in absolute value, in order to estimate the error in the integrals approximation by Simpson's quadrature for stochastic process.

2 Preliminary Notes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. A function $X : \Omega \to \mathbf{R}$ is a *random variable* if it is $\mathcal{A}-$measurable. A function $X : I \times \Omega \to \mathbf{R}$, where $I \subseteq \mathbf{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

A stochastic process $X : I \times \Omega \to \mathbf{R}$ is:

1. *Jensen-convex* if, for every $a, b \in I$, the following inequality is: satisfied:

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{X(a,\cdot) + X(b,\cdot)}{2}, \quad (a.e). \tag{6}$$

2. *convex* if, for every $a, b \in I$, $t \in (0, 1)$, the following inequality is tales place:

$$X(ta + (1-t)b, \cdot) \leq tX(a,\cdot) + (1-t)X(b,\cdot), \quad (a.e). \tag{7}$$

3. *quasi-convex* if, for every $a, b \in I$, $t \in (0, 1)$, the following inequality is satisfied:

$$X(ta + (1-t)b, \cdot) \leq \max\{X(a,\cdot), X(b,\cdot)\}, \quad (a.e). \tag{8}$$

4. *s-convex in the first sense* if, for some fixed $s \in (0, \infty]$ and for every $a, b > 0$, $a, b \in I$ and $\alpha, \beta > 0$ with $\alpha^s + \beta^s = 1$, the following inequality holds:

$$X(\alpha a + \beta b, \cdot) \leq \alpha^s X(a, \cdot) + \beta^s X(b, \cdot), \quad (a.e). \tag{9}$$

This class of stochastic process is denoted by C_s^1.

5. *s-convex in the second sense* if, for some fixed $s \in (0, \infty]$ and for every $a, b > 0$, $a, b \in I$ and $t \in (0, 1)$, the following inequality holds:

$$X(ta + (1-t)b, \cdot) \leq t^s X(a, \cdot) + (1-t)^s X(b, \cdot), \quad (a.e). \tag{10}$$

Also, a stochastic process $X : I \times \Omega \to \mathbf{R}$ is:

1. *continuous in probability* in the interval I, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability.

2. *mean-square continuous* in I, if for all $t_0 \in I$

$$\lim_{t \to t_0} \mathbf{E}[(X(t, \cdot) - X(t_0, \cdot))^2] = 0,$$

where $\mathbf{E}[X(t, \cdot)]$ denotes the expectation value of the random variable $X(t, \cdot)$.

3. *differentiable* at a point $t \in I$ if there is a random variable $X'(t, \cdot) : I \times \Omega \to \mathbf{R}$:

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

4. *mean-square differentiable* at a point $t \in I$ if there is a random variable $X'(t, \cdot) : I \times \Omega \to \mathbf{R}$:

$$\lim_{t \to t_0} \mathbf{E}\left[\left(\frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0} - X'(t, \cdot)\right)^2\right] = 0.$$

Note that mean-square continuity implies continuity in probability, but the converse is not true.

Fixed $X : I \times \Omega \to \mathbf{R}$ a stochastic process with $\mathbf{E}[X(t)^2] < \infty$ for all $t \in I$, $[a, b] \subseteq I$, $a = t_0 < t_1 < ... < t_n = b$ a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., n$, a random variable $Y : \Omega \to \mathbf{R}$ is called the *mean-square integral* of the process X on $[a, b]$, if for a normal sequence of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$ we have

$$\lim_{n\to\infty} \mathbf{E}\left[\left(\sum_{k=1}^{n} X(\Theta_k,\cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0.$$

In such case, we write

$$Y(\cdot) = \int_a^b X(s,\cdot)ds \quad (a.e).$$

For the existence of the mean-square integral is enough to assume the mean-square continuity of the stochastic process X. Basic properties of the mean-square integral can be read in [23].

The Simpson's rule for stochastic process is similar to the one for functions and reads as follow:

Theorem 2.1 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square continuous stochastic process, then the mean-square integral on $[a,b]$ is approximated as:*

$$\int_a^b X(t,\cdot)dt \approx \frac{b-a}{6}\left[X(a,\cdot) + 4X\left(\frac{a+b}{2},\cdot\right) + X(b,\cdot)\right], \quad (a.e). \tag{11}$$

This is known as Simpson's rules by Lagrange form with $n = 2$, for stochastic process.

Proof. We want to approximate $X(t,\cdot)$ as follow:

$$\begin{aligned} P_2(t) &= X(a,\cdot)\frac{(t-t_m)(t-b)}{(a-t_m)(a-b)} + X(t_m,\cdot)\frac{(t-a)(t-b)}{(t_m-a)(t_m-b)} \\ &\quad + X(b,\cdot)\frac{(t-a)(t-t_m)}{(b-a)(b-t_m)}. \end{aligned}$$

Denote $h = \dfrac{b-a}{2} = t_m - a = b - t_m$, then

$$P_2(t) = \frac{X(a,\cdot)}{2h^2}(t-t_m)(t-b) - \frac{X(t_m,\cdot)}{2h^2}(t-a)(t-b) + \frac{X(b,\cdot)}{2h^2}(t-a)(t-t_m).$$

Integrating on $[a,b]$,

$$\begin{aligned} \int_a^b P_2(t,\cdot)dx &= \frac{X(a,\cdot)}{2h^2}\int_a^b (t-t_m)(t-h)dt - \frac{X(t_m,\cdot)}{h^2}\int_a^b (t-a)(t-b)dx \\ &\quad + \frac{X(b,\cdot)}{2h^2}\int_a^b (t-a)(t-t_m)dt. \end{aligned}$$

We define:

$$\begin{aligned} I_1 &= \int_a^b (t-t_m)(t-b)dt, \\ I_2 &= \int_a^b (t-a)(t-b)dt, \\ I_3 &= \int_a^b (t-a)(t_m)dt. \end{aligned}$$

$$\begin{aligned} I_1 &= \int_a^b (t-t_m)(t-b)dx \\ &= (t-t_m)\frac{(t-b)^2}{2} - \frac{(t-b)^3}{6}\Big|_a^b \\ &= -(a-t_m)\frac{(a-b)^2}{2} + \frac{(a-b)^3}{6} = -(-h)\frac{(-2h)^2}{2} + \frac{(-2h^3)}{6} \\ &= 2h^3 - \frac{4}{3}h^3 \\ &= \frac{2}{3}h^3, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_a^b (t-a)(t-b)dt \\ &= (t-a)\frac{(t-b)^2}{2} - \frac{(t-b)^3}{6}\Big|_a^b \\ &= \frac{(a-b)^3}{b} \\ &= \frac{(-2h)^3}{6} \\ &= \frac{4}{3}h^3, \end{aligned}$$

