

Formulas of Fredholm Type for Fredholm Linear Equations in Fréchet Spaces

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Abstract

We consider Fredholm bounded linear operators $S+T$ acting from one Fréchet space X into another one Y , where S is Fredholm and T is nuclear. We obtain formulas for solutions of the induced equations: $(S+T)x = y_0$, $y^*(S+T) = x_0^*$. These formulas are abstract analogues of the classical formulas for solutions of Fredholm integral equations in the space of continuous functions on $[a, b]$. In this approach the main tools are provided by the theory of determinant systems. The effective formulas for determinant systems for nuclear perturbations of Fredholm operators in Fréchet spaces play the crucial role and lead to formulas of Fredholm type.

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1 Introduction

The primary concern in the classical Fredholm theory [11] is obtaining solutions of the equation $(I + \lambda T)x = x_0$ in the space $C[a, b]$, where T is an integral endomorphism with the kernel continuous on $[a, b] \times [a, b]$. The theory has been modified and generalized by many authors [19, 16, 14, 1, 15, 20] to make it applicable to certain classes of integral endomorphisms of concrete Banach spaces. The bibliography on determinants both in concrete and abstract Banach spaces is very rich and we mention here a small part of it only [18, 23, 22, 27, 17, 12, 13].

R. Sikorski [24, 25] has derived formulas of Fredholm type for solutions of a Fredholm linear equation $(I + T)x = x_0$ and for the adjoint equation $\xi(I + T) = \xi_0$, T being a quasinuclear endomorphism of a Banach space. Later A. Buraczewski [3, 4] extended the Sikorski's result to the formulas for Fredholm equations induced by endomorphisms of a Banach space of the form $S + T$, where S is Fredholm and T is quasinuclear. The further generalization [6] of these formulas was obtained for equations

$$(S + T)x = y_0 \quad (1)$$

and

$$\omega(S + T) = \xi_0, \quad (2)$$

where S is any fixed Fredholm operator, of non-negative index, from a Banach space X into another one Y and T is any quasinuclear operator from X into Y . The formulas of Fredholm type are abstract analogues of the classical formulas for solutions of Fredholm integral equations in $C[a, b]$.

So far determinants have been applied for solving linear Fredholm equations in Banach spaces. The purpose of this paper is to extend results related to Fredholm operators acting in Banach spaces over Fredholm operators acting in Fréchet spaces. In this approach we apply tools provided by the theory of determinant systems [24, 26, 2, 5, 8]. By means of terms of a determinant system we exhibit explicit solutions of a certain class of Fredholm linear equations in complete metrizable locally convex spaces. We focus on the case when the considered Fredholm equations (1), (2) are induced by nuclear perturbations of Fredholm bounded operators. We apply effective formulas for determinant systems for operators $S + T$, where S is a Fredholm operator from a Fréchet space X into another one Y and T is nuclear [8]. Our new results include formulas of Fredholm type for equations (1), (2) in Fréchet spaces.

2 Preliminaries

We begin with a brief review on the terminology used in the theory of determinant systems. The notation is adopted from papers [26, 2, 5, 7, 8, 9].

In what follows, X, Y denote fixed Fréchet spaces over the same field K of real or complex numbers with topologies determined by separating families of seminorms $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$, respectively, satisfying conditions: $p_n(x) \leq p_{n+1}(x)$, $q_n(y) \leq q_{n+1}(y)$ for $x \in X$, $y \in Y$. We endow the space X^* , of all continuous linear functionals on X , with the topology of the strict inductive limit of Banach spaces $\{(X_n^*, \|\cdot\|_n)\}_{n \in \mathbb{N}}$, where

$$X_n^* = \{x^* \in X^* : \exists c > 0 \forall x \in X, |x^*x| \leq cp_n(x)\},$$

$$\|x^*\|_n = \sup\{|x^*x| : p_n(x) \leq 1\} \quad \text{for } x^* \in X_n^*.$$

Given a $(\mu + m)$ -linear functional D on $(X^*)^\mu \times Y^m$, $D \begin{pmatrix} x_1^*, \dots, x_\mu^* \\ y_1, \dots, y_m \end{pmatrix}$ denotes its value at the point $(x_1^*, \dots, x_\mu^*, y_1, \dots, y_m) \in (X^*)^\mu \times Y^m$. The functional D is said to be *bi-skew symmetric* if it is skew symmetric both in variables x_1^*, \dots, x_μ^* and y_1, \dots, y_m . The set of all bi-skew symmetric functionals on $(X^*)^\mu \times Y^m$ is denoted by $bss_{\mu,m}(X^*, Y)$. D is called a (Y^*, X) -weakly continuous functional on $(X^*)^\mu \times Y^m$ if the following conditions are satisfied:

1. for any fixed points $x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_\mu^* \in X^*$ ($i = 1, \dots, \mu$) and $y_1, \dots, y_m \in Y$ there exists a point $x_i \in X$ such that

$$x^*x_i = D \begin{pmatrix} x_1^*, & \dots, & x_{i+1}^*, x^*, x_{i+1}^*, & \dots, & x_\mu^* \\ y_1, & & \dots, & & y_m \end{pmatrix} \quad \text{for every } x^* \in X^*.$$

