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3-Lie algebra Γ_{21} over the prime field Z_2

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Abstract

The 8-dimensional 3-Lie algebra Γ_{21} over the prime field Z_2 is constructed by 2-cubic matrix, and the structures of it is studied. It is proved that Γ_{21} is a solvable but non-nilpotent 3-Lie algebra. The inner derivation algebra $ad(\Gamma_{21})$ is a 12-dimensional solvable but non-nilpotent Lie algebra, and the derivation algebra $Der(\Gamma_{21})$ with dimension 17 is unsolvable. And the concrete expression of all derivations are given.

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1 Introduction

In papers [1, 2], authors constructed 3-Lie algebras by well known algebras and linear functions, derivations and involutions. In paper [3], 3-Lie algebras are constructed by N-cubic matrix over the field F with $chF \neq 2$. Bai, Guo, and Lin in papers [4, 5] discussed 3-Lie algebras J_{11} and J_{21} which are realized by 2-cubic matrix over a field F with characteristic $chF \neq 2$. In this paper, we pay our main attention to 8-dimensional 3-Lie algebras which are constructed by 2-cubic matrix in the prime field $Z_2 = \{0, 1\}$. In the following we suppose that $Z_2=\{0,1\}$ is the prime field with characteristic two, for a vector space V and a subset S, the subspace generated by S is denoted by (S).

2 Structure of 3-Lie algebras Γ_{21}

First we introduce some notations. An N-order cubic matrix $A=(a_{ijk})$ over the field Z_2 is an ordered object which the elements with 3 indices, and the element in the position (i,j,k) is $(A)_{ijk}=a_{ijk}$, $1\le i,j,k\le N$ and $a_{ijk}=0$ or 1. Denote the set of all cubic matrix over Z_2 by Ω. Then Ω is an N^3-dimensional vector space with $A+B=(a_{ijk}+b_{ijk})\in\Omega$, $\lambda A=(\lambda a_{ijk})\in\Omega$, for $\forall A=(a_{ijk}), B=(b_{ijk})\in\Omega$, $\lambda\in Z_2$, that is, $(A+B)_{ijk}=a_{ijk}+b_{ijk}$, $(\lambda A)_{ijk}=\lambda a_{ijk}$. Denote $E_{ijk}=(e_{h_1h_2h_3})$, where $e_{h_1h_2h_3}=\delta_{h_1i}\delta_{h_2j}\delta_{h_3k}$, that is when $h_1=i$, $h_2=j$, $h_3=k$, $e_{h_1h_2h_3}=1$, and elsewhere are zero. Then, $\{E_{ijk}|\ 1\le i,j,k\le n\}$ is a basis of Ω.

For all $A=(a_{ijk}), B=(b_{ijk})\in\Omega$, define the multiplication $*_{21}$ in Ω by

$$(A*_{21}B)_{ijk}=\sum_{p,q=1}^{n}a_{qjp}b_{ipk}, 1\le i,j,k\le n. \tag{1}$$

Denote $\langle A\rangle_1=\sum\limits_{p,q=1}^{N}(A)_{pqq}=\sum\limits_{p,q=1}^{N}a_{pqq}$, Then $\langle\ \rangle_1$ is a linear function from Ω to Z_2 and satisfies

$$\langle A*_{21}B\rangle_1=\langle B*_{21}A\rangle_1. \tag{2}$$

Define the multiplication $[,,]_{21}:\Omega\wedge\Omega\wedge\Omega\to\Omega$ as follows:

$$[A,B,C]_{21}=\langle A\rangle_1(B*_{21}C-C*_{21}B)$$

$$+\langle B\rangle_1(C*_{21}A-A*_{21}C)+\langle C\rangle_1(A*_{21}B-B*_{21}A). \tag{3}$$

Theorem 2.1[3] *The linear space Ω is a 3-Lie algebra in the multiplication $[,,]_{21}$, which is denoted by Γ_{21}, the multiplication $[,,]_{21}$ is simply denoted by $[,,]$.*

