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Some estimates on the Hermite-Hadamard inequality through convex and quasi-convex stochastic processes

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Abstract

We give some estimates of the left and right-hand side of the Hermite-Hadamard inequality for convex stochastic processes with convex or quasi-convex first and second derivatives at certain powers are presented.

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1 Introduction

Inequalities play a significant role in almost all fields of mathematics and several applications of them can be found in various areas of sciences such as, physical and engineering sciences, [1].

Many inequalities have been established for convex functions and one of most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications (see [5], [14]). For some of these inequalities, several authors have estimated the error in the approximation of its sides. The technique used consider derivatives of different orders and properties as convexity and quasi-convexity. In 1998, Dragomir and Agarwal [4], obtained inequalities for differentiable convex functions which are connected with the right-hand side of Hermite-Hadamard's (trapezoid) inequality. Then, in 2000, Pearce and Pečarić [13], presented an improvement to some error estimates of Dragomir and Agarwal, based on convexity, for the trapezoidal formula.

Corresponding estimates were established for the left-hand side of Hermite-Hadamard (midpoint) inequality. A parallel development was made based on concavity. This is a generalization of Dragomir and Agarwal's work. In the same way, as Dragomir and Agarwal approaches [4], inequalities for differentiable convex mapping which are connected with the left-hand side of Hermite-Hadamard (midpoint) inequality was proved by Kirmaci in 2004, (see [7]). In his work, some inequalities are presented in Kirmaci's work for differentiable convex functions, using Hermite-Hadamard's integral inequality holding for convex functions. Also, some applications to special means of real numbers were given and some error estimates for the midpoint formula were obtained. In the same year, D. A. Ion [6], established some estimates of the right-hand side of a Hermite-Hadamard type inequality in which some quasi-convex functions are involved. He also pointed out some applications to give estimates for the approximation error of the integral of a function $f(x)$ on $[a, b]$ in the trapezoidal formula and extend the initial results to functions of several variables. Later, in 2010, Sarakaya et al. [15] established several inequalities for twice differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Specifically, considering functions whose second derivatives in absolute values are convex and quasi-convex, he obtained inequalities related to the left-side of Hermite-Hadamard inequality. In the same year, Alomari *et al.* [2] obtained new refined inequalities of the right-hand side of Hermite-Hadamard type for functions whose second derivatives in absolute values are quasi-convex. For its part, Alomari in 2008, [1], in his PhD thesis provided a study of some famous and fundamental inequalities, particularly the Hermite-Hadamard via three types odd convex functions.

In 1980, K. Nikodem [11] established a research line about convex stochastic processes. Later, D. Kotrys in 2011 presented in [8] an inequality of Hermite-Hadamard type for Jensen-convex stochastic processes and N. Merentes *et al.*, proved in [3] a generalization for h-convex stochastic processes. In particular, with the function h equals to the identity, a Hermite-Hadamard inequality type for convex stochastic processes were obtained in [3].

The aim of this paper is establish a counterpart of the results for convex functions of Dragomir and Agarwal [4], Pearce and Peărić [13], Kirmaci [7], Ion [6], Sarakaya [15] and Alomari [1], [2], to convex functions for convex stochastic processes with convex and quasi-convex derivatives in absolute value, in order to estimate the error in the sides of the Hermite-Hadamard inequality type for convex stochastic processes proved in [3].

2 Preliminary Notes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. A function $X : \Omega \to \mathbf{R}$ is a *random variable* if it is $\mathcal{A}-$measurable. A function $X : I \times \Omega \to \mathbf{R}$, where $I \subseteq \mathbf{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

A stochastic process $X : I \times \Omega \to \mathbf{R}$ is

1. *Jensen-convex* if, for every $a, b \in I$, the following inequality is: satisfied:

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{X(a, \cdot) + X(b, \cdot)}{2}, \quad (a.e). \tag{1}$$

2. *convex* if, for every $a, b \in I$, $t \in (0, 1)$, the following inequality is tales place:

$$X(ta + (1-t)b, \cdot) \leq tX(a, \cdot) + (1-t)X(b, \cdot), \quad (a.e). \tag{2}$$

3. *quasi-convex* if, for every $a, b \in I$, $t \in (0, 1)$, the following inequality is satisfied:

$$X(ta + (1-t)b, \cdot) \leq \max\{X(a, \cdot), X(b, \cdot)\}, \quad (a.e). \tag{3}$$

Also, we say that a stochastic process $X : I \times \Omega \to \mathbf{R}$ is:

1. *continuous in probability* in the interval I, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability.

2. *mean-square continuous* in I, if for all $t_0 \in I$

$$\lim_{t \to t_0} \mathbf{E}[(X(t, \cdot) - X(t_0, \cdot))^2] = 0,$$

where $\mathbf{E}[X(t, \cdot)]$ denotes the expectation value of the random variable $X(t, \cdot)$.