2. for any fixed points $x_1^*, \dots, x_\mu^* \in X^*$ and $y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_m \in Y$ ($j = 1, \dots, m$) there exists a point $y_j^* \in Y^*$ such that

$$y_j^*y = D \begin{pmatrix} x_1^*, & \dots, & x_\mu^* \\ y_1, & \dots, & y_{j-1}, y, y_{j+1}, & \dots, & y_m \end{pmatrix} \quad \text{for every } y \in Y.$$

We denote by $L_{\mu,m}(X^*, Y)$ the space of all (Y^*, X) -weakly continuous functionals on $(X^*)^\mu \times Y^m$ equipped with the topology determined by a family of seminorms

$$\{p_{M_1^* \times \dots \times M_\mu^* \times M_1 \times \dots \times M_m}\}_{M_1^*, \dots, M_\mu^* \in \mathcal{M}_{X^*}, M_1, \dots, M_m \in \mathcal{M}_Y},$$

where \mathcal{M}_{X^*} contains all $\sigma(X^*, X)$ -bounded subsets of X^* , \mathcal{M}_Y contains all $\sigma(Y, Y^*)$ -bounded subsets of Y and

$$p_{M_1^* \times \dots \times M_\mu^* \times M_1 \times \dots \times M_m}(D) = \sup_{\substack{x_i^* \in M_i^*, i = 1, \dots, \mu \\ y_j \in M_j, j = 1, \dots, m}} \left| D \begin{pmatrix} x_1^*, \dots, x_\mu^* \\ y_1, \dots, y_m \end{pmatrix} \right|.$$

Each $D \in L_{1,1}(X^*, Y)$ is called an *operator on $X^* \times Y$* and can, simultaneously, be interpreted as a bilinear functional on $X^* \times Y$, x^*Dy being its value at $(x^*, y) \in X^* \times Y$ and as a linear mapping $D : Y \rightarrow X$, Dy being its value at y and also as a linear mapping $D : X^* \rightarrow Y^*$, x^*D being its value at x^* . It follows from the above that

$$x^*Dy = x^*(Dy) = (x^*D)y \quad \text{for all } x^* \in X^*, y \in Y.$$

We denote by $op(X^* \rightarrow Y^*, Y \rightarrow X)$ the space of all operators on $X^* \times Y$. Given non-zero points $x_0 \in X$, $y_0^* \in Y^*$, the operator $x_0 \cdot y_0^* \in op(X^* \rightarrow$

$Y^*, Y \rightarrow X$) whose value at a point $(x^*, y) \in X^* \times Y$ is equal to the product of the numbers x^*x_0 and y_0^*y , i.e. $x^*(x_0 \cdot y_0^*)y = x^*x_0 \cdot y_0^*y$, is said to be a *one-dimensional operator on $X^* \times Y$* .

For any $A \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ we introduce the following notation:

$$R(A) = \{Ax : x \in X\}, \quad N(A) = \{x \in X : Ax = 0\},$$

$$\mathcal{R}(A) = \{y^*A : y^* \in Y^*\}, \quad \mathcal{N}(A) = \{y^* \in Y^* : y^*A = 0\}.$$

A bilinear functional $A \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ such that $dimN(A) = n' < \infty$, $dim\mathcal{N}(A) = m' < \infty$, $R(A) = \mathcal{N}(A)^\perp$ and $\mathcal{R}(A) = N(A)^\perp$ is called a *Fredholm operator of order $r(A) = \min\{n', m'\}$ and index $d(A) = n' - m'$* . An operator $B \in op(X^* \rightarrow Y^*, Y \rightarrow X)$ is said to be a *generalized inverse of A* , provided $ABA = A$ and $BAB = B$.

A bounded linear functional $F : op(X^* \rightarrow Y^*, Y \rightarrow X) \rightarrow K$ is said to be a *quasinucleus on $op(X^* \rightarrow Y^*, Y \rightarrow X)$* , if there exists $T_F \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ such that $F(x \cdot y^*) = y^*T_Fx$ for $(y^*, x) \in Y^* \times X$. The space of all quasinuclei on $op(X^* \rightarrow Y^*, Y \rightarrow X)$ is denoted by $cn(Y^* \rightarrow X^*, X \rightarrow Y)$. For fixed non-zero $y_0 \in Y$, $x_0^* \in X^*$, the quasinucleus $x_0^* \otimes y_0$, defined by

$$(x_0^* \otimes y_0)(B) = x_0^*By_0 \text{ for } B \in op(X^* \rightarrow Y^*, Y \rightarrow X),$$

is said to be a *one-dimensional quasinucleus on $op(X^* \rightarrow Y^*, Y \rightarrow X)$* . Any finite sum $\sum_{i=1}^n x_i^* \otimes y_i$ of one-dimensional quasinuclei is said to be a *finitely dimensional quasinucleus on $op(X^* \rightarrow Y^*, Y \rightarrow X)$* . A quasinucleus $F \in cn(Y^* \rightarrow X^*, X \rightarrow Y)$ of the form:

