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Some remarks on generalized derivations

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Abstract

In this short note generalized derivations are briefly described. This is done first by noticing some link with algebraic approach to differential geometry. In particular, through introducing certain operators over modules. Secondly, a direct geometrical reference is shortly reminded. Finally, a remark in the context of the Lindblad decomposition is reminded.

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1 Introduction

It is assumed that basic algebraic and differential geometric notions are already known [1, 2]. If not stated otherwise, an algebra A is assumed to be commutative and unitary. The considerations presented herein emerge from some quest of extending differential geometry to singular spaces. In other words, to the more general context than smooth manifolds. One of the possibility is to do it *via* the help of algebraic approach [3, 4, 5, 6, 7].

2 *Der*-operator

Consider the following operator $\Box : P \to P$ where P is an A-module (A is a $\mathbb{K}$-algebra), such that:

1. $\Box$ is $\mathbb{K}$-linear,

2. there exists $X \in D(A)$ such that for every $a \in A$ and $p \in P$ $\Box(ap) = X(a)p + a\Box(p)$.

Definition 1. *Such an operator is called* Der-operator.

It is also an operator over X. Let also use the following notation $DerP := \bigcup_{X \in D(A)} Der_X P$.

$D(A)$ denotes the collection of all derivations of the algebra A, i.e., $\mathbb{K}$-linear mappings $d : A \to A$ satisfying the Leibniz rule ($\forall_{a,b \in A}\ d(ab) = d(a)b + ad(b)$).

By a *linear differential operator of order* j is understood $\triangle : P \to Q$ which is $\mathbb{K}$-linear and for all $a_0, \ldots, a_j \in A$ $[\ldots[\,[\triangle, a_0], a_1], \ldots], a_j] = 0$, where P and Q are A-modules. The collection of all linear differential operators of order j form P to Q is denoted by $Diff_j(P, Q)$.

Fact 1. $DerP \subseteq Diff_1(P, P)$.

Fact 2. *If P is faithful (i.e., $\forall_{p \in P}\ ap = 0 \Rightarrow a = 0$), then $Der_X P \cap Der_Y P = \emptyset$ for $X \neq Y$.*

Fact 3. *Let V be a finite dimension vector space. Then, $P = A \otimes_{\mathbb{K}} V$ is a free A-module. Let $D_X := X \otimes id_V$, i.e., $D_X(a \otimes v) := X(a) \otimes v$. Then, $D_X \in Der_X P$.*

Definition 2. *Let $\alpha : DerP \to D(A)$, such that $\alpha(\Box) \mapsto X$.*

Fact 4. *The below statements hold:*

1. *$DerP$ is an A-module.*
2. *$DerP$ is a Lie algebra. Let $\Box_1 \in Der_X P$, $\Box_2 \in Der_Y P$, then $[\Box_1, \Box_2] \in Der_{[X,Y]} P$.*
3. *α is A-linear, i.e., $\Box \in Der_X P \Rightarrow a\Box \in Der_{aX} P$.*
4. *$\Box_1, \Box_2 \in DerP, a \in A \Rightarrow [a\Box_1, \Box_2] = a[\Box_1, \Box_2] - Y(a)\Box_1$.*

In other words, $DerP$ is a Lie-Rinehart algebra [8, 9].

Fact 5. *$\alpha^{-1}(X) = Der_X P$ is an affine subspace of $DerP$ modelled over $End_A P$.*

Fact 6. $\Box_1, \Box_2 \in Der_X P \Rightarrow \Box_1, \Box_2 \in Der_0 P = Hom_A(P, P) = End_A(P)$.

Example 1. *Let $A = C^\infty(M)$, where M is a manifold. Let $P = \Gamma(\pi)$, i.e., P is the module of sections. Let $\{e_1, \ldots, e_m\}$ be a local basis of P and $\{x_1, \ldots, x_n\}$ be a local basis of M. Let us (define) denote $\Box(e_\alpha) =: \Box_\alpha^\beta e_\beta$. Then, $\Box(p) = \Box(p^\alpha e_\alpha) = X(p^\alpha)e_\alpha + p^\alpha \Box(e_\alpha) = X(p^\beta)e_\beta + p^\alpha \Box_\alpha^\beta e_\beta$.*

Therefore, $DerP \cong \{$all linear vector fields over $E \xrightarrow{\pi} M\}$.