3. *differentiable* at a point $t \in I$ if there is a random variable $X'(t, \cdot) : I \times \Omega \to \mathbf{R}$:

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

Note that mean-square continuity implies continuity in probability, but the converse is not true.

Fixed $X : I \times \Omega \to \mathbf{R}$ a stochastic process with $\mathbf{E}[X(t)^2] < \infty$ for all $t \in I$, $[a, b] \subseteq I$, $a = t_0 < t_1 < ... < t_n = b$ a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$

for all $k = 1, ..., n$, a random variable $Y : \Omega \to \mathbf{R}$ is called the *mean-square integral* of the process X on $[a, b]$, if for a normal sequence of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$ we have

$$\lim_{n\to\infty} \mathbf{E}\left[\left(\sum_{k=1}^{n} X(\Theta_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0.$$

In such case, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds \quad (a.e).$$

For the existence of the mean-square integral is enought to assume the mean-square continuity of the stochastic process X. Basic properties of the mean-square integral can be read in [18].

In [8], Hermite-Hadamard inequality for Jensen-convex stochastic process was proved. This result was extended for h-convex stochastic processes in [3] as follows:

Theorem 2.1 *Let $h : (0, 1) \to \mathbf{R}$ be a non-negative function, $h \not\equiv 0$ and $X : I \times \Omega \to \mathbf{R}$ a non negative, $h-$convex, mean square integrable stochastic process. For every $a, b \in I$, $(a < b)$, the following inequality is satisfied almost everywhere:*

$$\frac{1}{2h\left(\frac{1}{2}\right)} X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{(b-a)} \int_a^b X(t, \cdot)dt \leq (X(a, \cdot) + X(b, \cdot)) \int_0^1 h(z)dz. \tag{4}$$

As a corollary, the following inequality of Hermite-Hadamard type for convex stochastic processes holds almost everywhere for $a, b \in I$, with $a < b$:

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{(b-a)} \int_a^b X(u, \cdot)du \leq \frac{X(a, \cdot) + X(b, \cdot)}{2}, \quad (a.e.). \tag{5}$$

In [4], Dragomir and Agarwal proved that if $f : I \subset \mathbf{R} \to \mathbf{R}$ is a differentiable mapping on I°, where $a, b \in I$, $a < b$ with $|f'|$ convex on $[a, b]$, then the forecoming inequality is true:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(x)dx\right| \leq \frac{(b-a)}{8} \{|f'(a)| + |f'(b)|\}. \tag{6}$$

In [13], Pearce and Pečarić generalized the previous results. They proved: If $f : I \subset \mathbf{R} \to \mathbf{R}$ a differentiable mapping on I°, where $a, b \in I$, $a < b$ with $|f'|^q$ is convex on $[a, b]$, for some $q > 1$, the following inequality are valued:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left\{\frac{|f'(a)|^q+|f'(b)|^q}{2}\right\}^{1/q}. \quad (7)$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left\{\frac{|f'(a)|^q+|f'(b)|^q}{2}\right\}^{1/q}. \quad (8)$$

D. A. Ion [6], obtained two inequalities of the right-hand side of Hermite-Hadamard's type for functions whose derivative in absolute values are quasi-convex functions, i.e., if $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\max\{|f'(a)|,|f'(b)|\}. \quad (9)$$

Moreover, if $|f'|^q$ is quasi-convex on $[a,b]$ with $p>1$, then the following inequality takes place:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2(p+1)^{1/p}}\max\{|f'(a)|^q,|f'(b)|^q\}^{1/q}, \quad (10)$$

where $q=p/(p-1)$.

Note that the previous results are connected with the right-hand side if Hermite-Hadamard inequality. In order to estimate the error in the left-hand side U. S. Kirmaci showed that, if $|f'|$ is convex on $[a,b]$, then:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\{|f'(a)|+|f'(b)|\}. \quad (11)$$

Nevertheless, using functions whose second derivatives absolute values are convex and quasi-convex, Sarikaya *el al.* in 2010 [15], obtained inequalities related to the left side of Hermite-Hadamard inequality. For these results they consider $f: I\subset \mathbf{R}\to\mathbf{R}$ twice differentiable function on I° with $f''\in L_1[a,b]$, and $|f''|$ a convex function on [a,b], then:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left\{\frac{|f''(a)|+|f''(b)|}{2}\right\}. \quad (12)$$

Further, if $|f''|^q$ is convex on $[a,b]$, $q>1$, then the following inequality holds:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8(2p+1)^{1/p}}\left\{\frac{|f''(a)|^q+|f''(b)|^q}{2}\right\}^{1/q}, \tag{13}$$

where $q = p/(p-1)$.