$$F = \sum_{i=1}^{\infty} \lambda_i \tilde{x}_i^* \otimes \tilde{y}_i, \tag{3}$$

where $\lambda_i \in K$, $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$, $(\tilde{x}_i^*)_{i \in \mathbb{N}}$, $(\tilde{y}_i)_{i \in \mathbb{N}}$ are bounded sequences in X^* and Y , respectively, is called a *nucleus on $op(X^* \rightarrow Y^*, Y \rightarrow X)$* . In view of [8], the nucleus (3) determines the bounded nuclear operator [21, 10]

$$T_F = \sum_{i=1}^{\infty} \lambda_i \tilde{y}_i \cdot \tilde{x}_i^* .$$

Assume $F \in cn(Y^* \rightarrow X^*, X \rightarrow Y)$ and $D \in bss_{\mu,m}(X^*, Y) \cap L_{\mu,m}(X^*, Y)$. We fix all the variables x_2^*, \dots, x_μ^* and y_2, \dots, y_m and consider $D \begin{pmatrix} x_1^*, \dots, x_\mu^* \\ y_1, \dots, y_m \end{pmatrix}$ as the function of the variables x_1^*, y_1 only, i.e. as the operator from $op(X^* \rightarrow Y^*, Y \rightarrow X)$. We denote the value of F at the operator by

$$F \square D \begin{pmatrix} x_1^*, \dots, x_\mu^* \\ y_1, \dots, y_m \end{pmatrix}. \tag{4}$$

The symbol $F \square D$ stands for the function which assigns, to $x_2^*, \dots, x_\mu^* \in X^*$, $y_2, \dots, y_m \in Y$, the number (4). Thus $F \square D \in bss_{\mu-1, m-1}(X^*, Y)$. Assume, for $\mu, m > 1$, that $F \square D \in L_{\mu-1, m-1}(X^*, Y)$ and then repeat the above procedure to define $F \square F \square D \in bss_{\mu-2, m-2}(X^*, Y)$. By iterating the procedure k -times, $k = \min\{\mu, m\}$, we define $F \square D, F \square F \square D, \dots, \underbrace{F \square F \square \dots F \square D}_{k\text{-times}}$, provided $\underbrace{F \square F \square \dots F \square D}_{i\text{-times}} \in L_{\mu-i, m-i}(X^*, Y)$ ($i < k$). We also write

$$F^{\square k} = \frac{1}{k!} \underbrace{F \square F \square \dots F \square}_{k\text{-times}}, F \square \text{ being the function which assigns } F \square D \text{ to every } D \in L_{\mu, m}(X^*, Y) \cap bss_{\mu, m}(X^*, Y).$$

A sequence $(D_n)_{n \in N_0}$ fulfilling the conditions:

- (d1) $D_n \in bss_{\mu_n, m_n}(X^*, Y)$, where $\mu_n, m_n \in N_0, \mu_n = \mu_0 + n, m_n = m_0 + n, \min\{\mu_0, m_0\} = 0$;
- (d2) $D_n \in L_{\mu_n, m_n}(X^*, Y)$;
- (d3) there exists $r \in N_0$ such that $D_r \neq 0$;
- (d4) the following identities hold for $n \in N_0$:

$$(1) \quad D_{n+1} \begin{pmatrix} x_0^* & x_1^* & \dots & x_{\mu_n}^* \\ Ax & y_1 & \dots & y_{m_n} \end{pmatrix} = \sum_{i=0}^{\mu_n} (-1)^i x_i^* x D_n \begin{pmatrix} x_0^* & \dots & x_{i-1}^* & x_{i+1}^* & \dots & x_{\mu_n}^* \\ y_1 & \dots & \dots & \dots & \dots & y_{m_n} \end{pmatrix},$$

$$(2) \quad D_{n+1} \begin{pmatrix} y^* A & x_1^* & \dots & x_{\mu_n}^* \\ y_0 & y_1 & \dots & y_{m_n} \end{pmatrix} = \sum_{j=0}^{m_n} (-1)^j y^* y_j D_n \begin{pmatrix} x_1^* & \dots & x_{\mu_n}^* \\ y_0 & \dots & y_{j-1} & y_{j+1} & \dots & y_{m_n} \end{pmatrix},$$

where $x \in X, y^* \in Y^*, x_i^* \in X^*, y_j \in Y, (i = 0, 1, \dots, \mu_n, j = 0, 1, \dots, m_n)$, is said to be a *determinant system* for $A \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ [5]. The least $r \in N_0$, such that $D_r \neq 0$ and the difference $\mu_0 - m_0$ are called the *order* and the *index* of $(D_n)_{n \in N_0}$, respectively.

3 Main Results

In this section we examine a class of linear Fredholm equations, in Fréchet spaces, induced by nuclear perturbations of Fredholm operators. We prove the main theorem providing formulas for solutions of such equations. We begin with the considerations leading to some auxiliary results.

