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Structures of Γ_{11} over the prime field Z_2

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Abstract

The 8-dimensional 3-Lie algebra Γ_{11} over the prime field Z_2 is realized by 2-cubic matrix, and the structures of it is studied. It is proved that Γ_{11} is a solvable but non-nilpotent 3-Lie algebra (Theorem 2.3). The inner derivation algebra $ad(\Gamma_{11})$ is a 10-dimensional two-step nilpotent Lie algebra (Theorem 2.5), and the derivation algebra $Der(\Gamma_{11})$ with dimension 13 is solvable but non-nilpotent (Theorem 2.6). And the concrete expression of all derivations are given.

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1 Introduction

Authors in papers [1, 2] constructed 3-Lie algebras by Lie algebras, associative algebras, pre-Lie algebras and commutative associative algebras and their derivations and involutions. In paper [3], fifteen kinds of multiplications of N-cubic matrix are provided, and four non-isomorphic N^3-dimensional 3-Lie algebras are constructed. Bai, Guo, and Lin in papers [4, 5] discussed 3-Lie algebras J_{11} and J_{21} which are realized by 2-cubic matrix over a field with characteristic p, $p \neq 2$. In this paper, we pay our main attention to

8-dimensional 3-Lie algebras which are constructed by 2-cubic matrix in the prime field $Z_2 = \{0, 1\}$. In the following we suppose that $Z_2 = \{0, 1\}$ is the prime field with characteristic two, for a vector space V and a subset S, the subspace generated by S is denoted by (S).

2 Structure of 3-Lie algebra Γ_{11}

First we introduce some notations. An N-order cubic matrix $A = (a_{ijk})$ over the field Z_2 is an ordered object which the elements with 3 indices, and the element in the position (i, j, k) is $(A)_{ijk} = a_{ijk}$, $1 \leq i, j, k \leq N$ and $a_{ijk} = 0$ or 1. Denote the set of all cubic matrix over Z_2 by Ω. Then Ω is an N^3-dimensional vector space with $A + B = (a_{ijk} + b_{ijk}) \in \Omega$, $\lambda A = (\lambda a_{ijk}) \in \Omega$, for $\forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega$, $\lambda \in Z_2$, that is, $(A + B)_{ijk} = a_{ijk} + b_{ijk}$, $(\lambda A)_{ijk} = \lambda a_{ijk}$.

Denote $E_{ijk} = (e_{h_1h_2h_3})$, where $e_{h_1h_2h_3} = \delta_{h_1i}\delta_{h_2j}\delta_{h_3k}$, that is when $h_1 = i$, $h_2 = j$, $h_3 = k$, $e_{h_1h_2h_3} = 1$, and elsewhere are zero. Then, $\{E_{ijk} | \ 1 \leq i, j, k \leq n\}$ is a basis of Ω.

For all $A = (a_{ijk}), B = (b_{ijk}) \in \Omega$, define the multiplication $*_{11}$ in Ω by

$$(A *_{11} B)_{ijk} = \sum_{p=1}^{n} a_{ijp}b_{ipk}, 1 \leq i, j, k \leq n. \tag{1}$$

Denote $\langle A\rangle_1 = \sum\limits_{p,q=1}^{N} (A)_{pqq} = \sum\limits_{p,q=1}^{N} a_{pqq}$, Then $\langle \ \rangle_1$ is a linear function from the vector space Ω to Z_2 and satisfies

$$\langle A *_{11} B\rangle_1 = \langle B *_{11} A\rangle_1. \tag{2}$$

Define the multiplication $[, ,]_{11} : \Omega \wedge \Omega \wedge \Omega \to \Omega$ as follows:

$$[A, B, C]_{11} = \langle A\rangle_1(B *_{11} C + C *_{11} B)$$

$$+\langle B\rangle_1(C *_{11} A + A *_{11} C) + \langle C\rangle_1(A *_{11} B + B *_{11} A). \tag{3}$$

Theorem 2.1[3] *The linear space Ω is a 3-Lie algebra in the multiplication $*_{11}$, which is denoted by Γ_{11}, and the multiplication $[, ,]_{11}$ is simply denoted by $[, ,]$.*

In the following we suppose that Ω_2 is vector space over $Z_2 = \{0, 1\}$ which consists of 2-cubic matrix. Then the dimension of Ω_2 is eight and with a basis

$$\{E_{111}, E_{112}, E_{121}, E_{122}, E_{211}, E_{212}, E_{221}, E_{222}\}.$$

And for all $A \in \Omega_2$, $A = \sum\limits_{i,j,k=1}^{2} \lambda_{ijk}E_{ijk}$, $\lambda_{ijk} = 1, 0 \in Z_2$.

