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Positive solutions for boundary value problems of the third-order q-symmetric difference equations with parameter

Qi GE, Chengmin HOU

Department of Mathematics, Yanbian University, Yanji 133002, Jilin, China

Abstract

In this paper, we study the existence and the nonexistence of positive solutions for boundary value problems of the third-order q-symmetric difference equations with parameter. By using the properties of the Green function and Guo-Krasnoselskii fixed point theorem on cones, we prove the existence of positive solutions to this equation when the parameter belongs to different interval. And by apagoge, we prove the nonexistence of positive solutions to this equation when the parameter belongs to different interval.

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Keywords: q-symmetric difference equations,existence and the nonexistence of positive solutions,Fixed point theorem

1 Introduction

In recent years, Quantum calculus is a very interesting field in mathematics. As calculus without limits, quantum calculus plays an important role in several fields of physics such as cosmic strings and black holes [1], conformal quantum mechanics [2], nuclear and high energy physics [3]. For details, We can refer the reader to [4-7].

Quantum calculus has two types: the q-calculus and the h-calculus. The q-symmetriqc quantum calculus is a type of the q-calculus . In the q-symmetric quantum calculus, for a fixed $q \in (0,1)$ and $t \neq 0$ the q-symmetric derivative of a function f at point t is defined by

$$\frac{f(qt) - f(q^{-1}t)}{(q - q^{-1})t}.$$

The q-symmetric quantum calculus has proven to be useful in several fields, in particular in quantum mechanics [8]. As noticed in [8], the q-symmetric derivative let the q-exponential function have unique properties.

In 2012,Artur M.C. Brito da Cruz [9] has introduced a wealth of knowledge about q-symmetric variational calculus, which laid a good foundation to the continue study.

In 2015, Li Wang and Chengmin Hou[10] studied the application of differential transformation method to nonlinear q-symmetric damped systems. At present, the research about the existence of positive solutions for boundary value problems of the q-symmetric difference equations are scarce.

In this paper we investigate the existence of solutions for the following boundary value problem of the third-order q-symmetric difference equations with parameter:

$$\begin{cases} \widetilde{D}_q^3[u](t) = -\lambda g(t) f(u(t)), t \in (0,1); \\ u(0) = \alpha u(\eta), \widetilde{D}_q[u](0) = \widetilde{D}_q^2[u](1) = 0, \end{cases} \tag{1}$$

where $0 < q < 1$, $0 < \eta < 1$, $0 < \alpha < 1$, and λ is a positive parameter. Throughout this paper, we assume that the following conditions are satisfied:
$(H_1) f : [0, \infty) \to [0, \infty)$ is continuous;
$(H_2) g \in C([0, 1], [0, \infty))$ is increasing, and is not identically zero on any subinterval of $[0, 1]$;
$(H_3) 0 < \int_0^1 g(s)\widetilde{d}_q s < \infty$.

2 Preliminary notes

For the convenience of the reader, we give some background materials from q-symmetric calculus theory to facilitate analysis of problem (1).

Let $q \in (0, 1)$ and let I be an interval (bounded or unbounded) of $\mathbb{R}$ containing 0. We will denote by I^q the set $I^q := qI := \{qx : x \in I\}$. Note that $I^q \subseteq I$.

Definition 2.1 [9] *Let f be a real function defined on I. The q-symmetric difference operator of f is defined by*

$$\widetilde{D}_q[f](x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}, t \in I^q \setminus \{0\},$$

and $\widetilde{D}_q[f](0) := f'(0)$, provided f is differentiable at 0. We usually call $\widetilde{D}_q[f]$ the q-symmetric derivative of f.

The q-symmetric derivatives of higher order:

$$\widetilde{D}_q^0[f](x) = f(x), \widetilde{D}_q^n[f](x) = \widetilde{D}_q \widetilde{D}_q^{n-1}[f](x), n \in \mathbb{N}^+.$$

By the definition of the q-symmetric derivative, for any constant k, we have

$$\widetilde{D}_q(kt) = k, \widetilde{D}_q(kt^2) = k(q + q^{-1})t.$$

The q-symmetric difference operator has the following properties.