Let $S \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ be a fixed Fredholm operator of the order $r(S) = \min\{n', m'\} = r$ and the index $d(S) = n' - m'$. We denote by $\{s_1^*, \dots, s_{m'}^*\}, \{s_1, \dots, s_{n'}\}$ complete systems of solutions of the equations $y^* S = 0$ and $Sx = 0$, respectively. Given a generalized inverse $U \in op(X^* \rightarrow Y^*, Y \rightarrow X)$

of S , there exist points $u_1^*, \dots, u_{n'}^*$ and $u_1, \dots, u_{m'}$ such that

$$US = I - \sum_{i=1}^{n'} s_i \cdot u_i^* \quad \text{and} \quad SU = J - \sum_{i=1}^{m'} u_i \cdot s_i^*,$$

where $u_i^* s_j = \delta_{ij}$ ($i, j = 1, \dots, n'$), $s_i^* u_j = \delta_{ij}$ ($i, j = 1, \dots, m'$) and $I \in op(X^* \rightarrow X^*, X \rightarrow X)$, $J \in op(Y^* \rightarrow Y^*, Y \rightarrow Y)$ are identity operators. We define the following operator belonging to $op(X^* \rightarrow Y^*, Y \rightarrow X)$:

$$Q = U + \sum_{i=1}^r s_i \cdot s_i^*.$$

Let $(D_n)_{n \in N_0}$ be a determinant system for S , defined by the formula

$$D_n \begin{pmatrix} x_1^* \dots, & x_{n+n'-r}^* \\ y_1 \dots, & y_{n+m'-r} \end{pmatrix} = \begin{vmatrix} x_1^* U y_1 & \dots & x_1^* U y_{n+m'-r} & x_1^* s_1 & \dots & x_1^* s_{n'} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^* U y_1 & \dots & x_{n+n'-r}^* U y_{n+m'-r} & x_{n+n'-r}^* s_1 & \dots & x_{n+n'-r}^* s_{n'} \\ s_1^* y_1 & \dots & s_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ s_{m'}^* y_1 & \dots & s_{m'}^* y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix}, \quad (5)$$

where $x_i^* \in X^*$, $y_j \in Y$, $i = 1, \dots, n + n' - r$, $j = 1, \dots, n + m' - r$. Given $F \in cn(Y^* \rightarrow X^*, X \rightarrow Y)$, we denote

$$D_n(F) = \sum_{k=0}^{\infty} D_{n,k}(F), \quad n \in N_0, \quad (6)$$

where

$$D_{n,0}(F) = D_n \quad \text{and} \quad D_{n,k}(F) = F^{\square k} D_{n+k} \quad \text{for } k \in N. \quad (7)$$

In the sequel the following lemma plays an essential role.

Lemma 3.1. *Suppose that:*

- (a) $(D_n)_{n \in N_0}$ is the determinant system, defined by (5), for a Fredholm operator $S \in op(Y^* \rightarrow X^*, X \rightarrow Y)$ of order r ;
- (b) $F \in cn(Y^* \rightarrow X^*, X \rightarrow Y)$ is a nucleus defined by (3);
- (c) $r(S + T_F) = r'$;
- (d) $(D_n(F))_{n \in N_0}$ is the sequence defined by (6), (7).

Then

$$D_n(F) \begin{pmatrix} x_1^* M, & \dots, & x_{n+n'-r}^* M \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix} = (-1)^{n'+m'} D_n(F) \begin{pmatrix} x_1^*, & \dots, & x_{n+n'-r}^* \\ N y_1, & \dots, & N y_{n+m'-r} \end{pmatrix} \quad (8)$$

for $n \in N_0$, where

$$M = Q(S + T_F) - I, \quad N = (S + T_F)Q - J. \tag{9}$$

Proof. In view of the main result of [8], $(D_n(F))_{n \in N_0}$ is a determinant system for $S + T_F \in op(Y^* \rightarrow X^*, X \rightarrow Y)$, where the series on the right-hand side of (6) converges to $D_n(F)$ in the space $L_{n+n'-r, n+m'-r}(X^*, Y)$ ($n \in N_0$). Since $r(S + T_F) = r'$, there exist points $x'_1, \dots, x'_{r'+n'-r}$ and $y'_1, \dots, y'_{r'+m'-r}$ such that

$$\delta = D_{r'}(F) \begin{pmatrix} x'_1 & \cdots & x'_{r'+n'-r} \\ y'_1 & \cdots & y'_{r'+m'-r} \end{pmatrix} \neq 0.$$

Suppose that $\{z_1, \dots, z_{r'+n'-r}\}$ and $\{z^*_1, \dots, z^*_{r'+m'-r}\}$ are bases of $N(S + T_F)$ and $\mathcal{N}(S + T_F)$, respectively, where

$$x^* z_i = \frac{1}{\delta} D_{r'}(F) \begin{pmatrix} x'_1 & \cdots & x'_{i-1} & x^* & x'_{i+1} & \cdots & x'_{r'+n'-r} \\ y'_1 & & & \cdots & & & y'_{r'+m'-r} \end{pmatrix} \tag{10}$$

for $x^* \in X^*$ ($i = 1, \dots, r' + n' - r$) and

$$z^*_j y = \frac{1}{\delta} D_{r'}(F) \begin{pmatrix} x'_1 & & \cdots & & x'_{r'+n'-r} \\ y'_1 & \cdots & y'_{j-1} & y & y'_{j+1} & \cdots & y'_{r'+m'-r} \end{pmatrix} \tag{11}$$

for $y \in Y$ ($j = 1, \dots, r' + m' - r$). Denote by B a generalized inverse of $S + T_F$ defined by the formula