In the following we denote Ω_2 the vector space over $Z_2=\{0,1\}$ which consists of 2-cubic matrix. Then the dimension of Ω_2 is eight and with a basis $\{E_{111},E_{112}, E_{121},E_{122}, E_{211},E_{212}, E_{221},E_{222}\}$. And for all $A\in\Omega_2$, $A=\sum\limits_{i,j,k=1}^{2}\lambda_{ijk}E_{ijk}$, $\lambda_{ijk}=1,0\in Z_2$.

Theorem 2.2. *The multiplication of the 3-Lie algebra Γ_{21} in the basis*

$\{E_{111}, E_{112}, E_{121}, E_{122}, E_{211}, E_{212}, E_{221}, E_{222}\}$ *is as follows*

$$\begin{cases} [E_{111}, E_{112}, E_{121}] = E_{111} + E_{122}, & [E_{111}, E_{122}, E_{211}] = E_{111} + E_{211}, \\ [E_{112}, E_{121}, E_{122}] = E_{111} + E_{122}, & [E_{211}, E_{212}, E_{221}] = E_{211} + E_{222}, \\ [E_{212}, E_{221}, E_{222}] = E_{211} + E_{222}, & [E_{211}, E_{222}, E_{111}] = E_{211} + E_{111}, \\ [E_{111}, E_{121}, E_{211}] = E_{221} + E_{121}, & [E_{112}, E_{121}, E_{211}] = E_{111} + E_{122}, \\ [E_{111}, E_{121}, E_{212}] = E_{222} + E_{111}, & [E_{111}, E_{122}, E_{212}] = E_{112} + E_{212}, \\ [E_{121}, E_{122}, E_{212}] = E_{111} + E_{222}, & [E_{111}, E_{112}, E_{221}] = E_{211} + E_{122}, \\ [E_{111}, E_{122}, E_{221}] = E_{221} + E_{121}, & [E_{121}, E_{122}, E_{211}] = E_{121} + E_{221}, \\ [E_{112}, E_{122}, E_{221}] = E_{122} + E_{211}, & [E_{111}, E_{112}, E_{222}] = E_{212} + E_{112}, \\ [E_{111}, E_{122}, E_{222}] = E_{222} - E_{122}, & [E_{112}, E_{121}, E_{222}] = E_{111} + E_{122}, \\ [E_{211}, E_{221}, E_{111}] = E_{121} + E_{221}, & [E_{212}, E_{221}, E_{111}] = E_{211} + E_{222}, \\ [E_{221}, E_{222}, E_{111}] = E_{121} + E_{221}, & [E_{112}, E_{122}, E_{222}] = E_{212} + E_{112}, \\ [E_{211}, E_{221}, E_{112}] = E_{122} + E_{211}, & [E_{211}, E_{222}, E_{112}] = E_{212} + E_{112}, \\ [E_{211}, E_{212}, E_{121}] = E_{111} + E_{222}, & [E_{211}, E_{222}, E_{121}] = E_{121} + E_{221}, \\ [E_{212}, E_{222}, E_{121}] = E_{222} + E_{111}, & [E_{221}, E_{222}, E_{112}] = E_{211} + E_{112}, \\ [E_{211}, E_{212}, E_{122}] = E_{112} + E_{212}, & [E_{211}, E_{222}, E_{122}] = E_{122} + E_{222}, \\ [E_{212}, E_{221}, E_{122}] = E_{211} + E_{222}, & [E_{212}, E_{222}, E_{122}] = E_{112} + E_{212}, \end{cases} \tag{4}$$

where the zero product of the basis vectors are omitted.

Proof The result follows from the direct computation according to the definition of $*_{21}$ and Eqs.(1), (2) and (3).

For studying the structure of the 3-Lie algebra Γ_{21}, we need to find a special basis to simplify its multiplication.