Indeed, consider $X(p^\beta)e_\beta + p^\alpha \Box_\alpha^\beta e_\beta = \Box \mapsto Y = X^i \frac{\partial}{\partial x^i} + e_\alpha \Box_\alpha^\beta \frac{\partial}{\partial e_\beta}$. Therefore, Y is projectable on M.

The above part is based on [10].

3 Linear connection

By the *linear connection* is understood an operator $\nabla : D(A) \to DerP$, such that

1. $\nabla_{X+Y} = \nabla_X + \nabla_Y$,
2. $\nabla_{aX} = a\nabla_X$,
3. $\nabla_X(ap) = X(a)p + a\nabla_X(p)$.

Herein, the following notation is used $\nabla(X) =: \nabla_X$.

Fact 7. $\nabla - \nabla' \in End_A P$.

Fact 8. *If P is faithful, then ∇ is an injection.*

Fact 9. *Linear connections constitute an affine subspace of $Hom_A(D(M), Der(P))$ modelled over $Hom_A(D(M), End_A(P))$.*

Definition 3. *Let $\Gamma^\beta_{\gamma\alpha} e_\beta := \nabla_{\frac{\partial}{\partial x_\gamma}} e_\beta$. They are called* Christoffel symbols*. Therefore, $\nabla_X p = \nabla_{X^i \frac{\partial}{\partial x^i}} p^\alpha e_\alpha = X^i \nabla_{\frac{\partial}{\partial x^i}} p^\alpha e_\alpha = X^i(\frac{\partial p^\beta}{\partial x^i} + p^\alpha \Gamma^\beta_{\gamma\alpha}) e_\beta$.*

Definition 4. *Let $R^\nabla(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Of course, $R^\nabla(X,Y) \in End_A P$. It is said that ∇ is* flat*, if $R^\nabla = 0$.*

The above part is based on [10].

4 Cochain complex

Let $(\Lambda(M), d)$ be a de Rham differential algebra with a gradation. Then, $\Lambda(M) \otimes_A P$ is a $\Lambda(M)$-module with a gradation. Let $\omega \in \Lambda^i(M)$ and let $\Omega \in \Lambda^i(M) \otimes_A P$.

Then, $d_\nabla(\omega \wedge \Omega) := d\omega \wedge \Omega + (-1)^\omega \omega \wedge d_\nabla \Omega$ can be defined. Equivalently, it can be done in the following way. Let $d^0_\nabla p := \nabla_X p$ and let

$$(d^k_\nabla \Omega)(X_1, \ldots, X_{k+1}) := \sum_i (-1)^{i+1} \nabla_{X_i}(\Omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) + \sum_{i,j} (-1)^{i+j} \Omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \quad ,$$

where $\hat{X}_i$ denotes omitting of the element X_i.

Fact 10. *d_∇ is a Der-operator in $\Lambda(M) \otimes_A P$ over d.*

Fact 11. *$\Omega \in \Lambda^2(M) \otimes_A P$ and $(d_\nabla \circ d_\nabla)(\Omega) = R^\nabla \wedge \Omega$.*

Fact 12. ∇ *is flat, if and only if* $d^2_\nabla = 0$.

Fact 13. $0 \to P \overset{d_\nabla}{\to} \Lambda^1(M) \otimes_A P \overset{d_\nabla}{\to} \Lambda^2(M) \otimes_A P \overset{d_\nabla}{\to} \ldots, \overset{d_\nabla}{\to} \Lambda^n(M) \otimes_A P \overset{d_\nabla}{\to} 0$ *is a cochain complex ([11]), if* ∇ *is flat.*

The above part is based on [10].

5 Some remarks

Two remarks can be formulated. First, in the context of Definition 1. Let $E \overset{\pi}{\to} M$, i.e., E is a vector bundle over M. Let $X \in \mathfrak{X}(M)$ be a smooth vector field over M. Then, let $\hat{X} \in \mathfrak{X}(E)$ be such that $\pi_* \hat{X} = X$. $\hat{X}$ is linear, or, in other words, infinitesimal translations of fibres π are linear. $\hat{X}$ generates infinitesimal translation of section $p = \pi(x)$. $\Box(p)$ is the velocity of changes of this translation. $p \mapsto \Box(p)$ is a Der-operator.