$$x^* B y = \frac{1}{\delta} D_{r'+1}(F) \begin{pmatrix} x^* & x'_1 & \cdots & x'_{r'+n'-r} \\ y & y'_1 & \cdots & y'_{r'+m'-r} \end{pmatrix} \tag{12}$$

for $(x^*, y) \in X^* \times Y$. In view of (10) and (11),

$$B(S + T_F) = I - \sum_{i=1}^{r'+n'-r} z_i \cdot x'^*_i, \tag{13}$$

where $x'^*_i z_j = \delta_{ij}$ ($i, j = 1, \dots, r' + n' - r$) and

$$(S + T_F)B = J - \sum_{i=1}^{r'+m'-r} y'_i \cdot z^*_i, \tag{14}$$

where $z^*_i y'_j = \delta_{ij}$ ($i, j = 1, \dots, r' + m' - r$). It is obvious that $B(S + T_F)B = B$ and $(S + T_F)B(S + T_F) = S + T_F$. By using (13), bearing in mind (9), we obtain

$$BN = B(S + T_F)Q - B = \left(I - \sum_{i=1}^{r'+n'-r} z_i \cdot x'^*_i \right) Q - B = Q - B - \sum_{i=1}^{r'+n'-r} z_i \cdot x'^*_i Q. \tag{15}$$

Similarly, multiplying M in (9) on the right by B and using (14), we obtain

$$MB = Q(S + T_F)B - B = Q\left(J - \sum_{i=1}^{r'+m'-r} y'_i \cdot z_i^*\right) - B = Q - B - \sum_{i=1}^{r'+m'-r} Qy'_i \cdot z_i^*. \tag{16}$$

Introducing the notation

$$L = \sum_{i=1}^{r'+n'-r} z_i \cdot x_i^* Q, \quad K = \sum_{i=1}^{r'+m'-r} Qy'_i \cdot z_i^*, \tag{17}$$

and taking into account (15) and (16),

$$MB - L = BN - K. \tag{18}$$

Let $(D'_n)_{n \in N_0}$ be the determinant system for $S + T_F$ defined by the formula

$$D'_n \begin{pmatrix} x_1^*, & \dots, & x_{n+n'-r}^* \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix} = \begin{vmatrix} x_1^* B y_1 & \dots & x_1^* B y_{n+m'-r} & x_1^* z_1 & \dots & x_1^* z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^* B y_1 & \dots & x_{n+n'-r}^* B y_{n+m'-r} & x_{n+n'-r}^* z_1 & \dots & x_{n+n'-r}^* z_{r'+n'-r} \\ z_1^* y_1 & \dots & z_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix},$$

$x_i^* \in X^*$ ($i = 1, \dots, n + n' - r$), $y_j \in Y$ ($j = 1, \dots, n + m' - r$). It follows from the theory of determinant systems [26, 3] that there exists a scalar $c \neq 0$ such that $D'_n = cD_n(F)$ for $n \in N_0$. Therefore,

$$D_n(F) \begin{pmatrix} x_1^* M, & \dots, & x_{n+n'-r}^* M \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix} = \tag{19}$$

$$= c \begin{vmatrix} x_1^* M B y_1 & \dots & x_1^* M B y_{n+m'-r} & x_1^* M z_1 & \dots & x_1^* M z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^* M B y_1 & \dots & x_{n+n'-r}^* M B y_{n+m'-r} & x_{n+n'-r}^* M z_1 & \dots & x_{n+n'-r}^* M z_{r'+n'-r} \\ z_1^* y_1 & \dots & z_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix}.$$

Having (9), we easily obtain $Mz_i = -z_i$ ($i = 1, \dots, r' + n' - r$). Hence, applying well-known properties of classical determinants, we transform the right-

hand side of (19) into the determinant

$$\det \begin{bmatrix} x_1^* M B y_1 & \dots & x_1^* M B y_{n+m'-r} & x_1^* z_1 & \dots & x_1^* z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^* M B y_1 & \dots & x_{n+n'-r}^* M B y_{n+m'-r} & x_{n+n'-r}^* z_1 & \dots & x_{n+n'-r}^* z_{r'+n'-r} \\ z_1^* y_1 & \dots & z_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{n+m'-r} & 0 & \dots & 0 \end{bmatrix} \quad (20)$$

multiplied by $c(-1)^{r'+n'-r}$. Furthermore, by multiplying the $(n + m' - r + i)$ -th column of the matrix in (20), by $-x_i^* Q y_j$, and adding to the j -th column ($i = 1, \dots, r' + n' - r, j = 1, \dots, n + m' - r$), we preserve the determinant of the matrix. In view of (17), since $x_k^*(MB - L)y_j = x_k^* M B y_j - \sum_{i=1}^{r'+n'-r} x_k^* z_i \cdot x_i^* Q y_j$ ($k = 1, \dots, n + n' - r$), we obtain the equivalent form of (20):

$$\begin{vmatrix} x_1^*(MB - L)y_1 & \dots & x_1^*(MB - L)y_{n+m'-r} & x_1^* z_1 & \dots & x_1^* z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^*(MB - L)y_1 & \dots & x_{n+n'-r}^*(MB - L)y_{n+m'-r} & x_{n+n'-r}^* z_1 & \dots & x_{n+n'-r}^* z_{r'+n'-r} \\ z_1^* y_1 & \dots & z_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix} \quad (21)$$