Theorem 2.3 *The* 3*-Lie algebra* Γ_{21} *is a non-nilpotent indecomposable* 3*-Lie algebra with a basis* $e_1 = E_{111}, e_2 = E_{112}, e_3 = E_{121}, e_4 = E_{111} + E_{122}$, $e_5 = E_{211} + E_{111}, e_6 = E_{212} + E_{112}, e_7 = E_{221} - E_{121}$, $e_8 = E_{122} + E_{222}$. *And the multiplication in it is as follows:*

$$\begin{cases} [e_1, e_2, e_3] = e_4, [e_1, e_4, e_5] = e_5, [e_1, e_3, e_6] = e_8, \\ [e_1, e_3, e_5] = e_7, [e_1, e_4, e_6] = e_6, [e_1, e_2, e_8] = e_6, \\ [e_1, e_2, e_7] = e_5, [e_1, e_4, e_7] = e_7, [e_1, e_4, e_8] = e_8. \end{cases} \tag{5}$$

Proof It is clear that $\{e_1, \cdots, e_8\}$ is a basis of Ω_2. By the definition of $*_{21}$, we obtain Eq.(5). Since Γ_{21} can not be written as the direct sum of two proper ideals, Γ_{21} is indecomposable.

From $\Gamma_{21}^1 = [\Gamma_{21}, \Gamma_{21}, \Gamma_{21}] = (e_4, e_5, e_6, e_7, e_8)$, $\Gamma_{21}^2 = [\Gamma_{21}^1, \Gamma_{21}, \Gamma_{21}] = (e_5, e_6, e_7, e_8)$, and $\Gamma_{21}^3 = [\Gamma_{21}^2, \Gamma_{21}, \Gamma_{21}] = (e_5, e_6, e_7, e_8)$, then for all positive integer $s > 1$, we have $\Gamma_{21}^s = \Gamma_{21}^2 \neq 0$. Therefore, Γ_{21} is non-nilpotent.

Theorem 2.4 *The subalgebra* $H = (e_1, e_2, e_3, e_4)$ *is a Cartan subalgebra of the* 3*-Lie algebra* Γ_{21}. *And the decomposition of* Γ_{21} *associate to* H *is* $\Gamma_{21} = H \dot{+} \Gamma_\alpha$, *and* $\Gamma_\alpha = (e_5, e_6, e_7, e_8) = \{e \in \Gamma_{21} \mid (ad(h_1, h_2) + \alpha(h_1, h_2))^2(e) = 0, \forall h_1, h_2 \in H\}$, *where* $\alpha \in (H \otimes H)^*$, $\alpha(e_1, e_4) = 1$, *and others are zero.*

Proof From Theorem 2.3, $H = (e_1, e_2, e_3, e_4)$ is a Cartan subalgebra of Γ_{21}. Denote $\alpha : H \otimes H \to Z_2$, $\alpha(e_1, e_4) = 1$, $\alpha(e_1, e_2) = \alpha(e_1, e_3) = \alpha(e_2, e_3) = \alpha(e_2, e_4) = \alpha(e_3, e_4) = 0$, we have $ad(e_1, e_4)(e_5) = e_5$, $ad(e_1, e_4)(e_6) = e_6$, $ad(e_1, e_4)(e_7) = e_7$, $ad(e_1, e_4)(e_8) = e_8$, $ad^2(e_1, e_2)e_i = ad^2(e_1, e_3)(e_i) = ad^2(e_2, e_3)(e_i) = 0$, $ad^2(e_2, e_4)(e_i) = ad^2(e_3, e_4)(e_i) = 0$, $i = 5, 6, 7, 8$. We obtain the result.

Now we study the inner derivation algebra $ad(\Gamma_{21})$. For $e_i, e_j \in \Omega_2$, denote $ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l$, where $a_{kl}^{ij} = a_{kl}^{ji} = 0$ or $1 \in Z_2$. Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl} are the matrix units.

