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Construction of the fundamental system of solutions for an operator differential equation with a rapidly increasing at infinity block triangular potential

Aleksandr M. Kholkin

Department of higher and applied Mathematics,
Pryazovskii State Technical University
7 Universitetsa Str., Mariupol 87500, Ukraine
a.kholkin@gmail.com

Abstract

In this paper we consider Sturm-Liouville equation with block-triangular operator potential that fast increasing at the infinity. For him the fundamental system of solutions built and installed their asymptotics at infinity.

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1 Introduction

In the study of the connection between spectral and oscillation properties of non-self-adjoint differential operators with block-triangular operator coefficients the question arises of the structure of the spectrum of such operators. In [?] that the discrete spectrum of a differential operator with potential decreasing at infinity, have bounded the first moment, consists of a finite number of negative eigenvalues and essential spectrum covers the positive half. For the operator with block - triangular matrix potential that increases at infinity these questions are discussed in [?]. It uses the fundamental system of solutions, one of which is decreasing at infinity and the second increasing.

In this paper we construct a fundamental system of solutions of equation with block - triangular operator potential, increasing at infinity more quickly than x^2.

2 Preliminary Notes

Let $H_k,\ k=1,2,...,r$ finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$ and norm $|\cdot|$, $\dim H_k \le \infty$. Denote by $\mathbf{H} = H_1 \oplus H_2 \oplus ... \oplus H_r$. Element $\bar{h} \in \mathbf{H}$ will be written in the form $\bar{h} = col\left(\bar{h}_1, \bar{h}_2, ..., \bar{h}_r\right)$, where $\bar{h}_k \in H_k,\ k=\overline{1,r},\ I_k,\ I$- identity operators in H_k and $\mathbf{H}$ accordingly.

Let us consider the equation with block-triangular operator potential

$$l\left[\overline{y}\right] = -\overline{y}'' + V\left(x\right)\overline{y} = \lambda\overline{y}, \quad 0 \le x < \infty, \tag{1}$$

where

$$V\left(x\right) = v\left(x\right)\cdot I + U\left(x\right), \quad U\left(x\right) = \begin{pmatrix} U_{11}\left(x\right) & U_{12}\left(x\right) & ... & U_{1r}\left(x\right) \\ 0 & U_{22}\left(x\right) & ... & U_{2r}\left(x\right) \\ ... & ... & ... & ... \\ 0 & 0 & ... & U_{rr}\left(x\right) \end{pmatrix}, \tag{2}$$

$v\left(x\right)$ is a real scalar function, that $0 < v\left(x\right) \to \infty$ monotonically, as $x \to \infty$, and it has monotone absolutely continuous derivative. Also, $U\left(x\right)$ is a relatively small perturbation, e. g. $\left|U\left(x\right)\right|\cdot v^{-1}\left(x\right) \to 0$ as $x \to \infty$ or $\left|U\right| v^{-1} \in L^{\infty}\left(R_+\right)$. The diagonal blocks $U_{kk}\left(x\right),\quad k=\overline{1,r}$ are assumed as bounded self-adjoint operators in H_k.

Let

$$v\left(x\right) \ge Cx^{2\alpha},\ C>0,\ \alpha>1. \tag{3}$$

Condition (3) is performed, for example, quickly increasing functions e^x, $\exp\left\{e^x\right\}$ etc.

Assume that the coefficients of equation (1) satisfy the conditions

$$\int\limits_0^{\infty} \left|U\left(t\right)\right|\cdot v^{-\frac{1}{2}}\left(t\right)dt < \infty \tag{4}$$

$$\int\limits_0^{\infty} v'^2\left(t\right)\cdot v^{-\frac{5}{2}}\left(t\right)dt < \infty\ ,\quad \int\limits_0^{\infty} v''\left(t\right)\cdot v^{-\frac{3}{2}}\left(t\right)dt < \infty\quad . \tag{5}$$

Rewrite the equation (1) in the form

$$-\overline{y}'' + \left(v\left(x\right)+q\left(x\right)\right)\overline{y} = \left(\left(\lambda + q\left(x\right)\right)I - U\left(x\right)\right)\overline{y}, \tag{6}$$

where $q\left(x\right)$determined by a formula (cf. with the monograph [**?**])

$$q\left(x\right) = \frac{5}{16}\left(\frac{v'\left(x\right)}{v\left(x\right)}\right)^2 - \frac{1}{4}\frac{v''\left(x\right)}{v\left(x\right)}. \tag{7}$$

Now let us denote

$$\gamma_0(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(-\int_0^x \sqrt{v(u)}du \right), \tag{8}$$

$$\gamma_\infty(x) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(\int_0^x \sqrt{v(u)}du \right). \tag{9}$$

