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On inequalities of Fejér and Hermite-Hadamard types for strongly m-convex functions

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Abstract

In this paper we continue developing the concept of a strongly m-convex function recently introduced. We also establish some properties and show Fejér and Hermite-Hadamard types inequalities for these functions inspired basically on the concepts of m-convex and strongly convex functions and their corresponding inequalities.

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1 Introduction

We begin by recalling the concept of an m-convex function given in [2, 3, 7, 15] and some other references as well as the concept of a strongly convex function with modulus $c > 0$ introduced by Polyak in [13].

Definition 1.1 *A function $f : [0, b] \to \mathbf{R}$ is called m-convex, $0 \le m \le 1$, if for any $x, y \in [0, b]$ and $t \in [0, 1]$ we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1}$$

Remark 1.2 *It is important to point out that the above definition is equivalent to $f(mtx + (1-t)y) \le mtf(x) + (1-t)f(y)$, with x, y and t as before.*

Definition 1.3 *Let $I \subset \mathbf{R}$ be an interval and c be a positive number. A function $f : I \to \mathbf{R}$ is said to be strongly convex with modulus c if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \tag{2}$$

with $x, y \in I$ and $t \in [0, 1]$.

Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just "stronger versions" of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics ([1, 9, 10, 14]).

In [8], the two given definitions were combined to generate the concept of a strongly m-convex function, let $I \subseteq \mathbf{R}_+$, c be a positive number and $m \in [0, 1]$. As it is customary, sometimes either $m = 0$ or $m = 1$ is discarded.

Definition 1.4 *([8]) A function $f : I \to \mathbf{R}$ is called strongly m-convex with modulus $c > 0$ if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) - cmt(1-t)(x-y)^2, \tag{3}$$

with $x, y \in I$ and $t \in [0, 1]$.

Notice that for $m = 1$ the definition of strongly convex function is recasted, also any strongly m-convex function is, in particular, m-convex. Unless otherwise is stated c always will be a positive number.

In the forthcoming sections we shall state and prove some inequalities for these type of functions based upon the corresponding inequalities for m-convex and strongly convex functions.

2 Main Results

In this section we exhibit and prove inequalities similar to those for m-convex functions, most of them are inspired in [2, 3, 4, 5, 11]. Let us start with a result which, although easy, it will be used later.

Lemma 2.1 *If $0 \le a < b < +\infty$ and if $f : [0, +\infty) \to \mathbf{R}$ is an integrable function on $[a, b]$, then the following equalities hold*

$$\int_0^1 f(ta + (1-t)b)dt = \int_0^1 f(tb + (1-t)a)dt = \frac{1}{b-a}\int_a^b f(s)ds.$$

Proof. The proof is immediate if for the first equality we consider the change of variable $u = 1 - t$, and for the second equality we consider the change of variable $s = tb + (1-t)a$.

The following theorem improves the approximation of the average integral given in [2] for any strongly m-convex function.

Theorem 2.2 *Let $f : [0, +\infty) \to \mathbf{R}$ be a strongly m-convex function, $0 < m \le 1$, with modulus c; for $0 \le a < b < +\infty$ and $f \in L^1([a, b])$ the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(s)ds \le \min\Big\{\frac{1}{2}G(a,b) - \frac{c}{6m}(ma-b)^2, \frac{1}{2}G(b,a) - \frac{c}{6m}(mb-a)^2\Big\}$$

where $G(a, b) = f(a) + mf\left(\frac{b}{m}\right)$ and $G(b, a) = f(b) + mf\left(\frac{a}{m}\right)$.

Proof. By hypothesis

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) - cmt(1-t)(x-y)^2,$$

for arbitrary $x, y \in [0, +\infty)$ and $0 \le t \le 1$. If we choose $x = a$ and $y = \frac{b}{m}$ the above inequality becomes

$$f\left(ta + m(1-t)\frac{b}{m}\right) \le tf(a) + m(1-t)f\left(\frac{b}{m}\right) - cmt(1-t)\left(a - \frac{b}{m}\right)^2,$$

which turns, after reducing appropriately, into

$$f(ta + (1-t)b) \le tf(a) + m(1-t)f\left(\frac{b}{m}\right) - \frac{c}{m}t(1-t)(ma-b)^2. \qquad (4)$$