Secondly, in the context of Fact 9 $\nabla_X : P \to P$, such that $\nabla_X(\Box) = X \circ \Box$ is a Der-operator. $X \mapsto \nabla_X$ is a connection. Therefore, this connection induces the mapping $X \mapsto \hat{X}$, which is A-linear and injective. $\hat{X}$ is called the *lifting* of X.

The above part is based on [10].

6 Generalized derivations

In [12] generalized derivations were proposed.

Definition 5. *Let A be a $\mathbb{K}$-algebra. An additive mapping $g : A \to A$ is called a* generalized derivation*, if there exists a (classical) derivation $d : A \to A$, such that for every $a, b \in A$ $g(ab) = g(a)b + ad(b)$.*

The collection of all generalized derivations of A is denoted by $GD(A)$. Of course, $D(A) \subseteq GD(A)$.

If $l : A \to A$ is such that $\forall_{a,b \in A}\ l(ab) = l(a)b$, then l is called a *left multiplier*. The collection of all left multipliers of A is denoted by $LM(A)$. Of course, $LM(A) \subseteq GD(A)$.

By considering $P = A$, it is trivial to notice that

Proposition 1. $GD(A) = Der A$.

In similarity with Fact 2:

Fact 14. *If for every $a \in A$ $Aa = \{0\}$ implies that $a = 0$, then $GD(A) = LM(A) \oplus D(A)$.*

Generalized derivations are linear differential operators of order 1. If g is a generalized derivation, then $g(1_A)$ does not have to be equal to 0. However, if it is so, then g is a classical derivation [13, 14].

Differential geometry can be constructed over generalized derivations, in similarity to the classical case [15]. (This is no surprising, as in the context of the remark after Fact 4 generalized derivations can now be seen in connection with Lie algebroids.) In case of commutative algebra (as assumed herein), the procedure simplifies to considering a symmetric, A-blilinear mapping $m : GD(A) \times GD(A) \to A$. Moreover, it is assumed that $\hat{m} : GD(A) \to GD(A)^*$ is an isomorphism, where $\hat{m}(g_1)(g_2) := m(g_1, g_2)$ and $GD(A)^*$ is the dual of $GD(A)$, i.e., $GD(A)^* := Hom_A(GD(A), A)$. With the help of the Koszul formula the preconnection $\nabla^* : GD(A) \times GD(A) \to GD(A)^*$ is defined. Next, the connection $\nabla : GD(A) \times GD(A) \to GD(A)$ by $\nabla := (\hat{m})^{-1} \circ \nabla^*$. It is shown that

1. $\nabla_{g_1+g_2} = \nabla_{g_1} + \nabla_{g_2}$,
2. $\nabla_{ag} = a\nabla_g$,
3. $\nabla_{g_1}(ag_2) = \widetilde{g_1}(a)g_2 + a\nabla_{g_1}g_2$, where $g_1(a_1a_2) = g_1(a_1)a_2 + a_1\widetilde{g_1}(a_2)$ and $\widetilde{g_1}$ is a classical derivation corresponding to the generalized derivation g_1.

The above view on a connection is more general than those from Section 3 (i.e., the one called therein the linear connection).

For $A = C^\infty(M)$ it is shown that $GD(A) = \text{span}(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{\dim M}}, id_M)$. Therefore, clearly, this is attaching one extra "non-physical" dimension to M (i.e., not associated with any extra coordinate on M).

The above part is based on [15].

7 Quantum theory

For the position ($\hat{x}$) and the momentum ($\hat{p}$) operators the following relations [16] are known (ψ and φ denote wave functions in position and in momentum space, respectively):

$\hat{x}(\psi(x)) = x\psi(x)$, $\hat{p}(\psi(x)) = -i\hbar\psi'(x)$,
$\hat{x}(\varphi(p)) = i\hbar\varphi'(p)$, $\hat{p}(\varphi(p)) = p\varphi(p)$.

Having in mind that id is a generalized derivation, this might serve as a hint to use $GD(A)$ in a phase space description. However, the details remain an open problem.

Let $B(H)$ denote the bounded operators on the Hilbert space H. The *ultraweak topology* on $B(H)$ is defined as the weakest topology, for which all elements of predual space are continuous. By *predual* of some space is understood the one, whose dual is the initial space. In particular, the predual space of $B(H)$ is the space of trace class operators [17].