It follows from (18), that (21) is equal to

$$\begin{vmatrix} x_1^*(BN - K)y_1 & \dots & x_1^*(BN - K)y_{n+m'-r} & x_1^* z_1 & \dots & x_1^* z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^*(BN - K)y_1 & \dots & x_{n+n'-r}^*(BN - K)y_{n+m'-r} & x_{n+n'-r}^* z_1 & \dots & x_{n+n'-r}^* z_{r'+n'-r} \\ z_1^* y_1 & \dots & z_1^* y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix} \quad (22)$$

Taking into account the identities $x_k^*(BN - K)y_j = x_k^* B N y_j - \sum_{i=1}^{r'+m'-r} z_i^* y_j \cdot x_k^* Q y_i'$ ($k = 1, \dots, n + n' - r, j = 1, \dots, n + m' - r$), multiplying the

$(n + n' - r + i)$ -th ($i = 1, \dots, r' + m' - r$) row of the matrix

$$\begin{bmatrix} x_1^*BNy_1 & \dots & x_1^*BNy_{n+m'-r} & x_1^*z_1 & \dots & x_1^*z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^*BNy_1 & \dots & x_{n+n'-r}^*BNy_{n+m'-r} & x_{n+n'-r}^*z_1 & \dots & x_{n+n'-r}^*z_{r'+n'-r} \\ z_1^*y_1 & \dots & z_1^*y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^*y_1 & \dots & z_{r'+m'-r}^*y_{n+m'-r} & 0 & \dots & 0 \end{bmatrix},$$

by $-x_k^*Qy'_i$, and adding to its k -th row, (22) can be written in the form

$$\begin{vmatrix} x_1^*BNy_1 & \dots & x_1^*BNy_{n+m'-r} & x_1^*z_1 & \dots & x_1^*z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^*BNy_1 & \dots & x_{n+n'-r}^*BNy_{n+m'-r} & x_{n+n'-r}^*z_1 & \dots & x_{n+n'-r}^*z_{r'+n'-r} \\ z_1^*y_1 & \dots & z_1^*y_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^*y_1 & \dots & z_{r'+m'-r}^*y_{n+m'-r} & 0 & \dots & 0 \end{vmatrix}. \tag{23}$$

By virtue of (9), $z_i^*N = -z_i^*$ ($i = 1, \dots, r' + m' - r$). Hence, it follows from the well-known property of determinants, that (23) is equal to

$$\begin{vmatrix} x_1^*BNy_1 & \dots & x_1^*BNy_{n+m'-r} & x_1^*z_1 & \dots & x_1^*z_{r'+n'-r} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+n'-r}^*BNy_1 & \dots & x_{n+n'-r}^*BNy_{n+m'-r} & x_{n+n'-r}^*z_1 & \dots & x_{n+n'-r}^*z_{r'+n'-r} \\ z_1^*Ny_1 & \dots & z_1^*Ny_{n+m'-r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r'+m'-r}^*Ny_1 & \dots & z_{r'+m'-r}^*Ny_{n+m'-r} & 0 & \dots & 0 \end{vmatrix}$$

multiplied by $(-1)^{r'+m'-r}$. By the above, we conclude that

$$D_n(F) \begin{pmatrix} x_1^*M, & \dots, & x_{n+n'-r}^*M \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix} = c(-1)^{n'+m'} D'_n \begin{pmatrix} x_1^*, & \dots, & x_{n+n'-r}^* \\ Ny_1, & \dots, & Ny_{n+m'-r} \end{pmatrix}. \tag{24}$$

In view of the relationship between $(D'_n)_{n \in N_0}$ and $(D_n(F))_{n \in N_0}$, (24) yields (8), which completes the proof. \square

With the assumptions of Lemma 3.1, we can formulate the following its consequence.

Corollary 3.2.

$$D_{r'}(F) \begin{pmatrix} x_1^*M, & \dots, & x_{r'+n'-r}^*M \\ y_1, & \dots, & y_{r'+m'-r} \end{pmatrix} = (-1)^{r'+n'-r} D_{r'}(F) \begin{pmatrix} x_1^*, & \dots, & x_{n+n'-r}^* \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix}. \tag{25}$$

Proof. Since

$$D_{r'}(F) \begin{pmatrix} x_1^*, & \dots, & x_{r'+n'-r}^* \\ y_1, & \dots, & y_{r'+m'-r} \end{pmatrix} = c(-1)^{(r'+n'-r)(r'+m'-r)} \times$$

$$\times \begin{vmatrix} z_1^* y_1 & \dots & z_1^* y_{r'+m'-r} \\ \vdots & & \vdots \\ z_{r'+m'-r}^* y_1 & \dots & z_{r'+m'-r}^* y_{r'+m'-r} \end{vmatrix} \cdot \begin{vmatrix} x_1^* z_1 & \dots & x_1^* z_{r'+n'+m'} \\ \vdots & & \vdots \\ x_{r'+n'-m'}^* z_1 & \dots & x_{r'+n'-m'}^* z_{r'+n'-m'} \end{vmatrix},$$

substituting $n = r'$ in (20) and then using the well-known property of partitioned matrices, we obtain (25). □

We are now in a position to state and prove the main result of this paper, i.e. formulas of Fredholm type for linear equations (1), (2) induced by the operator $S + T_F$ in Fréchet spaces.