It is easy to see that $\gamma_0(x) \to 0,\ \gamma_\infty(x) \to \infty$ as $x \to \infty$. These solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))\,z = 0, \tag{10}$$

in such a way that for all $x \in [0, \infty)$ one has

$$W(\gamma_0, \gamma_\infty) := \gamma_0(x) \cdot \gamma'_\infty(x) - \gamma'_0(x) \cdot \gamma_\infty(x) = 1.$$

3 Main Results

Theorem 3.1 *Under conditions (3), (4), (5) equation (1) has a unique decreasing at infinity operator solution $\Phi(x, \lambda) \in B(H)$, satisfying the conditions*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x)} = I, \mathit{and} \lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma'_0(x)} = I. \tag{11}$$

Also, there exists increasing at infinity operator solution $\Psi(x, \lambda) \in B(H)$, satisfying the conditions

$$\lim_{x\to\infty} \frac{\Psi(x,\lambda)}{\gamma_\infty(x)} = I, \mathit{and} \lim_{x\to\infty} \frac{\Psi'(x,\lambda)}{\gamma'_\infty(x)} = I. \tag{12}$$

Proof. 1) Equation (6) equivalently to integral equation

$$\Phi(x,\lambda) = \gamma_0(x,\lambda)\, I + \int_x^\infty K(x,t,\lambda) \cdot \Phi(t,\lambda)\, dt, \tag{13}$$

where

$$K(x,t,\lambda) = C(x,t) \cdot [(\lambda + q(t))\, I - U(t)] \tag{14}$$

$$C(x,t) = \gamma_\infty(x) \cdot \gamma_0(t) - \gamma_\infty(t) \cdot \gamma_0(x), \tag{15}$$

with C(x; t) being the Cauchy function that in each variable satisfies equation (6) and the initial conditions

$$C\left(x,t\right)|_{x=t} = 0, \qquad C'_x\left(x,t\right)|_{x=t} = 1, \qquad C'_t\left(x,t\right)|_{x=t} = -1.$$

Set

$$\chi\left(x,\lambda\right) = \frac{\Phi\left(x,\lambda\right)}{\gamma_0\left(x\right)}$$

to rewrite equation (13) in form

$$\chi\left(x,\lambda\right) = I + \int\limits_x^\infty R\left(x,t,\lambda\right)\chi\left(t,\lambda\right)dt, \tag{16}$$

where

$$R\left(x,t,\lambda\right) = K\left(x,t,\lambda\right)\cdot\frac{\gamma_0\left(t\right)}{\gamma_0\left(x\right)}.$$

Thus

$$\left|C\left(x,t\right)\cdot\frac{\gamma_0\left(t\right)}{\gamma_0\left(x\right)}\right| = \left|\gamma_0^2\left(t\right)\cdot\frac{\gamma_\infty\left(x\right)}{\gamma_0\left(x\right)} - \gamma_0\left(t\right)\cdot\gamma_\infty\left(t\right)\right| =$$

$$= \left|\frac{1}{2\sqrt{v\left(t\right)}}\cdot\exp\left(-2\int\limits_0^t\sqrt{v\left(u\right)}du\right)\cdot\exp\left(2\int\limits_0^x\sqrt{v\left(u\right)}du\right) - \frac{1}{2\sqrt{v\left(t\right)}}\right| =$$

$$= \frac{1}{2\sqrt{v\left(t\right)}}\cdot\left|\exp\left(-2\int\limits_x^t\sqrt{v\left(u\right)}du\right) - 1\right|$$

and since with $x \le t$ one has $\exp\left(-2\int\limits_x^t\sqrt{v\left(u\right)}du\right) \le 1$, we deduce that

$$\left|C\left(x,t\right)\cdot\frac{\gamma_0\left(t\right)}{\gamma_0\left(x\right)}\right| \le \frac{1}{\sqrt{v\left(t\right).}} \tag{17}$$

Hence

$$\left|R\left(x,t,\lambda\right)\right| = \left|C\left(x,t\right)\cdot\frac{\gamma_0\left(t\right)}{\gamma_0\left(x\right)}\cdot\left[\left(\lambda+q\left(t\right)\right)I - U\left(t\right)\right]\right| \le$$

$$\le \frac{1}{\sqrt{v\left(t\right)}}\left(\left|\lambda\right| + \left|q\left(t\right)\right| + \left|U\left(t\right)\right|\right).$$

By virtue of (3)-(5), (7),

$$\frac{1}{\sqrt{v\left(t\right)}}\left(\left|\lambda\right| + \left|q\left(t\right)\right| + \left|U\left(t\right)\right|\right) \in L\left(0,\infty\right), \tag{18}$$

and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \le$ *const*. By (16), one has that $\lim\limits_{x\to\infty} \chi(x,\lambda) = I$, where the first part of formula (11) follows from.