As well, Sarikaya *et al.* [15], proved an inequality when $|f''|$ is quasi-convex on $[a,b]$. This result is as follows:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\max\{|f''(a)|+|f''(b)|\}. \tag{14}$$

Also, if $|f''|^q$ is quasi-convex on $[a,b]$, $q>1$:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8(2p+1)^{1/p}}\max\{|f''(a)|^q,|f''(b)|^q\}^{1/q}, \tag{15}$$

where $q = p/(p-1)$.

An improvement of the above result is showed in [15], its is the following:

$$\left|\frac{1}{(b-a)}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\max\{|f''(a)|^q,|f''(b)|^q\}^{1/q}, \tag{16}$$

where $q = p/(p-1)$.

In 2010, Alomari *et al.* [2], established new refined inequalities of the right-hand side of Hermite-Hadamard result for the class functions whose second derivatives at certain powers are quasi-convex functions. In this paper, Alomari *et al.* showed that if $f: I \subset \mathbf{R} \to \mathbf{R}$ twice differentiable mapping on I°, $a, b \in I$ with $a<b$, f'' is integrable on $[a,b]$ and $|f''|$ is a quasi-convex function on $[a,b]$ then:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\{|f''(a)|,|f''(b)|\}. \tag{17}$$

However, if $|f''|^q$ is quasi-convex on $[a,b]$ with $p>1$, then the following inequality holds:

$$\begin{aligned}\left|\frac{1}{(b-a)}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \qquad\qquad\qquad\qquad\qquad\qquad (18)\\ \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\max\{|f''(a)|^q,|f''(b)|^q\}^{1/q},\end{aligned}$$

where $q = p/(p-1)$.

Now, we present an equivalent of these results for stochastic processes whose first and second derivatives at certain powers are convex or quasi-convex.

3 Main Results

In order to prove an inequality for convex differentiable stochastic processes which are connected with the right-hand side of Hermite-Hadamard's inequality, we need to use the following lemma, a counterpart of an equality stated by Dragomir and Agarwal in [4]. Note that the next result is a Montgomery equality type [1].

Lemma 3.1 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $X'(t, \cdot)$ is mean-square integrable on $[a, b]$, then the following equality holds almost everywhere:*

$$\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du = \frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot))dt. \quad (19)$$

Proof. Let us calculate the integral on the right-hand side by integrating by parts

$$\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot)dt = \frac{1}{(b-a)}[X(a,\cdot)+X(b,\cdot)] - \frac{2}{(b-a)}\int_0^1 X(ta+(1-t)b,\cdot)dt, \quad (a.e).$$

Multiplying both sides of the integral by $(b-a)/2$, we have

$$\frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot)dt = \frac{X(a,\cdot)+X(b,\cdot)}{2} - \int_0^1 X(ta+(t-t)b,\cdot)dt, \quad (a.e).$$

Making $u = ta + (1-t)b$, in the integral of the right-hand side, we get the desired result.

The following theorem gives an error estimation for the right-hand side of the Hermite-Hadamard inequality for convex stochastic processes with convex first derivative in absolute value. A similar result was proved by Dragomir and Agarwal in [4] for convex functions.

Theorem 3.2 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t,\cdot)|$ is a convex stochastic process, then the following inequality takes place almost everywhere:*

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-a)}{8}\left[|X'(a,\cdot)| + |X'(b,\cdot)|\right], \quad (a.e).$$

Proof.

First, we point out that

$$\begin{aligned}\int_0^1 |1-2t|(1-t)dt &= \int_0^{1/2}(1-2t)(1-t)dt + \int_{1/2}^1 (2t-1)(1-t)dt = \frac{1}{4},\\ \int_0^1 |1-2t|tdt &= \int_0^{1/2}(1-2t)tdt + \int_{1/2}^1 (2t-1)tdt = \frac{1}{4}.\end{aligned} \tag{20}$$

Next, using the above information and Lemma 3.1 we obtain

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right|\\ &= \left|\frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot)dt\right|\\ &\leq \frac{(b-a)}{2}\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|dt\\ &\leq \frac{(b-a)}{2}\int_0^1 |1-2t|\left[t|X'(a,\cdot)| + (1-t)|X'(b,\cdot)|\right]dt\\ &= \frac{(b-a)}{2}\left[|X'(a,\cdot)|\int_0^1 |1-2t|tdt\right.\\ &\qquad \left.+|X'(b,\cdot)|\int_0^1 |1-2t|(1-t)dt\right]\\ &= \frac{(b-a)}{2}\left[\frac{1}{4}|X'(a,\cdot)| + \frac{1}{4}|X'(b,\cdot)|\right]\\ &= \frac{(b-a)}{8}\left[|X'(a,\cdot)| + |X'(b,\cdot)|\right], \quad (a.e).\end{aligned}$$

As the approximations established by Kirmaci in [7], inequalities for differentiable convex stochastic processes which are connected with the left-hand side of Hermite-Hadamard inequality are presented bellow, using the following Montgomery inequality type:

Lemma 3.3 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic processes mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $X'(t, \cdot)$ is mean-square integrable on $[a, b]$, then the following equality holds almost everywhere:*

$$\frac{1}{(b-a)}\int_a^b X(u,\cdot)du - X\left(\frac{a+b}{2},\cdot\right) = (b-a)\int_0^1 K(t)X'(ta+(1-t)b,\cdot))dt, \tag{21}$$

where

$$K(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right], \\ t-1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{22}$$

Proof.