It is known that $A = B(H)$ (together with addition, composition, norm and adjoint operation) constitutes a C^*-algebra. This serves as a classical tool to describe a quantum mechanical system [18]. In particular, $B(H)$ represents states of a quantum mechanical system.

Let A_{sa} denote the self adjoint part of A, i.e., $A_{sa} := \{a \in A \mid a^* = a\}$. In particular, $\{\text{observables}\} \subseteq A_{sa}$ [19].

$T : A \to A$ is called *hermitian preserving*, if $T(A_{sa}) \subseteq A_{sa}$. A hermitian preserving derivation of a C^*-algebra is called **-derivation*.

$a \in A$ is called *positive*, if $\{\lambda \in \mathbb{C} \mid \lambda 1_A - a \text{ is not invertible in } A\} \subseteq \mathbb{R}^+$.

Let $A_+ := \{a \in A \mid a \text{ is positive}\}$. $T : A \to A$ is called positive, if $T(A_+) \subseteq A_+$. Positive mappings extend to $T_n : M_n(A) \to M_n(A)$ by $T((a_{ij})) = (T(a_{ij}))$. A positive mapping T is called *completely positive*, if T_n is positive for all $n \in \mathbb{N}$.

The one parameter semigroup $(T_t)_{t\geq 0}$ describes the evolution of a quantum system. T_t is a completely positive operator on a C^*-algebra A. The mapping $A \ni a \mapsto T_t(a)$ with $t \geq 0$ represents the dynamics in the Heisenberg picture.

Supposing that A is unitary, $L : \mathbb{R}^+ \ni t \mapsto T_t \in B(A)$ is continuous. L is differentiable and $L'(0)$ is an infinitesimal generator of $(T_t)_{t\geq 0}$. According to Lindblad [20], completely positivity of all T_t is encoded in completely dissipativity of $L'(0)$, i.e., extensions of $L'_n(0)$ $(n \in \mathbb{N})$ to matrix algebras $M_n(A)$ are dissipative in a sense of the below definition for every $n \in \mathbb{N}$.

Definition 6. *$T : A \to A$ is* dissipative*, if $T(a^*a) + a^*(T1_A)a - a^*(Ta) - (Ta^*)a \geq 0$ for every $a \in A$.*

It can be proved that dissipativity induces continuity.

If the dynamics of C^*-algebra A is reversible, then the physical system is closed, and the semigroup $(T_t)_{t\geq 0}$ describing its evolution can be extended to the group $(\hat{T}_t)_{t\in\mathbb{R}}$. Each T_t is then invertible. $(\hat{T}_t)_{t\in\mathbb{R}}$ is a group of automorphisms of A. Hence, $L'(0)$ (the infinitesimal generator of $(T_t)_{t\geq 0}$, or, equivalently, derivative at 0 of $t \mapsto T_t$) is a *-derivation.

If the dynamic is irreversible, then the system is open (i.e., it is in interaction with the environment), and generalized derivations correspond to a "dissipation" of energy [21].

According to Lindblad [20], $L'(0)$ can be decomposed into its completely positive part and its derivative part. Moreover, let $g_{a,b}(x) := xa - bx$ (of course, $g_{a,b} \in GD(A)$), then

Theorem 1 ([21])**.** *Let A be a unitary C^*-algebra acting on the Hilbert space H. Then, for every completely dissipative operator $L : A \to A$ there exists a completely positive operator $T : A \to \overline{A^\sigma}$, such that $L = T + g_{k,-k^*}$, where $\overline{A^\sigma}$ denotes the ultraweak closure of A and $k \in \overline{A^\sigma}$.*

Indeed, every hermitian preserving generalized derivation is of the form $g_{k,-k^*}$ for some $k \in \overline{A^\sigma}$.

The above part is based on [21]. However, also [22, 20, 23, 24] can be useful.

8 Conclusions

The notion of a generalized derivation was shortly summarized. This concept is recently quite intensively explored mostly in abstract algebraic context (for example in [25, 26, 27, 28]). However, it was shown that it is also very useful in geometrical and physical applications. (Also, [29] can be of some interest in this context.) In particular, it was reminded that there is a strong link with differential geometry and certain differential operators. Moreover, a direct reference to the particular geometrical application was given. Finally, a remark in the context of the Lindblad decomposition in quantum theory was reminded.

It can be seen that a generalized derivation is a useful and interesting concept. However, there are still many interesting open problems connected with this concept.

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