Lemma 2.2 [9] *Let f and g be q-differentiable on I, let $\alpha, \beta \in \mathbb{R}$ and $t \in I^q$. One has*
1.$\widetilde{D}_q[f] \equiv 0$ *iff f is constant on I;*
2.$\widetilde{D}_q[\alpha f + \beta g](t) = \alpha \widetilde{D}_q[f](t) + \beta \widetilde{D}_q[g](t)$;
3.$\widetilde{D}_q[fg](t) = \widetilde{D}_q[f](t)g(qt) + f(q^{-1}t)\widetilde{D}_q[g](t)$;
4.$\widetilde{D}_q[\frac{f}{g}](t) = \frac{\widetilde{D}_q[f](t)g(q^{-1}t) - f(q^{-1}t)\widetilde{D}_q[g](t)}{g(qt)g(q^{-1}t)}$, *if $g(qt)g(q^{-1}t) \neq 0$.*

Definition 2.3 [9] *Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ and for $q \in (0,1)$ the q-symmetric integral of f from a to b is given by*

$$\int_a^b f(t)\widetilde{d}_q t = \int_0^b f(t)\widetilde{d}_q t - \int_0^a f(t)\widetilde{d}_q t,$$

where

$$\begin{aligned} \widetilde{I}_{q,0}[f](x) := \int_0^x f(t)\widetilde{d}_q t &= (q^{-1} - q)x \sum_{k=0}^{\infty} q^{2k+1} f(xq^{2k+1}) \\ &= (1 - q^2)x \sum_{k=0}^{\infty} q^{2k} f(xq^{2k+1}), x \in I, \end{aligned}$$

provided that the series converges at $x = a$ and $x = b$. In that case, f is called q-symmetric integrable on $[a, b]$. We say that f is q-integrable on I if it is q-integrable on $[a, b]$ for all $a, b \in I$.

As for q-symmetric derivatives, we can define an operator $\widetilde{I}_{q,0}^n$ by

$$\widetilde{I}_{q,0}^0[f](x) = f(x), I_{q,0}^n[f](x) = \widetilde{I}_{q,0}\widetilde{I}_{q,0}^{n-1}[f](x), n \in \mathbb{N}^+.$$

For operators defined in this manner, the following is valid:

$$\widetilde{D}_q\widetilde{I}_{q,0}[f](x) = f(x), \widetilde{I}_{q,0}\widetilde{D}_q[f](x) = f(x) - f(0).$$

By the definition of the q-symmetric integral, for any constant k, we have

$$\widetilde{I}_{q,0}(k) = \int_0^x k\widetilde{d}_q t = kx, \widetilde{I}_{q,0}(kx) = \int_0^x kt\widetilde{d}_q t = \frac{kq}{1+q^2}x^2.$$

On this basis, we have

$$\widetilde{I}_{q,0}^3\widetilde{D}_q^3[f](x) = f(x) + c_0 + c_1 x + c_2 x^2. \tag{2}$$

Lemma 2.4 [9] *Let $a, b \in I, a < b$ and $f : I \to \mathbb{R}$ continuous at 0. Then for $s \in [a, b]$ the sequence $(f(q^{2n+1}s))_{n \in \mathbb{N}}$ converges uniformly to $f(0)$ on I.*

Corollary 2.5 [9] *If $f: I \to \mathbb{R}$ is continuous at 0, then for $s \in [a,b]$ the series $\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1}s)$ is uniformly convergent on I, and, consequently, f is q-symmetric integrable on $[a,b]$.*

Lemma 2.6 [9] *Let $f, g: I \to \mathbb{R}$ be q-symmetric integrable on I, $a,b,c \in I$ and $\alpha, \beta \in \mathbb{R}$. Then*