$$\begin{aligned} I_3 &= \int_a^b (t-a)(t-t_m)dx \\ &= (t-a)\frac{(t-x_m)^2}{2} - \frac{(t-t_m)^3}{6}\Big|_a^b \\ &= (b-a)\frac{(b-x_m)^2}{2} - \frac{(b-t_m)^3}{6} + \frac{(a-x_m)^3}{6} \\ &= (2h)\frac{h^2}{2} - \frac{h^3}{6} + \frac{(-h)^3}{6} - h^3 - \frac{h^3}{3} - \frac{2h^3}{3}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_a^b P_2(t,\cdot)dx &= \frac{X(a,\cdot)}{2h^2}\left(\frac{2}{3}h^3\right) - \frac{X(t_m,\cdot)}{h^2}\left(-\frac{4}{3}h^3\right) + \frac{X(b,\cdot)}{2h^2}\left(\frac{2}{3}h^2\right) \\
&= X(a,\cdot)\frac{h}{3} + X(t_m,\cdot)\frac{4}{3}h + X(b,\cdot)\frac{h}{3} \\
&= \frac{h}{3}\left[X(a,\cdot) + 4X(t_m,\cdot) + X(b,\cdot)\right].
\end{aligned}$$

Setting $h = \dfrac{b-a}{2}$,

$$\int_a^b X(t,\cdot)dx \approx \frac{(b-a)}{6}\left[X(a,\cdot) + 4X(t_m,\cdot) + X(b,\cdot)\right].$$

Also, the Simpson's inequality has a counterpart for stochastic processes. In order to prove this inequality, we present the next result which represents the analogous weighted mean value theorem for integrals to stochastic processes.

Lemma 2.2 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square continuous stochastic process on $[a,b]$, with $a < b$ and let $f : \mathbf{R} \to \mathbf{R}$ be a monotonic integrable function on (a,b). Then, there is $\xi \in [a,b]$ such that*

$$\int_a^b X(t,\cdot)f(t)dt = X(\xi,\cdot)\int_a^b f(t)dt \tag{12}$$

almost everywhere.

Proof. Suppose $f(t) \geq 0$ on $[a,b]$. Because $X(t,\cdot)$ is a mean-square continuous stochastic process, there is $m(\cdot) = \inf_{t\in[a,b]} X(t,\cdot)$ and $M(\cdot) = \sup_{t\in[a,b]} X(t,\cdot)$ random variables. Then, $m(\cdot) \leq X(t,\cdot) \leq M(\cdot)$ for all $x \in [a,b]$. Multiplying this last inequality by $f(x)$ we get $m(\cdot)f(t) \leq X(t,\cdot)f(t) \leq M(\cdot)f(t)$ and integrating on $[a,b]$,:

$$m(\cdot)\int_a^b f(t)dt \leq \int_a^b X(t,\cdot)f(t)dt \leq M(\cdot)\int_a^b f(t)dt, \tag{13}$$

almost everywhere.

Let us call $I = \int_a^b f(t)dt$. If $I = 0$ then $\int_a^b X(t,\cdot)f(t)dt = 0$, almost everywhere. Hence, for each ξ, (12) is satisfied almost surely.

Otherwise, if $I > 0$ we can divide (13) by I and, because the fact that $X(t,\cdot)$ is mean-square continuous on $[a,b]$, there is ξ such that

$$m(\cdot) < X(\xi, \cdot) < M(\cdot), \qquad (a.e).$$

In consequence,

$$X(\xi, \cdot) = \frac{1}{I} \int_a^b X(t, \cdot) f(t) dt, \qquad (a.e).$$

Therefore,

$$\int_a^b X(t, \cdot) f(t) dt = X(\xi, \cdot) \int_a^b f(t) dt,$$

almost everywhere.

Theorem 2.3 *Suppose $X : I \times \Omega \to \mathbf{R}$ is a four times mean-square continuous differentiable stochastic process on I° and having the fourth derivative bounded on I°, that is, $||X^{(4)}(t, \cdot)||_\infty := \sup_{t \in I^\circ} |X^{(4)}(t, \cdot)| < \infty$, then the following inequality:*

$$\left| \frac{1}{6} \left[X(a, \cdot) + 4X\left(\frac{a+b}{2}, \cdot\right) + X(b, \cdot) \right] - \frac{1}{(b-a)} \int_a^b X(t, \cdot) dt \right| \leq \frac{1}{2880} ||X^{(4)}(t, \cdot)||_\infty (b-a)^5, \quad (14)$$

holds almost everywhere and it is the Simpson's inequality for stochastic processes.

Proof. Consider the Taylor polynomial of the processes $X(t, \cdot)$ around $t_1 = \frac{a+b}{2}$, as follow:

$$X(t, \cdot) = X(t_1, \cdot) + X'(t_1, \cdot)(t - t_1) + \frac{X''(t_1, \cdot)}{2}(t - t_1)^2 + \frac{X^{(3)}(t_1, \cdot)}{6}(t - t_1)^3 + \frac{X^{(4)}(\xi(t), \cdot)}{24}(t - t_1)^4.$$

Integrating on $[a, b]$, we have:

$$\int_a^b X(t, \cdot) dt = \left[X(t_1, \cdot)(t - t_1) \frac{X'(t_{1,}.)}{2}(t - t_1)^2 + \frac{X''(t_1, \cdot)}{6}(t - t_1)^3 + \frac{X^{(3)}(t_1, \cdot)}{24}(t - t_1)^4 \right]_{t=a}^{t=b} + \int_a^b \frac{X^{(4)}(\xi(t), \cdot)}{24}(t - t_1)^4 dt,$$

almost everywhere.

Applying the weighted mean value theorem for integral to stochastic processes,

$$\begin{aligned}\int_a^b \frac{X^{(4)}(\xi(t),\cdot)}{24}(t-t_1)^4dt &= \frac{X^{(4)}(\xi_1,\cdot)}{24}\int_a^b (t-t_1)^4dt \\ &= \frac{X^{(4)}(\xi_1,\cdot)}{24}(t-t_1)^5\Big|_a^b \\ &= \frac{X^{(4)}(\xi_1,\cdot)}{24}\left[(b-t_1)^5-(a-t_1)^5\right],\end{aligned}$$

with $\xi \in (a,b)$, almost surely.

Otherwise, we can develop $X''(t_1,\cdot)$ as follows:

$$X''(t_1,\cdot) = \frac{1}{h^2}\left[X(t_1-h,\cdot)-2X(t_1,\cdot)+X(t_1+h_1,\cdot)\right]-\frac{h^2}{12}X^{(4)}(\xi,\cdot),$$

almost everywhere, where $h=\dfrac{(b-a)}{2}$. Then

$$X''(t_1,\cdot) = \frac{1}{h^2}\left[X(a,\cdot)-2X(t_1,\cdot)+X(b,\cdot)\right]-\frac{h^2}{12}X^{(4)}(\xi,\cdot), \tag{15}$$

$\xi \in (a,b)$, almost everywhere.