Theorem 2.2. The multiplication of the 3-Lie algebra Γ_{11} in the basis $\{E_{111}, E_{112}, E_{121}, E_{122}, E_{211}, E_{212}, E_{221}, E_{222}\}$ is as follows

$$\begin{cases} [E_{111}, E_{112}, E_{121}] = E_{111} + E_{122}, \\ [E_{111}, E_{121}, E_{211}] = E_{121}, \\ [E_{111}, E_{211}, E_{212}] = E_{212}, \\ [E_{111}, E_{211}, E_{221}] = E_{221}, \\ [E_{111}, E_{112}, E_{211}] = E_{112}, \\ [E_{111}, E_{212}, E_{222}] = E_{212}, \\ [E_{111}, E_{221}, E_{222}] = E_{221}, \\ [E_{111}, E_{212}, E_{221}] = E_{211} + E_{222}, \\ [E_{112}, E_{121}, E_{122}] = E_{111} + E_{122}, \\ [E_{112}, E_{121}, E_{211}] = E_{111} + E_{122}, \\ [E_{112}, E_{122}, E_{211}] = E_{112}, \\ [E_{121}, E_{122}, E_{211}] = E_{121}, \\ [E_{211}, E_{212}, E_{221}] = E_{211} + E_{222}, \\ [E_{212}, E_{221}, E_{222}] = E_{211} + E_{222}, \end{cases} \tag{4}$$

where the zero product of the basis vectors are omitted.

Proof The result follows from a direct complication according to the definition of $*_{11}$ and Eqs.(1), (2) and (3).

For studying the structure of the 3-Lie algebra Γ_{11}, we need to find a special basis to simplify its multiplication.

Theorem 2.3 *1).* Γ_{11} *is an indecomposable* 3*-Lie algebra with two dimensional center, and the multiplication in the basis* $\{e_1 = E_{111},\ e_2 = E_{112},\ e_3 = E_{121},\ e_4 = E_{111} + E_{122},\ e_5 = E_{211} + E_{111},\ e_6 = E_{212},\ e_7 = E_{221},\ e_8 = E_{211} + E_{222}\}$ *is as follows*

$$\begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_5, e_6] = e_6, \\ [e_1, e_6, e_7] = e_8, \\ [e_1, e_2, e_5] = e_2, \\ [e_1, e_3, e_5] = e_3, \\ [e_1, e_5, e_7] = e_7. \end{cases} \tag{5}$$

2) The 3*-Lie algebra* Γ_{11} *is solvable but non-nilpotent.*

Proof It is clear that $\{e_1, \cdots, e_8\}$ is a basis of Ω_2. By Theorem 2.2 and a direct computation we have Eq.(4), and the center of Γ_{11} is $Z(\Gamma_{11}) = (e_4, e_8)$. Since Γ_{11} can not be written as the direct sum of two proper ideals, Γ_{11} is indecomposable. The result 1) follows.

From $\Gamma_{11}^{(1)} = [\Gamma_{11}, \Gamma_{11}, \Gamma_{11}] = (e_2, e_3, e_4, e_6, e_7, e_8)$, and $\Gamma_{11}^{(2)} = [\Gamma_{11}^{(1)}, \Gamma_{11}^{(1)}, \Gamma_{11}] = 0$, Γ_{11} is solvable. By $\Gamma_{11}^1 = [\Gamma_{11}, \Gamma_{11}, \Gamma_{11}] = (e_2, e_3, e_4, e_6, e_7, e_8)$,

$$\Gamma_{11}^2 = [\Gamma_{11}^1, \Gamma_{11}, \Gamma_{11}] = (e_2, e_3, e_4, e_6, e_7, e_8) = \Gamma_{11}^1,$$

so for all $s > 1$, we have $\Gamma_{11}^{s+1} = [\Gamma_{11}^{s}, \Gamma_{11}^{s}, \Gamma_{11}] = \Gamma_{11}^{1} \neq 0$. Then Γ_{11} is non-nilpotent. The proof is completed.

Theorem 2.4 *The subalgebra $H = (e_1, e_4, e_5, e_8)$ is a Cartan subalgebra of Γ_{11}. And the decomposition of Γ_{11} associate to H is*

$$\Gamma_{11} = H \dot{+} \Gamma_\alpha, \ \Gamma_\alpha = (e_2, e_3, e_6, e_7) = \{e \in \Gamma \mid [h_1, h_2, e] = \alpha(h_1, h_2)e\},$$

where $\alpha \in (H \otimes H)^*$, $\alpha(e_1, e_5) = 1$, $\alpha(e_1, e_j) = \alpha(e_i, e_j) = 0$, $i, j = 4, 5, 8$.