1. $\int_a^a f(t)\widetilde{d}_q t = 0$;
2. $\int_a^b f(t)\widetilde{d}_q t = -\int_b^a f(t)\widetilde{d}_q t$;
3. $\int_a^b f(t)\widetilde{d}_q t = \int_a^c f(t)\widetilde{d}_q t + \int_c^b f(t)\widetilde{d}_q t$;
4. $\int_a^b (\alpha f + \beta g)(t)\widetilde{d}_q t = \alpha \int_a^b f(t)\widetilde{d}_q t + \beta \int_a^b g(t)\widetilde{d}_q t$;
5. *Suppose that $f(t) \geq 0$, $\forall t \in \{q^{2n+1}c : n \in \mathbb{N}_0\} \cup \{0\}$. If $c \geq 0$, then*

$$\int_0^c f(t)\widetilde{d}_q t \geq 0,$$

In general it is not true that if f is a positive function on $[a,b]$, then

$$\int_a^b f(t)\widetilde{d}_q t \geq 0.$$

Lemma 2.7 $\widetilde{I}_{q,0}^3[f](x) = \frac{q}{1+q^2}\int_0^{q^2 x}(x - q^{-1}t)(x - qt)f(t)\widetilde{d}_q t.$

Proof By Definition 2.3, we have

$$\begin{aligned}
\widetilde{I}_{q,0}^2[f](x) &= \widetilde{I}_{q,0}[(1-q^2)x\sum_{n=0}^{\infty} q^{2n} f(xq^{2n+1})] \\
&= (1-q^2)^2 x^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{2m} q^{2n+2m+1} f(xq^{2n+2m+2}) \\
&= (1-q^2)^2 x^2 \sum_{m=0}^{\infty}\sum_{k=m}^{\infty} q^{2m} q^{2k+1} f(xq^{2k+2}) \\
&= (1-q^2)^2 x^2 \sum_{k=0}^{\infty}\sum_{m=0}^{k} q^{2m} q^{2k+1} f(xq^{2k+2}) \\
&= (1-q^2)^2 x^2 \sum_{k=0}^{\infty} \frac{1-q^{2k+2}}{1-q^2} q^{2k+1} f(xq^{2k+2}) \\
&= q(1-q^2)x \sum_{k=0}^{\infty} (x - q^{2k+2}x) q^{2k} f(xq^{2k+2}) \\
&= \int_0^{qx} (x-t) f(t)\widetilde{d}_q t,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{I}^3_{q,0}[f](x) &= \widetilde{I}_{q,0}\widetilde{I}^2_{q,0}[f](x) \\
&= q(1-q^2)^2x^3\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}q^{2k+2n+2}q^{2k}(q^{2k}-q^{2n+2k+2})f(xq^{2n+2k+3}) \\
&= q(1-q^2)^2x^3\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}q^{2k}q^{2n+2}(q^{2k}-q^{2n+2})f(xq^{2n+3}) \\
&= q(1-q^2)^2x^3\sum_{n=0}^{\infty}\sum_{k=0}^{n}q^{2k}q^{2n+2}(q^{2k}-q^{2n+2})f(xq^{2n+3}) \\
&= q(1-q^2)^2x^3\sum_{n=0}^{\infty}(\frac{1-q^{4n+4}}{1-q^4}-\frac{1-q^{2n+2}}{1-q^2}q^{2n+2})q^{2n+2}f(q^2xq^{2n+1}) \\
&= q^2x(1-q^2)q\sum_{n=0}^{\infty}\frac{(x-q^{2n+2}x)(x-q^{2n+4}x)}{1+q^2}q^{2n}f(q^2xq^{2n+1}) \\
&= \frac{q}{1+q^2}\int_0^{q^2x}(x-q^{-1}t)(x-qt)f(t)\widetilde{d}_qt.
\end{aligned}$$

The proof is complete.