Integrating by parts,

$$\begin{aligned} \int_0^1 K(t)X'(ta+(1-t)b,\cdot)dt &= \int_0^{1/2} tX'(ta+(1-t)b,\cdot)dt \\ &\quad + \int_{1/2}^1 (t-1)X'(ta+(1-t)b,\cdot)dt \\ &= \frac{1}{(a-b)}X\left(\frac{a+b}{2},\cdot\right) \\ &\quad - \frac{1}{(a-b)}\int_0^1 X(ta+(1-t)b,\cdot)dt, \quad (a.e). \end{aligned}$$

Multiplying both sides of the integral by $(b-a)$, we have

$$\begin{aligned} &(b-a)\int_0^1 K(t)X'(ta+(1-t)b,\cdot)dt \\ &\qquad = \int_0^1 X(ta+(1-t)b,\cdot)dt - X\left(\frac{a+b}{2},\cdot\right), \quad (a.e). \end{aligned}$$

By making the change of variable $u = ta + (1-t)b$ in the integral of the right-hand side, the desired result shows up.

Theorem 3.4 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t, \cdot)|$ is a convex stochastic process, then the following inequality holds almost everywhere:*

$$\left|X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \le \frac{(b-a)}{8}\left[|X'(a,\cdot)| + |X'(b,\cdot)|\right]. \tag{23}$$

Proof.

First, we notice that

$$\int_0^{1/2} t^2 dt = \frac{1}{24}, \qquad \int_0^{1/2} (1-t)^2 dt = \frac{1}{24},$$
$$\int_{1/2}^{1} t(1-t) dt = \frac{1}{12}, \qquad \int_0^{1/2} t(1-t) dt = \frac{1}{12}.$$

From Lemma 3.3 and using the well known Hölder integral inequality:

$$\begin{aligned}
&\left| X\left(\frac{a+b}{2}, \cdot\right) - \frac{1}{(b-a)} \int_a^b X(u,\cdot) du \right| \\
&\quad = \left| (b-a) \int_0^1 K(t) X'(ta + (1-t)b, \cdot)) dt \right| \\
&\quad \leq (b-a) \int_0^1 |K(t)||X'(ta + (1-t)b, \cdot)| dt \\
&\quad \leq (b-a) \left[\int_0^{1/2} t|X'(ta + (1-t)b, \cdot)| dt \right. \\
&\qquad \left. + \int_0^{1/2} (1-t)|X'(ta + (1-t)b, \cdot)| dt \right] \\
&\quad = (b-a) \left[\int_{1/2}^1 t[t|X'(a,\cdot)| + (1-t)|X'(b,\cdot)|] dt \right. \\
&\qquad \left. + \int_{1/2}^1 (1-t)|[t|X'(a,\cdot)| + (1-t)|X'(b,\cdot)|] dt \right] \\
&\quad = (b-a) \left[\int_0^{1/2} t^2 |X'(a,\cdot)| dt + \int_0^{1/2} (1-t)t|X'(b,\cdot)| dt \right. \\
&\qquad \left. + \int_{1/2}^1 (1-t)t|X'(a,\cdot)| dt + \int_{1/2}^1 (1-t)^2 |X'(b,\cdot)| dt \right] \\
&\quad = (b-a) \left[\frac{1}{12} |X'(a,\cdot)| + \frac{1}{24} (1-t)t|X'(b,\cdot)| \right. \\
&\qquad \left. + \frac{1}{24} (1-t)t|X'(a,\cdot)| + \frac{1}{12} (1-t)^2 |X'(b,\cdot)| \right] \\
&\quad = \frac{(b-a)}{8} \left[|X'(a,\cdot)| + |X'(b,\cdot)| \right], \quad (a.e).
\end{aligned}$$

The following result shows that Theorem 3.2 can be presented for stochastic processes with quasi-convex derivative in absolute valued as the one proved by Ion [6] for functions.

Theorem 3.5 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t, \cdot)|$ is a quasi-convex stochastic process, then the forecoming inequality holds almost everywhere:*

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-a)}{4}\max\{|X'(a,\cdot)| + |X'(b,\cdot)|\}. \tag{24}$$

Proof.