Differentiable (13) to get

$$\frac{\Phi'(x,\lambda)}{\gamma'_0(x)} = I + \int_x^\infty S(x,t,\lambda)\,\chi(t,\lambda)\,dt,$$

where

$$S(x,t,\lambda) = K'_x(x,t,\lambda)\,\frac{\gamma_0(t)}{\gamma'_0(x)} = C'_x(x,t)\cdot\frac{\gamma_0(t)}{\gamma'_0(x)}\cdot\left[(\lambda+q(t))\,I - U(t)\right].$$

We have similarly (17), that

$$\left|C'_x(x,t)\cdot\frac{\gamma_0(t)}{\gamma'_0(x)}\right| \le \frac{1}{\sqrt{v(t)}},$$

and therefore

$$|S(x,t,\lambda)| \le \frac{1}{\sqrt{v(t)}}\cdot\left[|\lambda| + |q(t)| + |U(t)|\right] \in L(0,\infty),$$

where the second part of formula (11) follows from.

2) Denote by $\hat{\Psi}(x,\lambda) \in B(H)$ block-triangular operator solution of equation (1) that increases at infinity, $\Psi_{kk}(x) \in B(H_k,H_k)$, $k=\overline{1,r}$ -its diagonal blocks. Now equation (6) is equivalent to the integral equation

$$\hat{\Psi}(x,\lambda) = \gamma_\infty(x)\cdot I - \int_0^x K(x,t,\lambda)\cdot\hat{\Psi}(t,\lambda)\,dt, \tag{19}$$

where, just as in (13), the kernel $K(x,t,\lambda)$ is given by (14). Now set

$$\chi(x,\lambda) = \frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x)}$$

to rewrite equation (19) in form

$$\chi(x,\lambda) = I - \int_0^x R(x,t,\lambda)\cdot\chi(t,\lambda)\,dt, \tag{20}$$

where

$$R(x,t,\lambda) = C(x,t,\lambda)\cdot\frac{\gamma_\infty(t)}{\gamma_\infty(x)}\cdot\left[(q(t)+\lambda)\cdot I - U(t)\right].$$

Similarly we can prove that the integral equation (20) has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \leq const$. Pass in (20) to a limit as $x \to \infty$ to get $\lim\limits_{x\to\infty} \chi(x,\lambda) = I + \tilde{C}(\lambda)$, where $\tilde{C}(\lambda)$ is block-triangular operator in $\mathbf{H}$, that is

$$\lim_{x\to\infty} \frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x)} = I + \tilde{C}(\lambda). \tag{21}$$

Now consider another block-triangular operator solution $\tilde{\Psi}(x,\lambda)$ that increases at infinity diagonal blocks which are defined by

$$\tilde{\Psi}_{kk}(x,\lambda) = \Phi_{kk}(x,\lambda) \int\limits_a^x \Phi_{kk}^{-1}(t,\lambda)\,(\Phi_{kk}^*(t,\lambda))^{-1} dt, \;\; k = \overline{1,r}, \;\; (a \geq 0),$$

$\Phi_{kk}(x,\lambda)$ are the diagonal blocks of operator solution $\Phi(x,\lambda)$ as in Section1. In view (15) and the definition of the functions $\gamma_0(x), \gamma_\infty(x)$ can be proved that

$$\lim_{x\to\infty} \frac{\tilde{\Psi}_{kk}(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_k, \quad k = \overline{1,r} \tag{22}$$

Since $\hat{\Psi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$ are the operator solutions of equation (1) that increase at infinity,

$$\hat{\Psi}(x,\lambda) = \tilde{\Psi}(x,\lambda) + \Phi(x,\lambda) \cdot C_0(\lambda), \tag{23}$$

where $C_0(\lambda)$ is some block-triangular operator. Thus

$$\lim_{x\to\infty} \frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x)} = \lim_{x\to\infty} \frac{\tilde{\Psi}(x,\lambda)}{\gamma_\infty(x)},$$

hence, by virtue (22),

$$\lim_{x\to\infty} \frac{\Psi_{kk}(x,\lambda)}{\gamma_\infty(x)} = I_k, \;\; k = \overline{1,r}$$

and in (21) has

$$\tilde{C}(\lambda) = \begin{pmatrix} 0 & C_{12}(\lambda) & \ldots & C_{1r}(\lambda) \\ 0 & 0 & \ldots & C_{2r}(\lambda) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

The solution $\Psi(x,\lambda)$ given by $\Psi(x,\lambda) = \hat{\Psi}(x,\lambda)\left(I + \tilde{C}(\lambda)\right)^{-1}$ is subject to first from condition (12).

Use (11) to differentiate (23), then find the asymptotes of $\widetilde{\Psi'}(x,\lambda)$ as $x \to \infty$ similarly to (18) to obtain the second part of formula (12). Theorem is proved.

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