Hence, as $h=b-t_1=t_1-a$, so

$$\int_a^b X(t,\cdot)dt = X(t_1,\cdot)2h+\frac{X''(t_1,\cdot)}{6}2h^3+\frac{X^{(4)}(\xi,\cdot)}{120}2h^5.$$

almost everywhere.

Replacing (15) in the last expression,

$$\int_a^b X(t,\cdot) = \frac{h}{3}\left[X(a,\cdot)-4X(t_1,\cdot)+X(b,\cdot)\right]-\frac{h^5}{90}X^{(4)}(\xi,\cdot),$$

almost surely.

Therefore,

$$\begin{aligned}\left|\frac{h}{3}\left[X(a,\cdot)-4X(t_1,\cdot)+X(b,\cdot)\right]-\int_a^b X(t,\cdot)dt\right| &= \left|\frac{h^5}{90}X^{(4)}(\xi,\cdot)\right| \\ &= \left|\frac{(b-a)^5}{2880}X^{(4)}(\xi,\cdot)\right| \\ &\le \frac{(b-a)^5}{2880}||X^{(4)}(\xi,\cdot)||_\infty\end{aligned}$$

almost everywhere.

Now, we present an equivalent results presented by Alomari *et al.* in [1] and [2] for stochastic processes whose first derivative at certain powers are s-convex and whose second and third derivatives at certain powers are quasi-convex.

3 Main Result

3.1 Inequalities for s-convex stochastic processes

In order to prove our main theorems for s-convex stochastic processes, let us begin with the following lemma which is a generalization of lemma presented by Alomari *et. al* in [1] for s-convex functions.

Lemma 3.1 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable on $[a, b]$, $a, b \in I$ with $a < b$. If $X'(t, \cdot)$ is mean-square integrable on $[a, b]$, then the following equality holds almost everywhere:*

$$\begin{aligned} \frac{1}{6}\left[X(a,\cdot) + 4X\left(\frac{a+b}{2},\cdot\right) + X(b,\cdot)\right] - \frac{1}{b-a}\int_a^b X(t,\cdot)dt \qquad (16) \\ = (b-a)\int_a^b p(t)X'(at+(1-t)b,\cdot)dt, \end{aligned}$$

where

$$p(t) = \begin{cases} t - \dfrac{1}{6}, & t \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{5}{6} - t, & t \in \left[\dfrac{1}{2}, 1\right). \end{cases}$$

Proof. Integrating by part

$$\begin{aligned} I &= \int_0^1 p(t)X'(at+(1-t)b,\cdot)dt \\ &= \int_0^{1/2}\left(t - \frac{1}{6}\right)X'(at+(1-t)b,\cdot)dt \\ &\quad + \int_{1/2}^1\left(t - \frac{5}{6}\right)X'(at+(1-t)b,\cdot)dt \\ &= \left(t - \frac{1}{6}\right)\frac{X(at+(1-t)b,\cdot)}{b-a}\bigg|_0^{1/2} \end{aligned}$$

$$
\begin{aligned}
&- \int_0^{1/2} \frac{X(at+(1-t)b,\cdot)}{b-a}dt \\
&+ \left(t-\frac{5}{6}\right)\frac{X(at+(1-t)b,\cdot)}{b-a}\Bigg|_{1/2}^{1} \\
&- \int_{1/2}^{1} \frac{X(at+(1-t)b,\cdot)}{b-a}dt \\
= \;& \frac{1}{6(b-a)}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right] \qquad (17)\\
&- \int_0^{1} \frac{X(at+(1-t)b,\cdot)}{b-a}dt, \quad (a.e).
\end{aligned}
$$

Setting $x = at + (1-t)b$, and $dx = (b-a)dt$, gives

$$
(b-a)\cdot I = \frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right] - \frac{1}{b-a}\int_a^b X(t,\cdot)dt., \quad (a.e),
$$

which gives the desired representation.

The next theorems gives a new refinements of the Simpson's inequality via s-convex stochastic processes and are the analogous to that obtained by [1] for s-convex functions.

Theorem 3.2 *Let $X : I\times\Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable on $[a,b]$, $a,b\in I$ with $a < b$. If $|X'|$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality takes place almost everywhere:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right] - \frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \qquad (18)\\
&\quad \le (b-a)\left(\frac{6^{-s}-9(2^{-s})+(5^{s+1})(6^{-s})+3s-12}{18(s^2+3s+2)}\right)\left[|X'(a,\cdot)|+|X'(b,\cdot)|\right].
\end{aligned}
$$

Proof. First we point out that

$$
\begin{aligned}
\int_0^{1/2}\left|t-\frac{1}{6}\right|t^s dt + \int_{1/2}^{1}\left|t-\frac{5}{6}\right|t^s dt &= \int_0^{1/2}\left|\frac{1}{6}-t\right|(1-t)^s dt \\
&\quad + \int_{1/2}^{1}\left|\frac{5}{6}-t\right|(1-t)^s dt \\
&= \frac{6^{-s}-9(2^{-s})+(5^{s+2})(6^{-s})+3s-12}{18(s^2+3s+2)}.
\end{aligned}
$$

Then, from Lemma 3.1 and since $|X'|$ is s-convex,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\leq (b-a)\int_0^1 |p(t)||X'(at+(1-t)b,\cdot)|dt \\
&\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left(t^s|X'(a,\cdot)|+(1-t)^s|X'(b,\cdot)|\right)dt \\
&\quad +(b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right|\left(t^s|X'(a,\cdot)|+(1-t)^s|X'(b,\cdot)|\right)dt \\
&\leq (b-a)|X'(a,\cdot)|\left[\int_0^{1/2}\left|t-\frac{1}{6}\right|t^s dt+\int_{1/2}^1\left|t-\frac{5}{6}\right|t^s dt\right] \\
&\quad +(b-a)|X'(b,\cdot)|\left[\int_0^{1/2}\left|t-\frac{1}{6}\right|(1-t)^s dt+\int_{1/2}^1\left|t-\frac{5}{6}\right|(1-t)^s dt\right], \\
&= (b-a)\left(\frac{6^{-s}-9(2^{-s})+(5^{s+1})(6^{-s})+3s-12}{18(s^2+3s+2)}\right)\left[|X'(a,\cdot)|+|X'(b,\cdot)|\right],
\end{aligned}$$

almost everywhere, which completes the proof.