Using the above information and Lemma 3.1 we obtain

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&= \left|\frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot)dt\right| \\
&\leq \frac{(b-a)}{2}\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|dt \\
&\leq \frac{(b-a)}{2}\int_0^1 |1-2t|\max\{|X'(a,\cdot)|, |X'(b,\cdot)|\}dt \\
&= \frac{(b-a)}{2}\max\{|X'(a,\cdot)|, |X'(b,\cdot)|]\left(\int_0^1 |1-2t|dt\right) \\
&= \frac{(b-a)}{4}\max\{|X'(a,\cdot)|, |X'(b,\cdot)|\}, \quad (a.e).
\end{aligned}$$

Now we present corresponding version of the previous result for stochastic processes whose first derivatives at certain powers are convex.

Theorem 3.6 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t, \cdot)|^{p/(p-1)}$ is a convex stochastic process, for $p > 1$, then the following inequality holds almost everywhere:*

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|X'(a,\cdot)|^q + |X'(b,\cdot)|^q}{2}\right]^{1/q}, \tag{25}$$

where $q = p/(p-1)$.

Proof.

Simply calculations show

$$\int_0^1 |1-2t|^p dt = 2\int_0^{1/2} (1-2t)^p dt = \frac{1}{p+1}, \tag{26}$$

Then, from Lemma 3.1 and using the well known Hölder integral inequality, we have successively:

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\quad = \left|\frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot))dt\right| \\
&\quad \le \frac{(b-a)}{2}\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|dt \\
&\quad \le \frac{(b-a)}{2}\left(\int_0^1 |X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q} \\
&\qquad \cdot \left(\int_0^1 |1-2t|^p dt\right)^{1/p} \\
&\quad = \frac{(b-a)}{2}\left(\int_0^1 |X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q}\left(\frac{1}{p+1}\right)^{1/p} \\
&\quad \le \frac{(b-a)}{2(p+1)^{1/p}}\left(\int_0^1 \{t|X'(a,\cdot)|^q + (1-t)|X'(b,\cdot)|^q\}\, dt\right)^{1/q} \\
&\quad = \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|X'(a,\cdot)|^q + |X'(b,\cdot)|^q}{2}\right]^{1/q}
\end{aligned}$$

The previous theorem can be improved as Pearce and Pečarić did in [13] for functions in [13], as follows:

Theorem 3.7 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t,\cdot)|^{p/(p-1)}$ is a convex stochastic process, for $p > 1$, then the incoming inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad \le \frac{(b-a)}{4}\left[\frac{|X'(a,\cdot)|^q + |X'(b,\cdot)|^q}{2}\right]^{1/q},
\end{aligned} \tag{27}$$

where $q = p/(p-1)$.

Proof.

It is readily stablished the two forecoming equalities:

$$\begin{aligned}\int_0^1 |1-2t|tdt &= \int_0^{1/2}(1-2t)tdt + \int_{1/2}^1 (2t-1)tdt = \frac{1}{4},\\ \int_0^1 |1-2t|(1-t)dt &= \int_0^{1/2}(1-2t)(1-t)dt + \int_{1/2}^1 (2t-1)(1-t)dt = \frac{1}{4}.\end{aligned}$$

Hence by Lemma 3.1:

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \qquad (28)\\ &\qquad = \frac{(b-a)}{2}\int_0^1 |(1-2t)||X'(ta+(1-t)b,\cdot))|dt,\end{aligned}$$

and by the power mean inequality

$$\begin{aligned}&\int_0^1 |(1-2t)||X'(ta+(1-t)b,\cdot))|dt\\ &\qquad \leq \left(\int_0^1 |1-2t|dt\right)^{1-1/q}\left(\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q}.\end{aligned}$$

Because of the convexity of $|X'|$, we obtain

$$\begin{aligned}&\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|^q dt\\ &\qquad \leq \int_0^1 |1-2t|\left\{t|X'(a,\cdot)|^q + (1-t)|X'(b,\cdot)|^q\right\}dt\\ &\qquad = |X'(a,\cdot)|^q\int_0^1 |1-2t|tdt + |X'(b,\cdot)|^q\int_0^1 |1-2t|(1-t)dt\\ &\qquad = \frac{|X'(a,\cdot)+X'(b,\cdot)|}{4}.\end{aligned}$$

Since $\int_0^1 |1-2t|dt = \frac{1}{2}$, from (28) and the displayed inequality,

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right|\\ &\qquad = \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{|X'(a,\cdot))|^q + |X'(b,\cdot))|^q}{4}\right\}^{1/q}\\ &\qquad = \frac{(b-a)}{4}\left\{\frac{|X'(a,\cdot))|^q + |X'(b,\cdot))|^q}{2}\right\}^{1/q}.\end{aligned}$$

Remark 1. Note that the previous result gives an improvement of the bound obtained in Theorem 3.6. Since $p > 1$ then $2^p > p + 1$ and accordingly

$$\frac{1}{4} < \frac{1}{2(p+1)^{1/p}}.$$

In order to prove an analogous result for the left-hand side of Hermite-Hadamard inequality whose first derivatives at certain powers are convex, we show the following theorem:

Theorem 3.8 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t,\cdot)|^{p/(p+1)}$ is a convex stochastic process, for $p > 1$, then next inequality is true almost everywhere:*

$$\left|X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \le \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|X'(a,\cdot)|^q + |X'(b,\cdot)|^q}{2}\right]^{1/q}, \tag{29}$$

where $q = p/(p-1)$.