In the same token, for $x = b$ and $y = \frac{a}{m}$, we get

$$f(tb + (1-t)a) \le tf(b) + m(1-t)f\left(\frac{a}{m}\right) - \frac{c}{m}t(1-t)(mb-a)^2. \qquad (5)$$

By integrating (4) and (5) on $[0,1]$ and using Lemma 2.1 we get

$$\frac{1}{b-a}\int_a^b f(s)ds \leq \frac{1}{2}\left[f(a)+mf\left(\frac{b}{m}\right)\right]-\frac{c}{6m}(ma-b)^2,$$

at the same time

$$\frac{1}{b-a}\int_a^b f(s)ds \leq \frac{1}{2}\left[f(b)+mf\left(\frac{a}{m}\right)\right]-\frac{c}{6m}(mb-a)^2$$

and conclusion follows.

The forthcoming result [2, Theorem 3] runs in the same way for a strongly m-convex function with modulus c. We state it, but the proof is omitted for obvious reasons.

Theorem 2.3 *Let $f:[0,+\infty)\to \mathbf{R}$ be a strongly m-convex function, $0<m\leq 1$, with modulus c. If $0\leq a<b<\infty$ and f is differentiable on $(0,+\infty)$, then one has the inequality*

$$\frac{f(mb)}{m}-\frac{b-a}{2}f'(mb)\leq \frac{1}{b-a}\int_a^b f(s)ds \leq \frac{(mb-a)f(a)-(ma-b)f(b)}{2(b-a)}.$$

Theorem 2.4 *Let $f:[0,+\infty)\to \mathbf{R}$ be a strongly m-convex function, $0<m\leq 1$, with modulus c. Then for $0\leq a<b<+\infty$ and $f\in L^1([a,b])$*

$$f\left(\frac{a+b}{2}\right)\leq \frac{1}{2(b-a)}\left[\int_a^b\left[f(s)+mf\left(\frac{s}{m}\right)\right]ds\right]-\frac{c}{4m}(ma-b)(mb-a) \quad (6)$$

and

$$\frac{1}{2(b-a)}\left[\int_a^b\left[f(s)+mf\left(\frac{s}{m}\right)\right]ds\right]\leq \frac{1}{4}\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)\right. \quad (7)$$
$$\left.+m^2f\left(\frac{b}{m^2}\right)\right]-\frac{(m+1)c}{12m^2}(ma-b)^2.$$

Proof. For $t=\frac{1}{2}$ and by taking $\frac{y}{m}$ instead y in Definition 1.4,

$$f\left(\frac{x+y}{2}\right)\leq \frac{1}{2}\left[f(x)+mf\left(\frac{y}{m}\right)\right]-\frac{c}{4m}(mx-y)^2. \quad (8)$$

By taking $x=ta+(1-t)b$ and $y=(1-t)a+tb$ for any $t\in[0,1]$, then

$$\begin{aligned} x+y &= a+b,\\ \frac{y}{m} &= (1-t)\frac{a}{m}+t\frac{b}{m},\\ mx-y &= (m+1)(a-b)t+mb-a, \end{aligned}$$

and
$$\int_0^1 (mx-y)^2 dt = \frac{(m+1)^2(b-a)^2}{3} + (ma-b)(mb-a).$$
Integration of (8) on $[0,1]$ with respect to t is performed for the above values and the following inequality comes out

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)}\left[\int_a^b \left[f(s)+mf\left(\frac{s}{m}\right)\right] ds\right] \\ &\quad -\frac{c}{4m}\left[\frac{(m+1)^2(b-a)^2}{3} + (ma-b)(mb-a)\right] \\ &\leq \frac{1}{2(b-a)}\left[\int_a^b \left[f(s)+mf\left(\frac{s}{m}\right)\right] ds\right] - \frac{c}{4m}(ma-b)(mb-a) \end{aligned}$$

so (6) is deduced. For the inequality (7) we notice that

$$f(ta+(1-t)b) \leq tf(a) + m(1-t)f\left(\frac{b}{m}\right) - \frac{ct(1-t)}{m}(ma-b)^2,$$

$$mf\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \leq m(1-t)f\left(\frac{a}{m}\right) + m^2tf\left(\frac{b}{m^2}\right) - \frac{ct(1-t)}{m^2}(ma-b)^2;$$
next by adding up these two inequalities

$$\begin{aligned} f(ta+(1-t)b)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) &\leq tf(a)+m(1-t)f\left(\frac{b}{m}\right) \\ &\quad +m(1-t)f\left(\frac{a}{m}\right)+m^2tf\left(\frac{b}{m^2}\right) \\ &\quad -\frac{(m+1)ct(1-t)}{m^2}(ma-b)^2. \end{aligned}$$