Lemma 2.8 [11] *Let X be a Banach space, and let $P\subset X$ be a cone in X. Assume Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $S:P\to P$ be a completely continuous operator such that, either*
(i) $\| Sw \|\leq\| w \|, w\in P\cap\Omega_1, \| Sw \|\geq\| w \|, w\in P\cap\Omega_2$, or
(ii) $\| Sw \|\geq\| w \|, w\in P\cap\Omega_1, \| Sw \|\leq\| w \|, w\in P\cap\Omega_2$.
Then S has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3 Green function and related lemmas

Let $E=C[0,1]$, $C^+[0,1]=\{u\in C[0,1]:u(t)\geq 0, t\in[0,1]\}$, then E is a Banach space with norm $\|u\|=\max_{0\leq t\leq 1}|u(t)|$.

Lemma 3.1 *Let $0<\eta<1, 0<\alpha<1$. Suppose that $h(t):[0,1]\to[0,+\infty)$ is continuous function, $0<\int_0^1 h(s)\widetilde{d}_qs<\infty$, then the boundary value problem*

$$\begin{cases} \widetilde{D}^3_q[u](t)=-h(t), t\in(0,1); \\ u(0)=\alpha u(\eta), \widetilde{D}_q[u](0)=\widetilde{D}^2_q[u](1)=0 \end{cases} \tag{3}$$

has the unique solution

$$u(t)=\int_0^1 G(t,s)h(s)\widetilde{d}_qs.$$

where

$$G(t,s) = K(t,s) + \frac{\alpha}{1-\alpha}K(\eta,s) \tag{4}$$

is corresponding Greens function, and

$$K(t,s) = \frac{1}{q+q^{-1}}\begin{cases} t^2, & 0 \le q^2 t \le s \le 1, \\ t^2 - (t-q^{-1}s)(t-qs), & 0 \le s \le q^2 t < 1. \end{cases} \tag{5}$$

Proof By (2), we have

$$u(t) = -\widetilde{I}^3_{q,0}h(t) + C_0 + C_1 t + C_2 t^2. \tag{6}$$

By $\widetilde{D}_q[u](0) = 0$, it follows that $C_1 = 0$, from

$$\widetilde{D}^2_q[u](1) = -\int_0^1 h(s)\widetilde{d}_q s + C_2(q+q^{-1}) = 0,$$

we get that $C_2 = \frac{1}{q+q^{-1}}\int_0^1 h(s)\widetilde{d}_q s$. thanks to $u(0) = \alpha u(\eta)$, in view of Lemma 2.7, we have

$$C_0 = -\frac{\alpha}{q+q^{-1}}\int_0^{q^2\eta}(\eta - q^{-1}s)(\eta - qs)h(s)\widetilde{d}_q s + \alpha C_0 + \frac{\alpha\eta^2}{q+q^{-1}}\int_0^1 h(s)\widetilde{d}_q s,$$

it follows that

$$C_0 = -\frac{\alpha}{(1-\alpha)(q+q^{-1})}\int_0^{q^2\eta}(\eta - q^{-1}s)(\eta - qs)h(s)\widetilde{d}_q s + \frac{\alpha\eta^2}{(1-\alpha)(q+q^{-1})}\int_0^1 h(s)\widetilde{d}_q s.$$