Corollary 3.3 *Let $X : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable, $a,b\in I$ with $a<b$. If $|X'|^{p/(p-1)}$ is convex on $[a,b]$ and $p>1$, then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\qquad\leq \frac{5(b-a)}{72}\left[|X'(a,\cdot)|+|X'(b,\cdot)|\right]. \quad (19)
\end{aligned}$$

Theorem 3.4 *Let $X : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable on $[a,b]$, $a,b\in I$ with $a<b$. If $|X'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality is true almost everywhere:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\qquad\leq \frac{(b-a)}{(1+s)^{1/q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(|X'(a,\cdot)|^q+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q\right)^{1/q}\right.
\end{aligned}$$

$$+\left(\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q+|X'(b,\cdot)|^q\right)^{1/q}\Bigg]. \quad (20)$$

Proof. From Lemma 3.1,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\le (b-a)\int_0^1 |p(t)||X'(at+(1-t)b,\cdot)|dt \\
&\le (b-a)\left[\int_0^{1/2}\left|t-\frac{1}{6}\right||X'(at+(1-t)b,\cdot)|dt\right. \\
&\qquad\left.+\int_{1/2}^1\left|t-\frac{5}{6}\right||X'(at+(1-t)b,\cdot)|dt\right] \\
&\le (b-a)\left[\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^p dt\right)^{1/p}\left(\int_0^{1/2}|X'(at+(1-t)b,\cdot)|^p dt\right)^{1/p}\right. \\
&\qquad\left.+\left(\int_{1/2}^1\left|t-\frac{5}{6}\right|^p dt\right)^{1/p}\left(\int_{1/2}^1|X'(at+(1-t)b,\cdot)|^p dt\right)^{1/p}\right] \\
&\le (b-a)\left[\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt\right)^{1/p}\right. \\
&\qquad\left(\int_0^{1/2}|X'(at+(1-t)b,\cdot)|^p dt\right)^{1/p} \\
&\qquad+\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^1\left(t-\frac{5}{6}\right)^p dt\right)^{1/p} \qquad (21) \\
&\qquad\left.\left(\int_0^{1/2}|X'(at+(1-t)b,\cdot)|^p dt\right)^{1/p}\right], \quad (a.e).
\end{aligned}$$

Since, $|X'|$ is s-convex by (**??**), we have

$$\begin{aligned}
\int_0^{1/2}|X'(at+(1-t)b,\cdot)|^q dt &= \frac{1}{b-a}\int_{\frac{a+b}{2}}^b |X'(t,\cdot)|^q dt \\
&\le \frac{1}{2}\left[\frac{|X'(b,\cdot)|^q+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q}{s+1}\right] \qquad (22) \\
&\le \frac{|X'(b,\cdot)|^q+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q}{s+1}, \quad (a.e).
\end{aligned}$$

and

$$\begin{aligned}\int_{1/2}^{1}|X'(at+(1-t)b,\cdot)|^q dt &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}}|X'(t,\cdot)|^q dt \\ &\leq \frac{1}{4}\left[\frac{|X'(b,\cdot)|^q+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q}{s+1}\right] \qquad (23)\\ &\leq \frac{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q+|X'(a,\cdot)|^q}{s+1}, \quad (a.e).\end{aligned}$$

Also,

$$\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt=\int_{5/6}^{1}\left(\frac{5}{6}-t\right)^p dt=\frac{1}{3}\left(\frac{3^{-p}}{p+1}\right), \qquad (24)$$

$$\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt=\int_{1/2}^{5/6}\left(t-\frac{5}{6}\right)^p dt=\frac{1}{6}\left(\frac{6^{-p}}{p+1}\right). \qquad (25)$$

Therefore, by (22)-(25), proof is complete.

If we put $s = 1$ in the Theorem 3.4, we have a corollary which result if $|X'|^{p/(p-1)}$ is asked to be convex on $[a, b]$. This is because s-convexity for stochastic processes is a generalization of convexity.

Corollary 3.5 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable on $[a, b]$, $a, b \in I$ with $a < b$. If $|X'|^{p/(p-1)}$ is convex on $[a, b]$ and $p > 1$, then the following inequality holds almost everywhere:*

$$\begin{aligned}&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\ &\leq \frac{(b-a)}{2^{1/q}}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{1/p}\left[\left(|X'(a,\cdot)|^q+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q\right)^{1/q}\right. \\ &\qquad \left.+\left(\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q+|X'(b,\cdot)|^q\right)^{1/q}\right]. \qquad (26)\end{aligned}$$

Theorem 3.6 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X' mean-square integrable on $[a, b]$, $a, b \in I$ with $a < b$. If $|X'|^{p/(p-1)}$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality is satisfied almost everywhere:*

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|$$
$$\leq \frac{(b-a)}{[216(s^2+3s+2)]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q} \tag{27}$$
$$\big\{([(3^{-s}(2^{1-s})+3s(2^{1-s})+3(2^{-s}))]|X'(b,\cdot)|^q$$
$$+[5^{s+2}3^{-s}2^{1-s}-6s(2^{-s})-21(2^{-s})+6s-24]|X'(a,\cdot)|^q)^{1/q}$$
$$+\big([(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})]|X'(a,\cdot)|^q$$
$$+[5^{s+2}3^{-s}2^{1-s}-6s(2^{-s})-21(2^{-s})+6s-24]|X'(b,\cdot)|^q\big)^{1/q}\Big\}.$$

Proof. From previous lemma and the power mean inequality,

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|$$
$$\leq (b-a)\int_0^1 |s(t)||X'(tb+(1-t)a,\cdot)|\,dt$$
$$\leq (b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||X'(tb+(1-t)a,\cdot)|dt$$
$$+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right||X'(tb+(1-t)a,\cdot)|dt$$
$$\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-1/q}$$
$$\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||X'(tb+(1-t)a,\cdot)|^q dt\right)^{1/q}$$
$$+(b-a)\left(\int_{\frac{1}{2}}^1\left|\left(t-\frac{5}{6}\right)\right|dt\right)^{1-1/q}$$
$$\left(\int_{\frac{1}{2}}^1\left|\left(t-\frac{5}{6}\right)\right||X'(tb+(1-t)a,\cdot)|^q dt\right)^{1/q}, \quad (a.e).$$

Since $|X'|$ is s-convex, we have

$$\int_0^{\frac{1}{2}}\left|\left(t-\frac{1}{6}\right)\right||X'(tb+(1-t)a,\cdot)|^q dt$$

$$\leq \int_0^{\frac{1}{6}} \left(\frac{1}{6}-t\right)(t^s|X'(b,\cdot)|^q+(1-t)^s|X'(a,\cdot)|^q)dt$$
$$+\int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right)(t^s|X'(b,\cdot)|^q+(1-t)^s|X'(a,\cdot)|^q)dt$$
$$=\left[\frac{(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})}{36(s^2+3s+2)}\right]|X'(b,\cdot)|^q$$
$$+\left[\frac{(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24}{36(s^2+3s+2)}\right]|X'(a,\cdot)|^q, \quad (a.e).$$

and

$$\int_{\frac{1}{2}}^{1} \left|\left(t-\frac{5}{6}\right)\right| |X'(tb+(1-t)a,\cdot)|^q dt$$
$$\leq \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right)(t^s|X'(b,\cdot)|^q+(1-t)^s|X'(a,\cdot)|^q)dt$$
$$+\int_{\frac{5}{6}}^{1} \left(t-\frac{5}{6}\right)(t^s|X'(b,\cdot)|^q+(1-t)^s|X'(a,\cdot)|^q)dt$$
$$=\left[\frac{(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})}{36(s^2+3s+2)}\right]|X'(a,\cdot)|^q$$
$$+\left[\frac{(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24}{36(s^2+3s+2)}\right]|X'(b,\cdot)|^q, \quad (a.e).$$

Also, we note that

$$\int_0^{\frac{1}{2}} \left|\left(t-\frac{1}{6}\right)\right| dt = \int_{\frac{1}{2}}^{1} \left|\left(t-\frac{5}{6}\right)\right| dt = \frac{5}{72}.$$

which completes the proof.