By integrating the last inequality on $[0,1]$ and using Lemma 2.1,

$$\begin{aligned} \frac{1}{2(b-a)}\int_a^b \left[f(s)+mf\left(\frac{s}{m}\right)\right] ds &\leq \frac{1}{4}\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)\right. \\ &\quad \left. +m^2f\left(\frac{b}{m^2}\right)\right] - \frac{(m+1)c}{12m^2}(ma-b)^2. \end{aligned}$$

This completes the proof.

Theorem 2.5 *If $f:[0,+\infty)\to\mathbf{R}$ is a strongly m-convex function, $0<m\leq 1$, with modulus c; $0\leq a<b<+\infty$ and $f\in L^1([ma,b])$, then*

$$\frac{1}{b-ma}\int_{ma}^b f(s)ds + \frac{1}{mb-a}\int_a^{mb} f(s)ds \leq \frac{(m+1)}{2}\big[f(a)+f(b)\big] - \frac{mc}{3}(a-b)^2.$$

Proof. By hypothesis and using a similar result from [2],

$$\begin{aligned}
f(ta+m(1-t)b) &\leq tf(a)+m(1-t)f(b)-mct(1-t)(a-b)^2\\
f(mtb+(1-t)a) &\leq (1-t)f(a)+mtf(b)-mct(1-t)(a-b)^2\\
f(tb+m(1-t)a) &\leq tf(b)+m(1-t)f(a)-mct(1-t)(a-b)^2\\
f(mta+(1-t)b) &\leq (1-t)f(b)+mtf(a)-mct(1-t)(a-b)^2.
\end{aligned}$$

Adding up the above four inequalities,

$$f(ta+m(1-t)b)+f(mtb+(1-t)a)+f(tb+m(1-t)a)+f(mta+(1-t)b)$$

$$\leq (m+1)(f(a)+f(b))-4mct(1-t)(a-b)^2.$$

Performing integration (on $[0,1]$) in both sides of above expression, using

$$\int_0^1 f(ta+m(1-t)b)dt=\int_0^1 f(mtb+(1-t)a)dt=\frac{1}{mb-a}\int_a^{mb} f(s)ds,$$

$$\int_0^1 f(tb+m(1-t)a)dt=\int_0^1 f(mta+(1-t)b)dt=\frac{1}{b-ma}\int_{ma}^{b} f(s)ds$$

(which are obtained by using Lemma 2.1 appropriately) and

$$4mc\int_0^1 t(1-t)(a-b)^2dt=\frac{2mc}{3}(a-b)^2,$$

it follows

$$\frac{1}{b-ma}\int_{ma}^{b} f(s)ds+\frac{1}{mb-a}\int_a^{mb} f(s)ds\leq\frac{(m+1)}{2}\big[f(a)+f(b)\big]-\frac{mc}{3}(a-b)^2.$$

Next two results are similar to others for m-convex functions given in [16]. First we recall a lemma shown in [6, Lemma 1].

Lemma 2.6 *Let $f: I\subset \mathbf{R}\to\mathbf{R}$ be a differentiable function on the interior of I; $a,b\in I$, $a<b$. If $f'\in L^1[a,b]$ then the following equality holds*

$$\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(s)ds$$

$$=\frac{(x-a)^2}{b-a}\int_0^1(1-t)f'(tx+(1-t)a)dt+\frac{(b-x)^2}{b-a}\int_0^1(1-t)f'(tx+(1-t)b)dt,$$

for all $x\in[a,b]$.