Thus, substituting C_0, C_2 into (6), we have

$$\begin{aligned}
u(t) &= -\frac{1}{q+q^{-1}}\int_0^{q^2 t}(t-q^{-1}s)(t-qs)h(s)\widetilde{d}_q s \\
&- \frac{\alpha}{(1-\alpha)(q+q^{-1})}\int_0^{q^2\eta}(\eta - q^{-1}s)(\eta - qs)h(s)\widetilde{d}_q s \\
&+ \frac{\alpha\eta^2}{(1-\alpha)(q+q^{-1})}\int_0^1 h(s)\widetilde{d}_q s + \frac{t^2}{q+q^{-1}}\int_0^1 h(s)\widetilde{d}_q s \\
&= \frac{1}{q+q^{-1}}[\int_0^1 t^2 h(s)\widetilde{d}_q s - \int_0^{q^2 t}(t-q^{-1}s)(t-qs)h(s)\widetilde{d}_q s] \\
&+ \frac{\alpha}{(1-\alpha)(q+q^{-1})}[\int_0^1 \eta^2 h(s)\widetilde{d}_q s - \int_0^{q^2\eta}(\eta - q^{-1}s)(\eta - qs)h(s)\widetilde{d}_q s]. \\
&= \int_0^1 K(t,s)h(s)\widetilde{d}_q s + \frac{\alpha}{1-\alpha}\int_0^1 K(\eta,s)h(s)\widetilde{d}_q s,
\end{aligned}$$

where $K(t,s)$ is given by (5). The proof is complete.

Lemma 3.2 *Let $0 < \eta < 1, 0 < \alpha < 1$, for $G(t,s)$ be given as (4), then we obtain the following results:*
(i) $0 \le G(t,s), \quad 0 \le t, s \le 1$;
(ii) $p(t)M(s) \le G(t,s) \le M(s), \quad 0 \le t, s \le 1$,
where

$$\begin{aligned} M(s) &= \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})}(1+\frac{\alpha q^2\eta}{1-\alpha}), \quad (7)\\ p(t) &= \frac{4q^4t^2(1-\alpha)}{(q+q^{-1})^2(1-\alpha+\alpha q^2\eta)}. \quad (8)\end{aligned}$$

Proof When $0 \le s \le q^2t < 1$, from (5), we have

$$(q+q^{-1})K(t,s) = (q+q^{-1})ts - s^2 = \frac{s(q^2t+t-qs)}{q} \ge \frac{s(s+t-qs)}{q} \ge 0.$$

When $0 \le q^2t \le s \le 1$, in view of (4) and (5), $G(t,s) \ge 0$ is easily checked. So

$$G(t,s) \ge 0, 0 \le t, s \le 1.$$

Next we will show that $G(t,s) \le M(s)$, $0 \le t, s \le 1$.
In fact, when $s \le q^2t$, for all $0 \le t, s \le 1$, it follows from (5) that

$$K(t,s) = \frac{s}{q+q^{-1}}[(q+q^{-1})t - s] \le \frac{s}{q+q^{-1}}[(q+q^{-1}) - s] \le \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})};$$

$$\begin{aligned} K(t,s) &= \frac{1}{q+q^{-1}}[(q+q^{-1})ts - s^2]\\ &= \frac{1}{q+q^{-1}}[t^2s^2 - t^2s^2 + (q+q^{-1})t^2s - (q+q^{-1})t^2s + (q+q^{-1})ts - s^2]\\ &= \frac{t^2}{q+q^{-1}}[(q+q^{-1})s - s^2] + \frac{s(1-t)}{q+q^{-1}}[(q+q^{-1})t - s - ts],\end{aligned}$$

thanks to

$$(q+q^{-1})t-s-ts = \frac{q^2t+t-qs-qts}{q} \ge \frac{s+t-qs-qts}{q} = \frac{s(1-q)+t(1-qs)}{q} \ge 0,$$

and $(q+q^{-1})^2 > 4$, we have

$$K(t,s) \ge \frac{t^2}{q+q^{-1}}[(q+q^{-1})s - s^2] \ge \frac{4t^2[(q+q^{-1})s - s^2]}{(q+q^{-1})^3}.$$

When $q^2t \leq s$, for all $0 \leq t, s \leq 1$, by (5) we have

$$K(t,s) = \frac{t^2}{q+q^{-1}} \leq \frac{1}{q+q^{-1}}\frac{s^2}{q^4} \leq \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})}.$$