Theorem 2.5 *the inner derivation algebra* $ad(\Gamma_{21})$ *is solvable but non-nilpotent Lie algebra with dimension* 12, *and* $X_1 = E_{34} + E_{75} + E_{86}$, $X_2 = E_{24} + E_{57} + E_{68}$, $X_3 = E_{55} + E_{66} + E_{77} + E_{88}$, $X_4 = E_{37} + E_{45}$, $X_5 = E_{38} + E_{46}$, $X_6 = E_{25} + E_{47}$, $X_7 = E_{26} + E_{48}$, $X_8 = E_{14}$, $X_9 = E_{15}$, $X_{10} = E_{16}$, $X_{11} = E_{17}$, $X_{12} = E_{18}$ *is a basis. And the multiplication in it is*

$$\begin{cases} [X_2, X_1] = X_3, \ [X_1, X_6] = X_4, \ [X_7, X_8] = X_{12}, \\ [X_1, X_7] = X_5, \ [X_1, X_{11}] = X_9, \ [X_3, X_{12}] = X_{12}, \\ [X_2, X_4] = X_6, \ [X_1, X_{12}] = X_{10}, \ [X_3, X_{10}] = X_{10}, \\ [X_2, X_5] = X_7, \ [X_2, X_{10}] = X_{12}, \ [X_3, X_{11}] = X_{11}, \\ [X_3, X_4] = X_4, \ [X_2, X_9] = X_{11}, \ [X_3, X_6] = X_6, \\ [X_3, X_5] = X_5, \ [X_3, X_7] = X_7, \ [X_3, X_9] = X_9, \\ [X_4, X_8] = X_9, \ [X_6, X_8] = X_{11}, \ [X_5, X_8] = X_{10}. \end{cases} \tag{6}$$

Proof By a direct computation according to Eq.(5) we have $ad(e_1, e_2) = E_{34} + E_{75} + E_{86}$, $ad(e_1, e_3) = E_{24} + E_{57} + E_{68}$, $ad(e_1, e_4) = E_{55} + E_{66} + E_{77} + E_{88}$, $ad(e_1, e_7) = E_{25} + E_{47}$, $ad(e_1, e_6) = E_{38} + E_{46}$, $ad(e_1, e_5) = E_{37} + E_{45}$, $ad(e_1, e_8) = E_{26} + E_{48}$, $ad(e_2, e_3) = E_{14}, ad(e_2, e_7) = E_{15}$, $ad(e_2, e_8) = E_{16}, ad(e_3, e_5) = E_{17}, ad(e_3, e_6) = E_{18}$. Then $\{X_1, \cdots, X_{12}\}$ is a basis of $ad(\Gamma_{21})$. From $[ad(e_i, e_j), ad(e_k, e_l)] = ad([e_i, e_j, e_k], e_l) + ad(e_k, [e_i, e_j e_l])$, we obtain Eq.(6). And

$ad^1(\Gamma_{21}) = [ad(\Gamma_{21}), ad(\Gamma_{21})] = (X_3, X_4, X_5, X_6, X_7, X_9, X_{10}, X_{11}, X_{12})$,

$ad^2(\Gamma_{21}) = (X_4, X_5, X_6, X_7, X_9, X_{10}, X_{11}, X_{12})$, $ad^s(\Gamma_{21}) = ad^2(\Gamma_{21}) \neq 0$, for $s > 2$, then $ad(\Gamma_{21})$ is non-nilpotent.

By $ad^{(1)}(\Gamma_{21}) = ad^1(\Gamma_{21})$, $ad^{(2)}(\Gamma_{21}) = (X_4, X_5, X_6, X_7, X_9, X_{10}, X_{11}, X_{12})$, $ad^{(3)}(\Gamma_{21}) = (X_9, X_{10}, X_{11}, X_{12})$, $ad^{(4)}(\Gamma_{21}) = 0$, $ad(\Gamma_{21})$ is solvable. The proof is completed.