Theorem 2.7 *Let $f : I \subset \mathbf{R}_+ \to \mathbf{R}$ be a differentiable function on the interior of I; $a, b \in I$, $a < b$. For f' being integrable on $[a, b]$ and $|f'|$ strongly m-convex function with modulus c, $m \in (0, 1]$ fixed; the following inequality holds for $x \in [a, b]$,*

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq (x-a)^2\left[\frac{|f'(x)|+2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right]+(x-b)^2\left[\frac{|f'(x)|+2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right]$$

$$-\frac{c}{12m}\left[[(x-a)(mx-a)]^2+[(x-b)(mx-b)]^2\right].$$

Proof. By Lemma 2.6,

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq (x-a)^2\int_0^1(1-t)|f'(tx+(1-t)a)|dt+(x-b)^2\int_0^1(1-t)|f'(tx+(1-t)b)|dt. \quad (9)$$

The strong m-convexity of $|f'|$ allows us

$$|f'(tx+(1-t)a)| \leq t|f'(x)|+m(1-t)\left|f'\left(\frac{a}{m}\right)\right|-\frac{ct(1-t)}{m}(mx-a)^2,$$

$$|f'(tx+(1-t)b)| \leq t|f'(x)|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|-\frac{ct(1-t)}{m}(mx-b)^2.$$

Multiplying these two foregoing inequalities by $1-t$ and integrating on $[0, 1]$ grants us

$$\int_0^1(1-t)|f'(tx+(1-t)a)|dt \leq \frac{1}{6}|f'(x)|+\frac{m}{3}\left|f'\left(\frac{a}{m}\right)\right|-\frac{c}{12m}(mx-a)^2$$

and

$$\int_0^1(1-t)|f'(tx+(1-t)b)|dt \leq \frac{1}{6}|f'(x)|+\frac{m}{3}\left|f'\left(\frac{b}{m}\right)\right|-\frac{c}{12m}(mx-b)^2,$$

respectively. Therefore the right hand side of (9) becomes less than or equal to

$$(x-a)^2\left[\frac{|f'(x)|+2m\left|f'\left(\frac{a}{m}\right)\right|}{6}\right]+(x-b)^2\left[\frac{|f'(x)|+2m\left|f'\left(\frac{b}{m}\right)\right|}{6}\right]$$

$$-\frac{c}{12m}\left[\left[(x-a)(mx-a)\right]^2+\left[(x-b)(mx-b)\right]^2\right].$$

If we set $x=\dfrac{a+b}{2}$ in the foregoing theorem the above inequality becomes

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(s)ds \leq \frac{(b-a)^2}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+m\left[\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right]\right]$$

$$-\frac{c(b-a)^2}{192m}\left[((m-2)a+mb)^2+((m-2)b+ma)^2\right].$$

Theorem 2.8 *Let $f : I \subset \mathbf{R}_+ \to \mathbf{R}$ be a differentiable function on the interior of I; $a, b \in I$, $a < b$ and $m \in (0, 1]$. For f' being integrable on $[a, b]$ and $|f'|^q$ strongly m-convex function with modulus c, the following inequality holds for all $x \in [a, b]$*

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq \left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[(x-a)^2\left(|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{3m}(mx-a)^2\right)^{\frac{1}{q}}\right.$$

$$\left.+(x-b)^2\left(|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{3m}(mx-b)^2\right)^{\frac{1}{q}}\right],$$

where $p > 1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Proof. We use Lemma 2.6 one more time and, as done in the foregoing theorem, we bound

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

by (9). Now we use Hölder inequality and strong m-convexity of $|f'|^q$ to get

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq (x-a)^2\left(\int_0^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$

$$+(x-b)^2\left(\int_0^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq (x-a)^2\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q}{2}-\frac{c}{6m}(mx-a)^2\right)^{\frac{1}{q}}$$

$$+(x-b)^2\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}-\frac{c}{6m}(mx-b)^2\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[(x-a)^2\left(|f'(x)|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{3m}(mx-a)^2\right)^{\frac{1}{q}}\right.$$

$$\left.+(x-b)^2\left(|f'(x)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{3m}(mx-b)^2\right)^{\frac{1}{q}}\right].$$

If we pick $x=\dfrac{a+b}{2}$, the corresponding inequality now is

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(s)ds$$

$$\leq\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^2\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{12m}((m-2)a\right.\right.$$

$$\left.\left.-mb)^2\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{12m}((m-2)b-ma)^2\right)^{\frac{1}{q}}\right].$$