Corollary 3.7 *Let $X : I\times\Omega \to \mathbf{R}$ be as in Theorem 3.6, then the following inequality holds almost everywhere:*

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|$$
$$\leq \frac{(b-a)}{[216(s^2+3s+2)]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}(|X'(a,\cdot)|+|X'(b,\cdot)|)$$
$$\Big[\left[(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})\right]^{1/q}$$
$$+\left[(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24\right]^{1/q}\Big].$$

Moreover, if $s = 1$,

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\ \leq \frac{5}{72}\left[\frac{1668}{6480}\right]^{1/q}(b-a)\left(|X'(a,\cdot)|+|X'(b,\cdot)|\right)$$

Proof. We consider in the previous inequality for $p > 1,\ q = p/(p-1)$ and we called:

$$\begin{aligned} a_1 &= \left[(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})\right]|X'(b,\cdot)|^q \\ b_1 &= \left[(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24\right]|X'(a,\cdot)|^q \\ a_2 &= \left[(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24\right]|X'(b,\cdot)|^q \\ b_2 &= \left[(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24\right]|X'(b,\cdot)|^q. \end{aligned}$$

Here, $0 < 1/q < 1$, for $q > 1$. Using the fact

$$\sum_{i=1}^{n}(a_i+b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=0}^{n} b_i^r,$$

for $0 < r < 1, a_i, a_2, \ldots a_n \geq 0$ and $b_1, b_2, \ldots, b_n \geq 0$, we obtain

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ \leq \frac{(b-a)}{[216(s^2+3s+2)]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left\{\left[(3^{-s})(2^{1-s})+3s(2^{1-s})+3(2^{-s})\right]^{1/q} \right. \\ \left. +\left[(5^{s+2})(3^{-s})(2^{1-s})-6s(2^{-s})-21(2^{-s})+6s-24\right]^{1/q}\right\} \\ \cdot\left(|X'(a,\cdot)+|X'(b,\cdot)|\right)$$

which completes the proof.

Corollary 3.8 *Let* $X : I \times \Omega \to \mathbf{R}$ *be as Theorem 3.6, and let* $s = 1$, *therefore the forecoming inequality holds almost everywhere for convex functions:*

$$\left|\frac{1}{6}\left[X'(a,\cdot)+4X'\left(\frac{a+b}{2},\cdot\right)+X'(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\ \leq \frac{(b-a)}{(1296)^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[(20|X'(b,\cdot)|^q+61|X'(a\cdot)|^q)^{1/q} \right. \\ \left. +(61|X'(b,\cdot)|^q+29|X'(a,\cdot)|^q)^{1/q}\right].$$

Moreover, if $|X'| \leq M$, *for any* $x \in I$,

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X'\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \leq \frac{5(b-a)}{36}M, \quad (a.e).$$

Proof. Calculating as in the first theorem

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||X'(ta+(1-t)b,\cdot)|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right||X'(ta+(1-t)a,\cdot)|dt\\
\leq\ & \int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)t^s|X'(a,\cdot)|^q+\left(\frac{1}{6}-t\right)(1-t)^s|X'(b,\cdot)|^q dt\\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)t^s|X'(a,\cdot)|^q+\left(t-\frac{1}{6}\right)(1-t)^s|X'(b,\cdot)|^q dt\\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)t^s|X(a,\cdot)|^q dt+\left(\frac{5}{6}-t\right)(1-t)^s|X'(b,\cdot)|^q dt\\
&+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)t^s|X'(a,\cdot)|^q+\left(t-\frac{5}{6}\right)(1-t)^s|X'(b,\cdot)|^q dt\\
=\ & |X'(a,\cdot)|^q\left[\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)t^s dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)t^s dt\right.\\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)t^s dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)t^s\right]\\
&+|X'(b,\cdot)|^q\left[\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)(1-t)^s dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)(1-t)^s dt\right.\\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t-\frac{5}{6}\right)(1-t)^s dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)(1-t)^s dt\right].
\end{aligned}$$

By Theorem 3.6 and replacing $s=1$,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|\\
&\leq\frac{(b-a)}{[1296]^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[(29|X'(b,\cdot)|^q+61|X(a,\cdot)|^q)^{1/q}\right.\\
&\left.+(29|X'(a,\cdot)|^q+61|X'(b,\cdot)|^q)^{1/q}\right].
\end{aligned}$$

Therefore, if $X'(u,\cdot)\leq M$, for any $u\in I$ we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[X'(a,\cdot)+4X'\left(\frac{a+b}{2},\cdot\right)+X'(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\quad\leq \frac{(b-a)}{(1296)^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[29M^q+61M^q\right]^{1/q}+\left[61M^q+29M^q\right]^{1/q} \\
&\quad\leq \frac{(b-a)}{(1296)^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[(90M^q)^{1/q}+(90M^q)^{1/q}\right] \\
&\quad= \frac{(b-a)}{(1296)^{1/q}}\left(\frac{5}{72}\right)^{1-1/q}\left[2(90)^{1/q}M\right] \\
&\quad= (b-a)\left(\frac{5}{72}\right)\left(\frac{5}{72}\right)^{-1/q}\left(2\left(\frac{5}{72}\right)^{1/q}M\right) \\
&\quad\leq \frac{5(b-a)}{36}M.
\end{aligned}$$

3.2 Inequalities for quasi-convex stochastic processes

In order to prove the result for quasi-convex stochastic processes, we need the following lemma which is a generalization of the result obtained by Alomari *et al.* in [2] to quasi-convex functions.