Proof.

First, point out that

$$\begin{aligned}\int_0^1 |K(t)|^p dt &= \int_0^{1/2} |t|^p dt + \int_{1/2}^1 |t-1|^p dt \\ &= \int_0^{1/2} t^p dt + \int_{1/2}^1 (1-t)^p dt = \frac{1}{(p+1)2^p}.\end{aligned} \tag{30}$$

From Lemma 3.3 and using the well known Hölder integral inequality, we obtain:

$$\begin{aligned}&\left|X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\ &\quad = \left|(b-a)\int_0^1 K(t)X'(ta+(1-t)b,\cdot))dt\right| \\ &\quad \le (b-a)\int_0^1 |K(t)||X'(ta+(1-t)b,\cdot)|dt \\ &\quad \le (b-a)\left(\int_0^1 |X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q} \\ &\qquad \cdot \left(\int_0^1 |K(t)|^p dt\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&= (b-a)\left(\int_0^1 |X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q}\left(\frac{1}{(p+1)2^p}\right)^{1/p}\\
&\le \frac{(b-a)}{2(p+1)^{1/p}}\left(\int_0^1 \{t|X'(a,\cdot)|^q+(1-t)|X'(b,\cdot)|^q\}\,dt\right)^{1/q}\\
&= \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|X'(a,\cdot)|^q+|X'(b,\cdot)|^q}{2}\right]^{1/q}, \quad (a.e).
\end{aligned}
$$

The next result improve Theorem 3.8.

Theorem 3.9 *Let $X : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a,b\in I$ with $a<b$. If $|X'(t,\cdot)|^{p/(p-1)}$ is a convex stochastic process, for $p>1$, then the following inequality holds almost everywhere:*

$$
\begin{aligned}
&\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \qquad (31)\\
&\qquad\qquad \le \frac{(b-a)}{4}\left[\frac{|X'(a,\cdot)|^q+|X'(b,\cdot)|^q}{2}\right]^{1/q},
\end{aligned}
$$

where $q=p/(p-1)$.

Proof.

First we must note that

$$
\begin{aligned}
\int_0^1 |K(t)|t\,dt &= \int_0^{1/2} t^2dt+\int_{1/2}^1 t(1-t)dt=\frac{1}{8},\\
\int_0^1 |K(t)|(1-t)dt &= \int_0^{1/2} t(1-t)dt+\int_{1/2}^1 (1-t)^2dt=\frac{1}{8}.
\end{aligned}
$$

From Lemma 3.1, we have

$$
\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \qquad (32)\\
&\qquad\qquad = (b-a)\int_0^1 |K(t)||X'(ta+(1-t)b,\cdot))|dt,
\end{aligned}
$$

and by the power mean inequality

$$
\begin{aligned}
&\int_0^1 |(1-2t)||X'(ta+(1-t)b,\cdot))|dt\\
&\qquad \le \left(\int_0^1 |1-2t|dt\right)^{1-1/q}\left(\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|^q dt\right)^{1/q}.
\end{aligned}
$$

Because $|X'|$ is convex, we obtain that

$$\begin{aligned}
&\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|^q dt \\
&\qquad \le \int_0^1 |1-2t| \left\{t|X'(a,\cdot)|^q + (1-t)|X'(b,\cdot)|^q\right\} dt \\
&\qquad = |X'(a,\cdot)|^q \int_0^1 |1-2t|tdt + |X'(b,\cdot)|^q \int_0^1 |1-2t|(1-t)dt \\
&\qquad = \frac{|X'(a,\cdot)+X'(b,\cdot)|}{4}.
\end{aligned}$$

Since $\int_0^1 |1-2t|dt = \frac{1}{2}$, we have from (32) and the displayed inequality that

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad = (b-a)\left(\frac{1}{4}\right)^{1-1/q} \left\{\frac{|X'(a,\cdot))|^q + |X'(b,\cdot))|^q}{8}\right\}^{1/q} \\
&\qquad = \frac{(b-a)}{4}\left\{\frac{|X'(a,\cdot))|^q + |X'(b,\cdot))|^q}{2}\right\}^{1/q}
\end{aligned}$$

Remark 2. The improvement of the constant gave in Theorem 3.8, is obtained because if $p > 1$ then $2^p > p+1$ and accordingly

$$\frac{1}{4} < \frac{1}{2(p+1)^{1/p}}.$$

In [6], Ion obtained a refined inequality of the right-hand side of Hermite-Hadamard's type for function, whose derivatives in absolute valued for certain powers are quasi-convex stochastic processes. Here, we present a counterpart of this result.