Theorem 2.9 *Let $f: I\subset \mathbf{R}_+\to\mathbf{R}$ be a differentiable function on the interior of I; $a,b\in I$, $a<b$ and $m\in(0,1]$ fixed. For $|f'|^q$ being strongly m-convex function with modulus c and f' integrable on $[a,b]$, $x\in[a,b]$ and $q\geq 1$, we have*

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[(x-a)^2\left(\frac{|f'(x)|^q}{2}+m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{4m}(mx-a)^2\right)^{\frac{1}{q}}\right.$$

$$\left.+\ (x-b)^2\left(\frac{|f'(x)|^q}{2}+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{4m}(mx-b)^2\right)^{\frac{1}{q}}\right].$$

Proof. By Hölder inequality,

$$\int_0^1 (1-t)|f'(tx+(1-t)a)|dt = \int_0^1 (1-t)^{1-\frac{1}{q}}\left[(1-t)^{\frac{1}{q}}|f'(tx+(1-t)a)|\right]dt$$

$$\leq \left[\int_0^1 (1-t)dt\right]^{1-\frac{1}{q}} \left[\int_0^1 (1-t)|f'(tx+(1-t)a)|^q dt\right]^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\int_0^1 (1-t)|f'(tx+(1-t)a)|^q dt\right]^{\frac{1}{q}}.$$

Now we use Lemma 2.6 and the strong m-convexity of $|f'|^q$ to obtain

$$(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)ds$$

$$\leq (x-a)^2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left[t(1-t)|f'(x)|^q + m(1-t)^2\left|f'\left(\frac{a}{m}\right)\right|^q\right.\right.$$

$$\left.\left.-\ \frac{ct(1-t)^2}{m}(mx-a)^2\right]dt\right)^{\frac{1}{q}} + (x-b)^2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 [t(1-t)|f'(x)|^q\right.$$

$$\left.+m(1-t)^2\left|f'\left(\frac{b}{m}\right)\right|^q - \frac{ct(1-t)^2}{m}(mx-b)^2\right]dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)^{\frac{1}{q}}\left[(x-a)^2\left(\frac{|f'(x)|^q}{2}+m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{4m}(mx-a)^2\right)^{\frac{1}{q}}+(x-b)^2\right.$$

$$\left.\times\ \left(\frac{|f'(x)|^q}{2}+m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{4m}(mx-b)^2\right)^{\frac{1}{q}}\right].$$

As we notice in previous results, if $x=\dfrac{a+b}{2}$ then the foregoing theorem looks like

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(s)ds$$

$$\leq\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^2\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2m\left|f'\left(\frac{a}{m}\right)\right|^q-\frac{c}{8m}((m-2)a+mb)^2\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2m\left|f'\left(\frac{b}{m}\right)\right|^q-\frac{c}{8m}((m-2)b+ma)^2\right)^{\frac{1}{q}}\right].$$

Next results are inspired in [12].

Theorem 2.10 *Suppose* $f, g : [a, b] \to [0, +\infty)$, $0 \leq a < b < \infty$, *are strongly* m_1*-convex and strongly* m_2*-convex functions, with modulus* c_1 *and* c_2 *respectively;* $m_1, m_2 \in (0, 1]$. *If* $f, g \in L^1([a, b])$, *then the following inequality is true,*

$$\frac{1}{b-a}\int_a^b [f(x)]^{\frac{x-a}{b-a}}[g(x)]^{\frac{b-x}{b-a}}\,dx$$
$$\leq \frac{1}{3}\left[f(b) + m_2 g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[g(b) + m_1 f\left(\frac{a}{m_1}\right)\right] - \frac{1}{12}\left[\frac{c_1}{m_1}(m_1 b - a)^2\right.$$
$$\left.+\frac{c_2}{m_2}(m_2 b - a)^2\right].$$