Proof From Theorem 2.3, the subalgebra $H = (e_1, e_4, e_5, e_8)$ is an abelian subalgebra, so it is also a nilpotent 3-Lie algebra. If the element e in Γ_{11} satisfies $[e, \Gamma_{11}, H] \subseteq H$, then $e \in H$, so H is a Cartan subalgebra of Γ_{11}. Thanks to Eq.(4), the decomposition of Γ_{11} associate to H is

$$\Gamma_{11} = H \dot{+} \Gamma_\alpha, \ \Gamma_\alpha = (e_2, e_3, e_6, e_7) = \{e \in \Gamma \mid [h_1, h_2, e] = \alpha(h_1, h_2)e\},$$

where $\alpha \in (H \otimes H)^*$, $\alpha(e_1, e_5) = 1$, $\alpha(e_1, e_j) = \alpha(e_i, e_j) = 0$, $i, j = 4, 5, 8$. The proof is completed.

Now we study the inner derivation algebra $ad(\Gamma_{11})$. For $e_i, e_j \in \Omega$, denote

$$ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l, \text{ where } a_{kl}^{ij} = a_{kl}^{ji} = 0, \ or \ 1 \in Z_2.$$

Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum\limits_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl}, $1 \leq k, l \leq 8$ are matrix units.

Theorem 2.5 The inner derivation algebra $ad(\Gamma_{11})$ satisfies the following

1) $\dim ad(\Gamma_{11}) = 10$, and $X_1 = E_{34} + E_{52}$, $X_2 = E_{24} + E_{53}$, $X_3 = E_{56} + E_{78}, X_4 = E_{57} + E_{68}$, $X_5 = E_{14}, X_6 = E_{12}, X_7 = E_{13}$, $X_8 = E_{16}, X_9 = E_{17}$, $X_{10} = E_{18}$ is a basis of $ad(\Gamma_{11})$, the multiplication in it is

$$\begin{cases} [X_1, X_7] = X_5, \\ [X_2, X_6] = X_5, \\ [X_3, X_9] = X_{10}, \\ [X_4, X_8] = X_{10}. \end{cases} \tag{5}$$

2) $ad(\Gamma_{11})$ is a two-step nilpotent Lie algebra, and the center $Z(ad(\Gamma_{11}))$ $= (X_4, X_{10}) = \Gamma_{11}^1$.

Proof By Theorem 2.3, for all e_i, e_j, $1 \leq i, j \leq 8$, the left multiplication $ad(e_i, e_j) : \Gamma_{11} \to \Gamma_{11}$, defined by $ad(e_i, e_j)e = [e_i, e_j, e]$, satisfy $ad(e_1, e_2)$ $= E_{34} + E_{52}$, $ad(e_1, e_3) = E_{24} + E_{53}$, $ad(e_1, e_6) = E_{56} + E_{78}$, $ad(e_1, e_7) = E_{57} + E_{68}$, $ad(e_2, e_3) = E_{14}$, $ad(e_2, e_5) = E_{12}$, $ad(e_3, e_5) = E_{13}$, $ad(e_5, e_6) = E_{16}$, $ad(e_5, e_7) = E_{17}$, $ad(e_6, e_7) = E_{18}$. Therefore, $\{X_1, \cdots, X_{10}\}$ is a basis of $ad(\Gamma_{11})$. From$[ad(e_i, e_j), ad(e_k, e_l)] = ad([e_i, e_j, e_k], e_l) + ad(e_k, [e_i, e_j, e_l])$ and Eq.(4), we have the result.

Theorem 2.6 *The derivation algebra* $Der(\Gamma_{11})$ *with dimension* 13*, and* $\{X_1, \cdots, X_{13}\}$*, where* $X_{11} = E_{11}+E_{44}+E_{55}+E_{88}$*,* $X_{12} = E_{54}+E_{58}$*,* $X_{13} = E_{15}$*, and* X_i *for* $1 \leq i \leq 10$ *expressed in Theorem 2.5, is a basis of* $Der(\Gamma_{11})$*, and the multiplication in it is*

$$\begin{cases} [X_1, X_7] = X_5, \ [X_2, X_6] = X_5, \\ [X_3, X_9] = X_{10}, [X_2, X_{11}] = X_2, \\ [X_4, X_8] = X_{10}, [X_1, X_{11}] = X_1, \\ [X_3, X_{11}] = X_3, [X_4, X_{11}] = X_4, \\ [X_1, X_{13}] = X_6, [X_2, X_{13}] = X_7, \\ [X_3, X_{13}] = X_8, [X_4, X_{13}] = X_9, \\ [X_6, X_{11}] = X_6, [X_{12}, X_{13}] = X_5 + X_{10}. \end{cases} \tag{6}$$

2) $Der(\Gamma_{11})$ *is solvable but non-nilpotent Lie algebra,* $Z(Der(\Gamma_{11})) = (X_5, X_{10})$*, and* $Der^1(\Gamma_{11}) = ad(\Gamma_{11})$*.*

Proof The result follows from Theorem 2.5 and a direct computation.

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