Lemma 3.9 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $X''(t,\cdot)$ is mean-square integrable on $[a,b]$, then the following equality holds almost everywhere:*

$$\begin{aligned}
\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right] \qquad (28) \\
= (b-a)^2\int_a^b p(t)X''(at+(1-t)b,\cdot)dt,
\end{aligned}$$

where

$$p(t)=\begin{cases}\frac{1}{6}t\,(3t-1)\,, & t\in\left[0,\frac{1}{2}\right], \\[2ex] \frac{1}{6}(t-1)\,(3t-1)\,, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Proof. We note that integrating by parts

$$
\begin{aligned}
I &= \int_0^1 p(t)X''(at+(1-t)b,\cdot)dt \\
&= \frac{1}{6}\int_0^{1/2} t\,(3t-1)\,X'(at+(1-t)b,\cdot)dt \\
&\quad +\frac{1}{6}\int_{1/2}^{1} (t-1)\,(3t-1)\,X'(at+(1-t)b,\cdot)dt. \\
&= \frac{1}{6}t\,(3t-1)\,\left.\frac{X(at+(1-t)b,\cdot)}{b-a}\right|_0^{1/2} \qquad (29)\\
&\quad -\left[\frac{1}{2}t+\frac{1}{6}(3t-1)\,\left.\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}\right|_0^{1/2}\right] \\
&\quad +\int_0^{1/2}\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}dt+\frac{1}{6}(t-1)(3t-2)\,\left.\frac{X(at+(1-t)b,\cdot)}{b-a}\right|_{1/2}^{1} \\
&\quad -\left[\frac{1}{2}(t-1)+\frac{1}{6}(3t-2)\right]\,\left.\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}\right|_{1/2}^{1} \\
&\quad +\int_{1/2}^{1}\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}dt \\
&= \frac{1}{24}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{b-a}-\frac{1}{3}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(b-a)^2}-\frac{1}{6}\frac{X'(a,\cdot)}{(b-a)^2} \\
&\qquad\qquad +\int_0^{1/2}\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}dt \\
&\quad -\frac{1}{6}\frac{X'(b,\cdot)}{(b-a)^2}-\frac{1}{24}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{b-a}-\frac{1}{3}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(b-a)^2} \\
&\qquad\qquad +\int_{1/2}^{1}\frac{X(at+(1-t)b,\cdot)}{(b-a)^2}dt \\
&= \frac{1}{(b-a)^2}\int_0^1 X(at+(1-t)b,\cdot)dt \\
&\qquad\qquad -\frac{1}{6(b-a)^2}\left[X(a,\cdot)+X(b,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)\right]. \quad (a.e).
\end{aligned}
$$

Setting $x=at+(1-t)b$, and $dx=(b-a)dt$, gives

$$
(b-a)\cdot I=\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right], \quad (a.e),
$$

which leads to the desired representation (28).

The next theorems gives a new refinement of Simpson's inequality for quasi-convex stochastic processes.

Theorem 3.10 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds almost everywhere:*

$$\begin{aligned}\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\ \leq \frac{(b-a)^2}{162}\left[\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\}\right. \\ \left.+\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(b,\cdot)|\right\}\right].\end{aligned} \tag{30}$$

Proof.

First, we notice that

$$\begin{aligned}\int_0^{1/3} t(1-3t)dt &= \int_{1/3}^1 t(3t-1)dt = \int_{1/3}^{2/3}(1-t)(2-3t)dt \\ &= \int_{2/3}^1 (1-t)(3t-2)dt = \frac{1}{54}.\end{aligned} \tag{31}$$

Then, by Lemma 3.9 and since $|X''|$ is quasi-convex,

$$\begin{aligned}&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\ &\leq (b-a)^2\int_0^1 |p(t)||X''(at+(1-t)b,\cdot)|dt \\ &\leq \frac{(b-a)^2}{6}\left[\int_0^{1/2} t\,|3t-1|\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\}dt\right. \\ &\quad\left.+\int_{1/2}^1 |t-1|\,|3t-2|\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(a,\cdot)|\right\}dt\right] \\ &= \frac{(b-a)^2}{6}\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\} \\ &\qquad\left[\int_0^{1/3} t(1-3t)dt+\int_{1/3}^{1/2} t(3t-1)dt\right]\end{aligned}$$

$$+\frac{(b-a)^2}{6}\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(a,\cdot)|\right\}$$
$$\left[\int_{1/2}^{2/3}(1-t)(2-3t)dt+\int_{2/3}^{1}(1-t)(3t-2)dt\right],$$
$$=\frac{(b-a)^2}{162}\left[\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\}\right.$$
$$\left.+\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(b,\cdot)|\right\}\right],\quad (a.e).$$

which completes the proof.

Corollary 3.11 *Let $X : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a,b\in I$ with $a<b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $X(s,\cdot)=X\left(\frac{a+b}{2},\cdot\right)=X(b,\cdot)$, then the following inequality is true almost everywhere:*

$$\left|\frac{1}{b-a}\int_a^b X(t,\cdot)dt-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\le\frac{(b-a)^2}{162}\left[\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\}\right. \tag{32}$$
$$\left.+\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(b,\cdot)|\right\}\right].$$

For instance, for $M>0$, if $|X''(t,\cdot)|<M$ for all $t\in[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b X(t,\cdot)dt-X\left(\frac{a+b}{2},\cdot\right)\right|\le\frac{(b-a)^2}{162}M,\quad (a.e). \tag{33}$$

The corresponding version for powers of the absolute value of second derivative is incorporated in the following result:

Theorem 3.12 *Let $X' : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a,b\in I$ with $a<b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality takes place almost everywhere:*

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|$$
$$\le\frac{(b-a)^2}{6}\left(\frac{\sqrt{\pi}12^{-p}\Gamma(p+1)}{6\Gamma\left(\frac{3}{2}+p\right)}+\frac{4(3^{-p})+3(2^{-p})(p-1)}{12(2+3p+p^2)}\right)^{1/p} \tag{34}$$

$$\left[\max\left\{|X''(a,\cdot)|^{p/(p-1)}, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^{p/(p-1)}\right\}^{(p-1)/p} + \max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^{p/(p-1)}, |X''(b,\cdot)|^{p/(p-1)}\right\}^{(p-1)/p}\right].$$

Proof.

From Lemma 3.9 and since $|X''|$ is quasi-convex,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\quad\le (b-a)\int_0^1 |p(t)||X''(at+(1-t)b,\cdot)|dt \\
&\quad\le \frac{(b-a)^2}{6}\left[\left(\int_0^{1/2}(t\,|3t-1|)^p dt\right)^{1/p}\left(\int_0^{1/2}|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_{1/2}^1(|t-1|\,|3t-2|)^p dt\right)^{1/p}\left(\int_{1/2}^1|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right] \\
&\quad\le \frac{(b-a)^2}{6}\left[\left(\int_0^{1/3}t^p(1-3t)^p dt+\int_{1/3}^{1/2}t^p(3t-1)^p dt\right)^{1/p}\right. \\
&\qquad\qquad\left(\int_0^{1/2}|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q} \\
&\qquad+\left(\int_{1/2}^{2/3}(t-1)^p(2-3t)^p dt+\int_{2/3}^{1}(t-1)^p(3t-2)^p dt\right)^{1/p} \\
&\qquad\qquad\left.\left(\int_{1/2}^{1}|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right], \quad (a.e).
\end{aligned}$$