Theorem 3.10 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X'(t,\cdot)|^{p/(p-1)}$ is a quasi-convex stochastic process, then the following inequality is true almost everywhere:*

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \qquad (33)\\
&\qquad \le \frac{(b-a)}{2(p+1)^{1/p}} \max\{|X'(a,\cdot)|^q + |X'(b,\cdot)|^q\}^{1/q}.
\end{aligned}$$

Proof.

Using the above information and Lemma 3.1 we obtain

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\quad= \left|\frac{(b-a)}{2}\int_0^1 (1-2t)X'(ta+(1-t)b,\cdot)dt\right| \\
&\quad\le \frac{(b-a)}{2}\int_0^1 |1-2t||X'(ta+(1-t)b,\cdot)|dt \\
&\quad\le \frac{(b-a)}{2}\left(\int_0^1 |X'(ta+(1-t)b,\cdot)|^q]dt\right)^{1/q} \\
&\qquad \cdot\left(\int_0^1 |1-2t|^p dt\right)^{1/p} \\
&\quad\le \frac{(b-a)}{2}\left(\int_0^1 \max\{|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\}dt\right)^{1/q}\left(\frac{1}{p+1}\right)^{1/p} \\
&\quad= \frac{(b-a)}{2(p+1)^{1/p}}(\max\{|X'(a,\cdot)|^q,|X'(b,\cdot)|^q\})^{1/q}, \quad (a.e).
\end{aligned}$$

In order to prove refined inequalities of the right-hand side of Hermite-Hadamard's type for stochastic processes whose second derivatives at certain powers are convex and quasi-convex stochastic processes, we present the following lemma which shows a Montgomery identity type.

Lemma 3.11 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $X''(t,\cdot)$ is mean-square integrable on $[a,b]$, then the following equality takes place almost everywhere:*

$$\begin{aligned}
&\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du \qquad (34)\\
&\qquad\qquad= \frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot))dt.
\end{aligned}$$

Proof.

Let us calculate the integral on the right-hand side integrating by parts

$$\begin{aligned}
\int_0^1 (1-2t)X''(ta+(1-t)b,\cdot)dt &= \frac{1}{(b-a)}\int_0^1 X'(ta+(1-t)b,\cdot)dt \\
&= \frac{1}{(b-a)^2}[X(a,\cdot)+X(b,\cdot)]
\end{aligned}$$

$$-\frac{2}{(b-a)^2}\int_0^1 X(ta+(1-t)b,\cdot)dt, \quad (a.e).$$

By multiplying both sides of the integral by $(b-a)^2/2$, we have

$$\frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot)dt = \frac{X(a,\cdot)+X(b,\cdot)}{2} - \int_0^1 X(ta+(1-t)b,\cdot)dt, \quad (a.e).$$

Making the change of variable $u = ta+(1-t)b$ in the integral of the right-hand side, the desired result comes up.

The following result, shows an estimate of the right-hand side of the Hermite-Hadamard inequality by convex second derivate in absolute value.

Theorem 3.12 *Let $X : I\times\Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a,b\in I$ with $a<b$. If $|X''(t,\cdot)|$ is a convex stochastic process, then next inequality holds almost everywhere:*

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-a)}{6}\{|X''(a,\cdot)|+|X''(b,\cdot)|\}. \tag{35}$$

Proof.

Because for $t\in[0,1]$ implies $0\le t(1-t)^2 \le (1-t)^2$ and $0 \le (1-t)t^2 \le t^2$,

$$\begin{aligned}\int_0^1 t(1-t)^2dt &\leq \int_0^1 (1-t)^2dt = \frac{1}{3},\\ \int_0^1 (1-t)t^2dt &\leq \int_0^1 t^2dt = \frac{1}{3}.\end{aligned}$$

Next, using the above information and Lemma 3.11:

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right|\\ &\quad= \left|\frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot)dt\right|\\ &\quad\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)|X''(ta+(1-t)b,\cdot)|dt\end{aligned}$$

$$
\begin{aligned}
&leq\frac{(b-a)^2}{2}\int_0^1 t(1-t)\{t|X''(a,\cdot)|+(1-t)|X''(b,\cdot)|\}dt \\
&=\frac{(b-a)^2}{2}\int_0^1 t^2(1-t)|X''(a,\cdot)|dt+\int_0^1 t(1-t)^2|X''(b,\cdot)|dt \\
&=\frac{(b-a)^2}{2}|X''(a,\cdot)|\int_0^1 t^2(1-t)dt+|X''(b,\cdot)|\int_0^1 t(1-t)^2dt \\
&=\frac{(b-a)^2}{2}\left\{\frac{1}{3}|X''(a,\cdot)|+\frac{1}{3}|X''(b,\cdot)|\right\} \\
&=\frac{(b-a)^2}{6}\left\{|X''(a,\cdot)|+|X''(b,\cdot)|\right\}, \quad (a.e).
\end{aligned}
$$

In the next theorem, we present a refined inequality of the right-hand side of Hermite-Hadamard's type stochastic processes whose second derivative in absolute valued for certain powers are convex.