Proof. By hypothesis,

$$f(tb + (1-t)a) \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) - m_1 c_1 t(1-t)\left(b - \frac{a}{m_1}\right)^2,$$

$$g(tb + (1-t)a) \leq tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) - m_2 c_2 t(1-t)\left(b - \frac{a}{m_2}\right)^2$$

for all $t \in [0, 1]$. Since f, g are non-negative,

$$[f(tb + (1-t)a)]^t[g(tb + (1-t)a)]^{1-t}$$
$$\leq \left[tf(b) + m_1(1-t)f(am_1) - c_1 m_1 t(1-t)(m_1 b - a)^2\right]^t \left[tg(b)\right.$$
$$\left.+ m_2(1-t)g\left(\frac{a}{m_2}\right) - \frac{c_2}{m_2}t(1-t)(m_2 b - a)^2\right]^{1-t}. \tag{10}$$

We now use the Cauchy Inequality on the right hand side of (10), which states that if $\alpha, \beta > 0$, $\alpha + \beta = 1$, then $\alpha x + \beta y \geq x^\alpha y^\beta$ for every positive real numbers x and y and get

$$[f(tb + (1-t)a)]^t[g(tb + (1-t)a)]^{1-t}$$
$$\leq t\left[tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) - \frac{c_1}{m_1}t(1-t)(m_1 b - a)^2\right] + (1-t)\left[tg(b)\right.$$
$$\left.+ m_2(1-t)g\left(\frac{a}{m_2}\right) - \frac{c_2}{m_2}t(1-t)(m_2 b - a)^2\right]$$
$$= t^2 f(b) + m_1 t(1-t)f\left(\frac{a}{m_1}\right) - \frac{c_1}{m_1}t^2(1-t)(m_1 b - a)^2 + t(1-t)g(b)$$
$$+ m_2(1-t)^2 g\left(\frac{a}{m_2}\right) - \frac{c_2}{m_2}t(1-t)^2(m_2 b - a)^2$$

for all $t \in [0,1]$. Conclusion follows by performing integration of last inequality with respect to t on $[0,1]$ and using the change $x = tb + (1-t)a$, that is,

$$\int_0^1 [f(tb+(1-t)a)]^t [g(tb+(1-t)a)]^{1-t}\,dt \leq \frac{1}{3}\left[f(b)+m_2 g\left(\frac{a}{m_2}\right)\right]+\frac{1}{6}\left[g(b)+m_1 f\left(\frac{a}{m_1}\right)\right]-\frac{1}{12}\left[\frac{c_1}{m_1}(m_1 b-a)^2 + \frac{c_2}{m_2}(m_2 b-a)^2\right].$$

If we choose $m_1 = m_2 = 1$ in Theorem 2.10, it turns into

$$\frac{1}{b-a}\int_a^b [f(x)]^{\frac{x-a}{b-a}}[g(x)]^{\frac{b-x}{b-a}}\,dx \leq \frac{1}{3}\left[f(b)+g(a)\right]+\frac{1}{6}\left[g(b)+f(a)\right]-\frac{c_1+c_2}{12}(b-a)^2.$$

Theorem 2.11 *Suppose that $f, g : [0,b] \to \mathbf{R}$, $b > 0$, are strongly m_1-convex and strongly m_2-convex functions, with modulus c_1 and c_2 respectively; $m_1, m_2 \in (0,1]$. If $f, g \in L^1([a,b])$, then*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)dx + m_2\frac{g\left(\frac{a}{m_2}\right)}{(b-a)^2}\int_a^b (b-x)f(x)dx \\
&+\frac{f(b)}{(b-a)^2}\int_a^b (x-a)g(x)dx + m_1\frac{f\left(\frac{a}{m_1}\right)}{(b-a)^2}\int_a^b (b-x)g(x)dx - \frac{c_2(m_2 b-a)^2}{m_2(b-a)^3} \\
&\times \int_a^b (x-a)(b-x)f(x)dx - \frac{c_1(m_1 b-a)^2}{m_1(b-a)^3}\int_a^b (x-a)(b-x)g(x)dx \\
&\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{3}f(b)g(b) + \frac{m_1}{6}f\left(\frac{a}{m_1}\right)g(b) \\
&+\frac{m_2}{6}f(b)g\left(\frac{a}{m_2}\right) + \frac{m_1 m_2}{3}f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right) + \frac{c_1 c_2}{30 m_1 m_2}(m_1 b-a)^2(m_2 b-a)^2 \\
&-\frac{1}{12}\left[\frac{c_2(m_2 b-a)^2}{m_2}\left(f(b)+m_1 f\left(\frac{a}{m_1}\right)\right) + \frac{c_1(m_1 b-a)^2}{m_1}\left(g(b)+m_2 g\left(\frac{a}{m_2}\right)\right)\right].
\end{aligned}$$