Since $|X''|$ is quasi-convex,

$$\int_0^{1/2}|X''(at+(1-t)b,\cdot)|^q dt \le \max\left\{|X''(a,\cdot)|^q, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}, \quad (35)$$

$$\int_{1/2}^{1}|X''(at+(1-t)b,\cdot)|^q dt \le \max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q, |X''(b,\cdot)|^2\right\}. \quad (36)$$

Besides, for $p>1$ we have to use the fact that

$$\int_{0}^{1/3} t^p(1-3t)^p dt = \int_{2/3}^{1} (t-1)^p(3t-2)^p dt = \frac{12^{-p}\sqrt{\pi}\Gamma(p+1)}{6\Gamma\left(\frac{3}{2}+p\right)}, \quad (37)$$

$$\int_{1/3}^{1/2} t^p(3t-1)^p dt = \int_{1/2}^{2/3} (t-1)^p(2-3t)^p dt = \frac{4(3^{-p})+3(2^{-p})(p-1)}{12(2+3p+p^2)}. \quad (38)$$

Then, because (35)-(38), the desired result shows up

Corollary 3.13 *Let $X' : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a, b \in I$ with $a < b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, and $X(a,\cdot) = X\left(\frac{a+b}{2},\cdot\right) = X(b,\cdot)$ then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b X(t,\cdot)dt - X\left(\frac{a+b}{2},\cdot\right)\right| \\
&\leq \frac{(b-a)^2}{6}\left(\frac{\sqrt{\pi}12^{-p}\Gamma(p+1)}{6\Gamma\left(\frac{3}{2}+p\right)} + \frac{4(3^{-p})+3(2^{-p})(p-1)}{12(2+3p+p^2)}\right)^{1/p} \quad (39)\\
&\left[\max\left\{|X''(a,\cdot)|^{p/(p-1)}, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^{p/(p-1)}\right\}^{(p-1)/p}\right. \\
&\left.+\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^{p/(p-1)}, |X''(b,\cdot)|^{p/(p-1)}\right\}^{(p-1)/p}\right].
\end{aligned}$$

The following theorem is a generalization of inequality (30).

Theorem 3.14 *Let $X' : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a, b \in I$ with $a < b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality is valid almost everywhere:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot) + 4X\left(\frac{a+b}{2},\cdot\right) + X(b,\cdot)\right] - \frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \quad (40)\\
&\leq \frac{(b-a)^2}{162}\left[\max\left\{|X''(a,\cdot)|^q, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}^{1/q}\right. \\
&\left.+\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q, |X''(b,\cdot)|^q\right\}^{1/q}\right].
\end{aligned}$$

Proof. From Lemma 3.9 and the power mean inequality,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\quad\leq (b-a)\int_0^1 |p(t)||X''(at+(1-t)b,\cdot)|dt \\
&\quad\leq \frac{(b-a)^2}{6}\left[\left(\int_0^{1/2} t\,|3t-1|\,dt\right)^{1-1/q}\right. \\
&\qquad\quad \left(\int_0^{1/2} t\,|3t-1|\,|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q} \\
&\qquad\quad +\left(\int_{1/2}^1 |t-1|\,|3t-2|\,dt\right)^{1-1/q} \qquad (41)\\
&\qquad\quad \left.\left(\int_{1/2}^1 |t-1|\,|3t-2|\,|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right] \\
&\quad\leq \frac{(b-a)^2}{6}\left[\left(\int_0^{1/3} t(1-3t)dt+\int_{1/3}^{1/2} t(3t-1)dt\right)^{1-1/q}\right. \\
&\qquad\quad \cdot\left(\int_0^{1/2} t\,|3t-1|\,|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q} \\
&\qquad\qquad +\left(\int_{1/2}^{2/3}(t-1)(2-3t)dt+\int_{2/3}^1 (t-1)(3t-2)dt\right)^{1-1/q} \\
&\qquad\quad \left.\cdot\left(\int_{1/2}^1 |t-1|\,|3t-2|\,|X''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right], \quad (a.e).
\end{aligned}$$

Since $|X''|$ is quasi-convex,

$$\begin{aligned}
&\int_0^{1/2} t\,|3t-1|\,|X''(at+(1-t)b,\cdot)|^q dt \\
&\qquad\qquad \leq \frac{1}{27}\max\left\{|X''(a,\cdot)|^q, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}, \quad (a.e), \qquad (42)\\
&\int_{1/2}^1 |t-1|\,|3t-2|\,|X''(at+(1-t)b,\cdot)|^q dt \\
&\qquad\qquad \leq \frac{1}{27}\max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q, |X''(b,\cdot)|^q\right\}, \quad (a.e), \qquad (43)
\end{aligned}$$

where we used the equality

$$\int_0^{1/3} t|3t-1|dt = \int_{1/2}^{1} |t-1|\,|3t-2|\,dt = \frac{1}{27}. \tag{44}$$

Then, by a combination of (42)-(44), gives the required result which completes the proof.

Corollary 3.15 *Let $X' : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X'' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $|X''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, and $X(a,\cdot) = X\left(\frac{a+b}{2},\cdot\right) = X(b,\cdot)$ then the following inequality holds almost everywhere:*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b X(t,\cdot)dt - X\left(\frac{a+b}{2},\cdot\right)\right| & \\ \le \frac{(b-a)^2}{162}\Bigg[\max\left\{|X''(a,\cdot)|^q, \left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}^{1/q} & \\ + \max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|^q, |X''(b,\cdot)|^q\right\}^{1/q}\Bigg]. &\end{aligned} \tag{45}$$

The following lemma is very useful to get another result for quasi-convex stochastic processes, but used the third derivative.

Lemma 3.16 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X''' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $X'''(t,\cdot)$ is mean-square integrable on $[a,b]$, then the following equality holds almost everywhere:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b X(t,\cdot)dt - \frac{1}{6}\left[X(a,\cdot) + 4X\left(\frac{a+b}{2},\cdot\right) + X(b,\cdot)\right] & \\ = (b-a)^4 \int_a^b p(t) X'''(at + (1-t)b,\cdot)dt, &\end{aligned} \tag{46}$$

where

$$p(t) = \begin{cases} \dfrac{1}{6}t^2\left(t - \dfrac{1}{2}\right), & t \in \left[0, \dfrac{1}{2}\right], \\[2ex] \dfrac{1}{6}(t-1)^2\left(t - \dfrac{1}{2}\right), & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