Theorem 3.13 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X''(t,\cdot)|^{p/p-1}$ is a convex stochastic process, then the following inequality holds almost everywhere:*

$$
\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad\le\frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left[\frac{|X''(a,\cdot)|^q+|X''(b,\cdot)|^q}{2}\right]^{1/q},
\end{aligned}
$$

where $q = p/(p-1)$.

Proof.

We use the equality

$$
\int_0^1 [t(1-t)]^p dt=\frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}. \tag{36}
$$

where $\Gamma(\cdot)$ represents the Gamma function.

By the above information and Lemma 3.11:

$$
\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad=\left|\frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot)dt\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|X''(ta+(1-t)b,\cdot)|dt\\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 |X''(ta+(1-t)b,\cdot)|^q dt\right)^{1/q} dt\\
&\qquad\qquad \cdot\left(\int_0^1 [t(1-t)]^p dt\right)^{1/p}\\
&\le \frac{(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\\
&\qquad\qquad \cdot\left(\int_0^1 t|X''(a,\cdot)|^q+(1-t)|X''(a,\cdot)|^q dt\right)^{1/q}\\
&= \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\\
&\qquad\qquad \cdot\left[\frac{|X''(a,\cdot)|^q+|X''(a,\cdot)|^q}{2}\right]^{1/q}, \quad (a.e).
\end{aligned}
$$

As a counterpart of a result exposed by Alomari *et al.* in [2], we show in the following theorem an inequality for stochastic processes with second derivatives in absolute values are quasi-convex.

Theorem 3.14 *Let $X : I\times\Omega\to\mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a,b\in I$ with $a<b$. If $|X''(t,\cdot)|$ is a quasi-convex stochastic process, then the following inequality holds almost everywhere:*

$$
\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad\qquad \le \frac{(b-a)^2}{12}\max\{|X''(a,\cdot)|,|X''(b,\cdot)|\}.
\end{aligned}
\tag{37}
$$

Proof.

It is clear that

$$
\int_0^1 t(1-t)dt=\frac{1}{6}. \tag{38}
$$

Hence by (38) and Lemma 3.11, we obtain

$$
\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right|
$$

$$\begin{aligned}
&= \left|\frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot)dt\right| \\
&\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)|X''(ta+(1-t)b,\cdot)|dt \\
&\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\max\{|X''(a,\cdot)|,|X''(b,\cdot)|\}dt \\
&= \frac{(b-a)^2}{2}\max\{|X''(a,\cdot)|,|X''(b,\cdot)|\}\int_0^1 t(1-t)dt, \\
&= \frac{(b-a)^2}{12}\max\{|X''(a,\cdot)|,|X''(b,\cdot)|\}, \quad (a.e).
\end{aligned}$$

In the next result, we present another counterpart Alomari's *et al.* work [2]. Here we give a refined inequality of the right-hand side of Hermite-Hadamard's type for stochastic processes whose second derivative in absolute valued for certain powers are quasi-convex.

Theorem 3.15 *Let $X : I \times \Omega \to \mathbf{R}$ be a stochastic process mean-square differentiable on I°, $a, b \in I$ with $a < b$. If $|X''(t,\cdot)|^{p/p-1}$ is a quasi-convex stochastic process, then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \qquad (39)\\
&\qquad \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\max\{|X''(a,\cdot)|^q,|X''(b,\cdot)|^q\}^{1/q}.
\end{aligned}$$

where $q = p/(p-1)$.

Proof.

In the same way that in the Theorem 3.13, we have to use the integral (36). Then, by using the above information and Lemma 3.11, we get

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{(b-a)}\int_a^b X(u,\cdot)du\right| \\
&\qquad = \left|\frac{(b-a)^2}{2}\int_0^1 t(1-t)X''(ta+(1-t)b,\cdot)dt\right| \\
&\qquad \leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)|X''(ta+(1-t)b,\cdot)|dt \\
&\qquad \leq \frac{(b-a)^2}{2}\left(\int_0^1 |X''(ta+(1-t)b,\cdot)|^q dt\right)^{1/q} dt
\end{aligned}$$

$$\cdot \left(\int_0^1 [t(1-t)]^p dt\right)^{1/p}$$

$$\leq \frac{(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}$$

$$\cdot \left(\int_0^1 \max\{|X''(a,\cdot)|^q, |X''(a,\cdot)|^q\} dt\right)^{1/q}$$

$$= \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}$$

$$\cdot \left[\max\{|X''(a,\cdot)|^q, |X''(a,\cdot)|^q\}\right]^{1/q}, \quad (a.e).$$

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