Theorem 3.3. *Under the assumptions of Lemma 3.1, let $(D_n^*)_{n \in N_0}$ be a determinant system for $S + T_F$, defined by the formula:*

$$D_n^* \begin{pmatrix} x_1^*, & \dots, & x_{n+n'-r}^* \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix} = D_n(F) \begin{pmatrix} x_1^* M, & \dots, & x_{n+n'-r}^* M \\ y_1, & \dots, & y_{n+m'-r} \end{pmatrix}.$$

If $x_1^*, \dots, x_{r'+n'-r}^*$ and $y_1, \dots, y_{r'+m'-r}$ are points satisfying the condition

$$\delta^* = D_{r'}^* \begin{pmatrix} x_1^*, & \dots, & x_{r'+n'-r}^* \\ y_1, & \dots, & y_{r'+m'-r} \end{pmatrix} \neq 0$$

and B^* is the operator defined by the formula

$$x^* B^* y = \frac{1}{\delta^*} D_{r'+1}^* \begin{pmatrix} x^*, & x_1^*, & \dots, & x_{r'+n'-r}^* \\ y, & y_1, & \dots, & y_{r'+m'-r} \end{pmatrix} \text{ for } (x^*, y) \in X^* \times Y, \quad (26)$$

then the following statements hold:

(a) $\{z_1, \dots, z_{r'+n'-r}\}$ and $\{z_1^*, \dots, z_{r'+m'-r}^*\}$ are bases for $N(S + T_F)$ and $\mathcal{N}(S + T_F)$, respectively, where

$$x^* z_i = \frac{1}{\delta^*} D_{r'}^* \begin{pmatrix} x_1^*, & \dots, & x_{i-1}^*, & x^*, & x_{i+1}^*, & \dots, & x_{r'+n'-r}^* \\ y_1, & & & \dots, & & & y_{r'+m'-r} \end{pmatrix} \quad (27)$$

for $x^* \in X^* (i = 1, \dots, r' + n' - r)$,

$$z_j^* y = \frac{1}{\delta^*} D_{r'}^* \begin{pmatrix} x_1^*, & & \dots, & & x_{r'+n'-r}^* \\ y_1, & \dots, & y_{j-1}, & y, & y_{j+1}, & \dots, & y_{r'+m'-r} \end{pmatrix} \quad (28)$$

for $y \in Y$ ($j = 1, \dots, r' + m' - r$).

(b) The equation $(S + T_F)x = y_0$ has a solution $x \in X$ if and only if

$$D_{r'}^* \begin{pmatrix} x_1^*, & \dots, & x_{r'+n'-r}^* \\ y_1', & \dots, & y_{r'+m'-r}' \end{pmatrix} = 0 \tag{29}$$

($j = 1, \dots, r' + m' - r$) and the general form of the solution is

$$x = (Q - B^*)y_0 + \alpha_1 z_1 + \dots + \alpha_{r'+n'-r} z_{r'+n'-r} \quad (\alpha_i \in K, \quad i = 1, \dots, r' + n' - r).$$

(c) The equation $y^*(S + T_F) = x_0^*$ has a solution $y^* \in Y^*$ if and only if

$$D_{r'}^* \begin{pmatrix} x_1^*, & \dots, & x_{i-1}^*, & x_0^*, & x_{i+1}^*, & \dots, & x_{r'+n'-r}^* \\ y_1', & & & \dots, & & & y_{r'+m'-r}' \end{pmatrix} = 0 \tag{30}$$

($i = 1, \dots, r' + n' - r$) and the general form of the solution is given by

$$y^* = x_0^*(Q - B^*) + \beta_1 z_1^* + \dots + \beta_{r'+m'-r} z_{r'+m'-r}^* \quad (\beta_j \in K, \quad j = 1, \dots, r' + m' - r).$$

Proof. Since (25) holds, the bi-skew symmetry of $D_{r'}(F)$ combined with (27) give rise to identities: $x_j^* z_i = \delta_{ij}$ ($i, j = 1, \dots, r' + n' - r$). Thus, $z_1, \dots, z_{r'+n'-r}$ are linearly independent. Denote $c = (-1)^{r'+n'+m'} \frac{1}{\delta^*}$. Since $D_n(F) = 0$ for $n < r'$, in view of (d4) (2), we obtain

$$\begin{aligned} & y^*(S + T_F)z_i = \\ &= cD_{r'}(F) \begin{pmatrix} x_1^*, \dots, & x_{i-1}^*, y^*(S + T_F), & x_{i+1}^*, \dots, & x_{r'+n'-r}^* \\ y_1', & \dots, & y_{r'+m'-r}' \end{pmatrix} = \\ &= c \sum_{j=1}^{r'+m'-r} (-1)^{i+j} y_j^* y_j' D_{r'-1}(F) \begin{pmatrix} x_1^*, \dots, & x_{i-1}^*, x_{i+1}^*, \dots, & x_{r'+n'-r}^* \\ y_1', \dots, & y_{j-1}', y_{j+1}', \dots, & y_{r'+m'-r}' \end{pmatrix} = 0. \end{aligned}$$