Proof. As we know,

$$f(tb+(1-t)a) \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) - m_1 c_1 t(1-t)\left(b-\frac{a}{m_1}\right)^2,$$

$$g(tb+(1-t)a) \leq tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right) - m_2 c_2 t(1-t)\left(b-\frac{a}{m_2}\right)^2.$$

Elementary properties of real numbers show that if $p, q, r, s \in \mathbf{R}$, $p \leq q$ and $r \leq s$ then $ps + qr \leq pr + qs$. Therefore

$$\begin{aligned}
&f(tb+(1-t)a)\left[tg(b)+m_2(1-t)g\left(\frac{a}{m_2}\right)-m_2c_2t(1-t)\left(b-\frac{a}{m_2}\right)^2\right]\\
&\quad+g(tb+(1-t)a)\left[tf(b)+m_1(1-t)f\left(\frac{a}{m_1}\right)-m_1c_1t(1-t)\left(b-\frac{a}{m_1}\right)^2\right]\\
&\leq f(tb+(1-t)a)g(tb+(1-t)a)+\left[tf(b)+m_1(1-t)f\left(\frac{a}{m_1}\right)-m_1c_1t(1-t)\right.\\
&\quad\times\left.\left(b-\frac{a}{m_1}\right)^2\right]\left[tg(b)+m_2(1-t)g\left(\frac{a}{m_2}\right)-m_2c_2t(1-t)\left(b-\frac{a}{m_2}\right)^2\right]
\end{aligned}$$

by a simple use of the mentioned property. Then

$$\begin{aligned}
&tf(tb+(1-t)a)g(b)+m_2(1-t)f(tb+(1-t)a)g\left(\frac{a}{m_2}\right)-\frac{c_2}{m_2}t(1-t)(m_2b-a)^2\\
&\quad\times f(tb+(1-t)a)+tf(b)g(tb+(1-t)a)+m_1(1-t)f\left(\frac{a}{m_1}\right)g(tb+(1-t)a)\\
&\quad-\frac{c_1}{m_1}t(1-t)(m_1b-a)^2g(tb+(1-t)a)\\
&\leq f(tb+(1-t)a)g(tb+(1-t)a)+t^2f(b)g(b)+m_2t(1-t)f(b)g\left(\frac{a}{m_2}\right)-\frac{c_2}{m_2}t^2\\
&\quad\times(1-t)(m_2b-a)^2f(b)+m_1t(1-t)f\left(\frac{a}{m_1}\right)g(b)+m_1m_2(1-t)^2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\\
&\quad-\frac{m_1c_2}{m_2}t(1-t)^2(m_2b-a)^2f\left(\frac{a}{m_1}\right)-\frac{c_1}{m_1}t^2(1-t)(m_1b-a)^2g(b)-\frac{m_2c_1}{m_1}\\
&\quad\times t(1-t)^2(m_1b-a)^2g\left(\frac{a}{m_2}\right)+\frac{c_1c_2}{m_1m_2}t^2(1-t)^2(m_1b-a)^2(m_2b-a)^2.
\end{aligned}$$

Conclusion shows up by integration with respect to t on $[0,1]$ and by using the change $x = tb + (1-t)a$.

Corollary 2.12 *If f, g are as in Theorem 2.11 and $m_1 = m_2 = 1$, then* *the above inequality turns into*

$$\frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)dx + \frac{g(a)}{(b-a)^2}\int_a^b (b-x)f(x)dx + \frac{f(b)}{(b-a)^2}$$
$$\times\int_a^b (x-a)g(x)dx + \frac{f(a)}{(b-a)^2}\int_a^b (b-x)g(x)dx - \frac{c_2}{b-a}\int_a^b (x-a)(b-x)f(x)dx$$
$$-\frac{c_1}{b-a}\int_a^b (x-a)(b-x)g(x)dx$$
$$\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) + \frac{1}{6}C(a,b),$$

with

$$M(a,b) = f(a)g(a) + f(b)g(b),\ N(a,b) = f(a)g(b) + f(b)g(a),$$

and

$$C(a,b) = \frac{c_1c_2}{5}(b-a)^4 - \frac{1}{2}\left[c_2(b-a)^2\left(f(a)+f(b)\right) + c_1(b-a)^2\left(g(a)+g(b)\right)\right].$$