Proof. We may integrate by parts, and get

$$
\begin{aligned}
I &= \int_0^1 p(t)X'''(at+(1-t)b,\cdot)dt \\
&= \frac{1}{6}\int_0^{1/2} t^2\left(t-\frac{1}{2}\right)X'''(at+(1-t)b,\cdot)dt \\
&\quad +\frac{1}{6}\int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)X'''(at+(1-t)b,\cdot)dt \\
&= \frac{1}{6}t^2\left(t-\frac{1}{2}\right)\left.\frac{X''(at+(1-t)b,\cdot)}{a-b}\right|_0^{1/2} \\
&\quad -\frac{1}{6}t(3t-1)\left.\frac{X'(at+(1-t)b,\cdot)}{(a-b)^2}\right|_0^{1/2} \\
&\quad +\left(t-\frac{1}{6}\right)\left.\frac{X(at+(1-t)b,\cdot)}{(a-b)^3}\right|_0^{1/2}-\int_0^{1/2}\frac{X(at+(1-t)b,\cdot)}{(a-b)^3}dt \\
&\quad +\frac{1}{6}(1-t)^2\left(t-\frac{1}{2}\right)\left.\frac{X''(at+(1-t)b,\cdot)}{a-b}\right|_{1/2}^{1} \\
&\quad -\frac{1}{6}(3t-2)(t-1)\left.\frac{X'(at+(1-t)b,\cdot)}{(a-b)^2}\right|_{1/2}^{1} \\
&\quad +\left(t-\frac{5}{6}\right)\left.\frac{X''(at+(1-t)b,\cdot)}{(a-b)^3}\right|_{1/2}^{1}-\int_{1/2}^1\frac{X''(at+(1-t)b,\cdot)}{(a-b)^3}dt \\
&= -\frac{1}{24}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(a-b)^2}+\frac{2}{6}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(a-b)^3}+\frac{1}{6}\frac{X'(b,\cdot)}{(a-b)^3} \\
&\qquad\qquad -\int_0^{1/2}\frac{X''(at+(1-t)b,\cdot)}{(a-b)^3}dt \\
&\quad +\frac{1}{24}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(a-b)^2}+\frac{2}{6}\frac{X'\left(\frac{a+b}{2},\cdot\right)}{(a-b)^3}+\frac{1}{6}\frac{X'(b,\cdot)}{(a-b)^3} \\
&\qquad\qquad -\int_{1/2}^1\frac{X''(at+(1-t)b,\cdot)}{(a-b)^3}dt, \quad (a.e).
\end{aligned}
$$

Setting $s = at+(1-t)b$, gives

$$
(b-a)^4\cdot I = \int_a^b X(s,\cdot)ds - \frac{(b-a)}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right], \quad (a.e).
$$

which gives (46).

Theorem 3.17 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X''' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $|X'''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds almost everywhere:*

$$\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \leq \frac{(b-a)^4}{1152}\left[\max\left\{|X''(a,\cdot)|,\left|X''\left(\frac{a+b}{2},\cdot\right)\right|\right\} + \max\left\{\left|X''\left(\frac{a+b}{2},\cdot\right)\right|,|X''(b,\cdot)|\right\}\right]. \tag{47}$$

Proof.

First, we have

$$\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)dt = \int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)dt = \frac{1}{192} \tag{48}$$

Then, by Lemma 3.16 and since $|X''|$ is quasi-convex,

$$\begin{aligned}
&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right| \\
&\leq (b-a)^4\int_0^1 |p(t)||X'''(at+(1-t)b,\cdot)|dt \\
&\leq \frac{(b-a)^4}{6}\left[\int_0^{1/2} t^2\left|t-\frac{1}{2}\right|\max\left\{|X'''(a,\cdot)|,\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|\right\}dt\right. \\
&\quad\left.+\int_{1/2}^1 (t-1)^2\left|\frac{1}{2}-t\right|\max\left\{\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|,|X'''(a,\cdot)|\right\}dt\right] \\
&= \frac{(b-a)^4}{6}\max\left\{|X'''(a,\cdot)|,\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|\right\}\left[\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)dt\right] \\
&\quad+\frac{(b-a)^4}{6}\max\left\{\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|,|X'''(a,\cdot)|\right\}\left[\int_{1/2}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right] \\
&= \frac{(b-a)^2}{1552}\left[\max\left\{|X'''(a,\cdot)|,\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|\right\}\right. \\
&\qquad\left.+\max\left\{\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|,|X'''(a,\cdot)|\right\}\right], \quad (a.e).
\end{aligned}$$

The corresponding version of the inequality (47) for power in terms of the third derivative is incorporated as follows:

Theorem 3.18 *Let $X' : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I° and X''' mean-square integrable on $[a,b]$, $a,b \in I$ with $a < b$. If $|X'''|$ is quasi-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality is true almost everywhere:*

$$\begin{aligned}&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|\\ &\quad\le \frac{(b-a)^2}{48(2^{1/p})}\left(\frac{\Gamma(p+1)\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}\left[\max\left\{|X'''(a,\cdot)|^q,\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}^{1/q}\right.\\ &\qquad\left.+\max\left\{\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|^q,|X'''(b,\cdot)|^q\right\}^{1/q}\right].\end{aligned}$$

Proof. From Lemma 3.16 and since $|X'''|$ is quasi-convex,

$$\begin{aligned}&\left|\frac{1}{6}\left[X(a,\cdot)+4X\left(\frac{a+b}{2},\cdot\right)+X(b,\cdot)\right]-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|\\ &\quad\le (b-a)\int_0^1 |p(t)||X'''(at+(1-t)b,\cdot)|dt\\ &\quad\le \frac{(b-a)^2}{6}\left[\left(\int_0^{1/2} t^2\left|t-\frac{1}{2}\right|dt\right)^{1/p}\left(\int_0^{1/2}|X'''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right.\\ &\qquad\left.+\left(\int_{1/2}^1 (t-1)^2\left|\frac{1}{2}-t\right|dt\right)^{1/p}\left(\int_{1/2}^1|X'''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right]\\ &\quad\le \frac{(b-a)^2}{6}\left[\left(\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)dt\right)^{1/p}\left(\int_0^{1/2}|X'''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right.\\ &\qquad+\left(\int_{1/2}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1/q}\\ &\qquad\qquad\left.\left(\int_{1/2}^1|X'''(at+(1-t)b,\cdot)|^q dt\right)^{1/q}\right],\quad (a.e).\end{aligned}$$

Since $|X'''|$ is quasi-convex, we have

$$\int_0^{1/2}|X'''(at+(1-t)b,\cdot)|^q dt \;\le\; \max\left\{|X'''(a,\cdot)|^q,\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|^q\right\}, \quad (49)$$

$$\int_{1/2}^{1} |X'''(at+(1-t)b,\cdot)|^q dt \;\leq\; \max\left\{\left|X'''\left(\frac{a+b}{2},\cdot\right)\right|^q, |X'''(b,\cdot)|^2\right\}, \quad (50)$$

almost everywhere. Hence, for $p>1$,

$$\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)dt = \int_{1/2}^{1}(t-1)^2\left(t-\frac{1}{2}\right)dt = \frac{2^{-1-3p}\Gamma(p+1)\Gamma(2p+1)}{\Gamma(3p+2)}. \quad (51)$$

Because (49)-(51), we obtain the desired result.

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