Consequently, $\{z_1, \dots, z_{r'+n'-r}\}$ is a complete system of solutions of the homogeneous equation $(S + T_F)x = 0$. Similarly, bearing in mind (28), it follows from the bi-skew symmetry of $D_{r'}(F)$ that $z_j^* y_i = \delta_{ij}$ ($i, j = 1, \dots, r' + m' - r$). This implies the linear independence of $z_1^*, \dots, z_{r'+m'-r}^*$. Furthermore, remembering that $(D_n^*)_{n \in \mathbb{N}_0}$ satisfies (d4) (1) and $r(S + T_F) = r'$, we conclude that

$$\begin{aligned} & z_j^*(S + T_F)x = \\ &= cD_{r'}(F) \begin{pmatrix} x_1^*, & \dots, & x_{r'+n'-r}^* \\ y_1', \dots, & y_{j-1}', (S + T_F)x, & y_{j+1}', \dots, & y_{r'+m'-r}' \end{pmatrix} = \\ &= c \sum_{i=1}^{r'+n'-r} (-1)^{i+j} x_i^* x_i' D_{r'-1}(F) \begin{pmatrix} x_1^*, \dots, & x_{i-1}^*, x_{i+1}^*, \dots, & x_{r'+n'-r}^* \\ y_1', \dots, & y_{j-1}', y_{j+1}', \dots, & y_{r'+m'-r}' \end{pmatrix} = 0. \end{aligned}$$

Thus $\{z_1^*, \dots, z_{r'+m'-r}^*\}$ is a basis for $\mathcal{N}(S + T_F)$, which proves statement (a).

Substituting $(S + T_F)x$ for y in (26) and then taking into account both (d4) (1) and Lemma 3.1, we obtain

$$\begin{aligned} x^* B^*(S + T_F)x &= \frac{1}{\delta^*} D_{r'+1}^* \begin{pmatrix} x^*, & x_1^*, \dots, & x_{r'+n'-r}^* \\ (S + T_F)x, & y_1', \dots, & y_{r'+m'-r}' \end{pmatrix} = \quad (31) \\ &= \frac{1}{\delta^*} (x^* Mx\delta^* + \\ &+ \sum_{i=1}^{r'+n'-r} (-1)^i x_i^* Mx D_{r'}^* \begin{pmatrix} x^*, x_1^*, & \dots, & x_{i-1}^*, x_{i+1}^*, & \dots, & x_{r'+n'-r}^* \\ y_1', & & \dots, & & y_{r'+m'-r}' \end{pmatrix}). \end{aligned}$$

Next, by applying (25), (27) and the bi-skew symmetry of $D_{r'}(F)$, we transform the right-hand side of (31) into the form

$$x^* Mx - \sum_{i=1}^{r'+n'-r} x_i^* Mx \cdot x^* z_i.$$

Consequently, $B^*(S + T_F) = M - \sum_{i=1}^{r'+n'-r} z_i \cdot x_i^* M$, which leads to

$$(Q - B^*)(S + T_F) = I + \sum_{i=1}^{r'+n'-r} z_i \cdot x_i^* M. \quad (32)$$

Similarly, substituting $y^*(S + T_F)$ for x^* in (26) and then using (d4) (2) and Lemma 3.1, we find

$$\begin{aligned} y^*(S + T_F)B^*y &= \frac{1}{\delta^*} D_{r'+1}^* \begin{pmatrix} y^*(S + T_F), & x_1^*, \dots, & x_{r'+n'-r}^* \\ y, & y_1', \dots, & y_{r'+m'-r}' \end{pmatrix} = \quad (33) \\ &= \frac{1}{\delta^*} (y^* Ny\delta^* + \\ &+ \sum_{j=1}^{r'+m'-r} (-1)^j y^* Ny_j' D_{r'}^* \begin{pmatrix} x_1^*, & & \dots, & & x_{r'+n'-r}^* \\ y, y_1', & \dots, & y_{j-1}', y_{j+1}', & \dots, & y_{r'+m'-r}' \end{pmatrix}). \end{aligned}$$

The bi-skew symmetry of $D_{r'}(F)$ combined with (25) and (28), yield the following form of the right-hand side of (33):

$$y^* Ny - \sum_{j=1}^{r'+m'-r} y^* y_j' \cdot z_j^* Ny^*.$$

Thus, $(S + T_F)B^* = N - \sum_{j=1}^{r'+m'-r} Ny'_j \cdot z_j^*$, which implies

$$(S + T_F)(Q - B^*) = J + \sum_{j=1}^{r'+m'-r} Ny'_j \cdot z_j^*. \quad (34)$$

Multiplying (34) by y_0 on the right and assuming (29) holds, we get

$$(S + T_F)(Q - B^*)y_0 = y_0,$$

which proves statement (b). Analogously, multiplying (32) by x_0^* on the left and assuming (30) holds, we arrive at

$$x_0^*(Q - B^*)(S + T_F) = x_0^*.$$

This shows that statement (c) is valid, and the proof is complete. □

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