Now,we discuss the derivation algebra $Der(\Gamma_{21})$.

Theorem 2.6 *The derivation algebra* $Der(\Gamma_{21})$ *is an unsolvable Lie algebra with dimension* 17, *and the multiplication in the basis* $\{X_1, \cdots, X_{17}\}$, *where* $X_1 = E_{34} + E_{75} + E_{86}$, $X_2 = E_{24} + E_{57} + E_{68}$, $X_3 = E_{55} + E_{66} + E_{77} + E_{88}$, $X_4 = E_{37} + E_{45}$, $X_5 = E_{38} + E_{46}$, $X_6 = E_{25} + E_{47}$, $X_7 = E_{26} + E_{48}$, $X_8 = E_{14}$, $X_9 = E_{15}$, $X_{10} = E_{16}$, $X_{11} = E_{17}$, $X_{12} = E_{18}$, $X_{13} = E_{11} + E_{44} + E_{77} + E_{88}$,

$X_{14} = E_{22}+E_{33}+E_{77}+E_{88}$, $X_{15} = E_{55}+E_{77}$, $X_{16} = E_{56}+E_{78}$, $X_{17} = E_{65}+E_{87}$ *is as follows*

$$\begin{cases} [X_2, X_1] = X_3, & [X_1, X_6] = X_4, \quad [X_1, X_{13}] = X_1, \ [X_2, X_{13}] = X_2, \\ [X_1, X_7] = X_5, & [X_1, X_{11}] = X_9, \quad [X_4, X_{13}] = X_4, \ [X_5, X_{13}] = X_5, \\ [X_2, X_4] = X_6, & [X_1, X_{12}] = X_{10}, \ [X_9, X_{13}] = X_9, \ [X_{10}, X_{13}] = X_{10}, \\ [X_2, X_5] = X_7, & [X_2, X_{10}] = X_{12}, \ [X_3, X_4] = X_4, \quad [X_2, X_9] = X_{11}, \\ [X_3, X_5] = X_5, & [X_3, X_6] = X_6, \quad [X_4, X_{15}] = X_4, \ [X_6, X_{15}] = X_6, \\ [X_3, X_7] = X_7, & [X_3, X_{11}] = X_{11}, \ [X_9, X_{15}] = X_9, \ [X_{11}, X_{15}] = X_{11}, \\ [X_3, X_9] = X_9, & [X_3, X_{10}] = X_{10}, \ [X_4, X_{16}] = X_5, \ [X_6, X_{16}] = X_7, \\ [X_4, X_8] = X_9, & [X_3, X_{12}] = X_{12}, \ [X_9, X_{16}] = X_{10}, \ [X_{11}, X_{16}] = X_{12}, \\ [X_6, X_8] = X_{11}, & [X_7, X_8] = X_{12}, \ [X_5, X_{17}] = X_4, \ [X_7, X_{17}] = X_6, \\ [X_5, X_8] = X_{10}, & [X_{10}, X_{17}] = X_9, \ [X_{12}, X_{17}] = X_{11}, \ [X_{15}, X_{16}] = X_{16}, \\ [X_{15}, X_{17}] = X_{17}, & [X_{16}, X_{17}] = X_3. \end{cases}$$

Proof By a direct computation according to the multiplication (5), X_1, $\cdots$, X_{17} is a basis of $Der(\Gamma_{21})$, and with the above multiplication. Since $Der^{(1)}(\Gamma_{21}) = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_9, X_{10}, X_{11}, X_{12}, X_{16}, X_{17})$, $Der^{(2)}(\Gamma_{21}) = (X_3, X_4, X_5, X_6, X_7, X_9, X_{10}, X_{11}, X_{12}, X_{16}, X_{17})$ $= Der^{(s)}(\Gamma_{21}) \neq 0$ for all $s > 2$, we obtain the result.

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