Corollary 2.13 *If hypotheses of Theorem 2.11 hold and f, g are increasing functions, then*

$$\frac{g(a)}{(b-a)^2}\left[\int_a^b (x-a)f(x)dx + \int_a^b (b-x)f(x)dx\right] + \frac{f(a)}{(b-a)^2}\left[\int_a^b (x-a)g(x)dx\right.$$
$$\left.+\int_a^b (b-x)g(x)dx\right] - \frac{c_2}{b-a}\int_a^b (x-a)(b-x)f(x)dx$$
$$-\frac{c_1}{b-a}\int_a^b (x-a)(b-x)g(x)dx$$
$$\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) + \frac{1}{6}C(a,b).$$

with $M(a,b)$, $N(a,b)$ and $C(a,b)$ as before.

Corollary 2.14 *If we choose $m_1 = m_2 = 1$ and $g(x) = 1$ in Theorem 2.11, then*

$$\frac{1}{(b-a)^2}\left[\int_a^b (x-a)f(x)dx + \int_a^b (b-x)f(x)dx\right] + \frac{f(a)+f(b)}{2} - \frac{c_2}{b-a}$$
$$\times\int_a^b (x-a)(b-x)f(x)dx - \frac{c_1}{6}(b-a)^2$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx + \frac{6-c_2(b-a)^2}{12}(f(a)+f(b)) + \frac{c_1c_2}{30}(b-a)^4 - \frac{c_1}{6}(b-a)^2.$$

Theorem 2.15 *Suppose that $f, g : [a, b] \to [0, +\infty)$, $0 \le a < b < \infty$, are strongly m_1-convex and strongly m_2-convex functions, with modulus c_1 and c_2 respectively; $m_1, m_2 \in (0, 1]$. If $f, g \in L^1([a, b])$ and $p \ge 1$, then the next inequality holds*

$$\frac{1}{b-a}\int_a^b [f(x) + g(x) + k(x-a)(b-x)]^p\, dx$$
$$\le \frac{2^{p-1}}{p+1}\left([f(b) + g(b)]^p + \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right),$$

where $k = \frac{c_1}{m_1}\left(\frac{m_1 b - a}{b-a}\right)^2 + \frac{c_2}{m_2}\left(\frac{m_2 b - a}{b-a}\right)^2$.

Proof. We begin by recalling that if r, s are non-negative real numbers and $p \ge 1$, then $(r+s)^p \le 2^{p-1}(r^p + s^p)$; on the other hand we know by hypothesis,

$$f(tb + (1-t)a) + m_1 c_1 t(1-t)\left(b - \frac{a}{m_1}\right)^2 \le tf(b) + m_1(1-t) f\left(\frac{a}{m_1}\right),$$

$$g(tb + (1-t)a) + m_2 c_2 t(1-t)\left(b - \frac{a}{m_2}\right)^2 \le tg(b) + m_2(1-t) g\left(\frac{a}{m_2}\right).$$

Now by adding up the last two inequalities and using the above mentioned property of nonnegative real numbers we come out with

$$\left[f(tb + (1-t)a) + g(tb + (1-t)a) + m_1 c_1 t(1-t)\left(b - \frac{a}{m_1}\right)^2 + m_2 c_2 t(1-t)\right.$$
$$\left.\times\left(b - \frac{a}{m_2}\right)^2\right]^p$$
$$\le 2^{p-1}\left(t^p[f(b) + g(b)]^p + (1-t)^p\left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right).$$

The proof concludes by integrating the above inequality with respect to t on $[0, 1]$ and by using the change $x = tb + (1-t)a$.

Corollary 2.16 *Under the assumptions of Theorem 2.15 with $m_1 = m_2 = 1$ and $p \to 1$,*

$$\int_a^b [f(x) + g(x)]dx \le \frac{b-a}{2}\left[f(a) + f(b) + g(a) + g(b) - \frac{c_1 + c_2}{3}(b-a)^2\right].$$

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