Thanks to $\frac{4[(q+q^{-1})s-s^2]}{(q+q^{-1})^2} < 1$, we have

$$K(t,s) = \frac{t^2}{q+q^{-1}} \geq \frac{4t^2[(q+q^{-1})s-s^2]}{(q+q^{-1})^3}.$$

Comprehensive the above content, we get

$$K(t,s) \leq \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})}, 0 \leq t, s \leq 1; \tag{9}$$

$$K(t,s) \geq \frac{4t^2[(q+q^{-1})s-s^2]}{(q+q^{-1})^3}, 0 \leq t, s \leq 1. \tag{10}$$

On the other hand, by (5), we have

$$K(\eta,s) = \frac{1}{q+q^{-1}}\begin{cases} \eta^2, & 0 \leq q^2\eta \leq s \leq 1, \\ \eta^2 - (\eta - q^{-1}s)(\eta - qs), & 0 \leq s \leq q^2\eta < 1. \end{cases}$$

If $q^2\eta \leq s$, for $0 \leq \eta, s \leq 1$, we have

$$\frac{q^4(q+q^{-1})K(\eta,s)}{s(q+q^{-1}-s)} = \frac{q^4\eta^2}{s(q+q^{-1}-s)} \leq \frac{q^4\eta^2}{(q+q^{-1}-1)s} \leq \frac{q^2\eta}{q+q^{-1}-1} < q^2\eta.$$

If $s \leq q^2\eta$, for $0 \leq \eta, s \leq 1$, in view of $\frac{(q+q^{-1})\eta-s}{q+q^{-1}-s} < \eta$, we have

$$\frac{q^4(q+q^{-1})K(\eta,s)}{s(q+q^{-1}-s)} = \frac{q^4[(q+q^{-1})\eta-s]}{q+q^{-1}-s} < q^4\eta \leq q^2\eta.$$

So we obtain that

$$K(\eta,s) \leq \frac{q^2\eta s(q+q^{-1}-s)}{q^4(q+q^{-1})}, 0 \leq \eta, s \leq 1. \tag{11}$$

From (10), it follows that

$$K(\eta,s) \geq \frac{4\eta^2[(q+q^{-1})s-s^2]}{(q+q^{-1})^3}, 0 \leq \eta, s \leq 1. \tag{12}$$

Hence, by (4), (9) and (11), we get

$$\begin{aligned} G(t,s) &\leq \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})} + \frac{\alpha}{1-\alpha}\cdot\frac{q^2\eta s(q+q^{-1}-s)}{q^4(q+q^{-1})} \\ &= \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})}(1+\frac{\alpha q^2\eta}{1-\alpha}) \\ &= M(s). \end{aligned}$$

So $G(t,s) \leq M(s), 0 \leq t, s \leq 1$.

The end, by (4), (10) and (12), for $0 \leq t, s \leq 1$, we have

$$\begin{aligned} G(t,s) &\geq \frac{4t^2[(q+q^{-1})s - s^2]}{(q+q^{-1})^3} \\ &= \frac{4q^4t^2(1-\alpha)}{(q+q^{-1})^2(1-\alpha+\alpha q^2\eta)} \frac{s(q+q^{-1}-s)}{q^4(q+q^{-1})}(1+\frac{\alpha q^2\eta}{1-\alpha}) \\ &= p(t)M(s). \end{aligned}$$

The proof is complete.

Choose a cone K in E as follows:

$$K = \{u \in C^+[0,1] : u(t) \geq p(t)\|u\|, t \in [0,1]\}.$$

Suppose that u is a positive solution of the boundary value problem (1), then

$$u(t) = \lambda \int_0^1 G(t,s)g(s)f(u(s))\widetilde{d}_q s, t \in [0,1].$$

Define an operator $A : C^+[0,1] \to C^+[0,1]$ by

$$Au(t) = \lambda \int_0^1 G(t,s)g(s)f(u(s))\widetilde{d}_q s.$$

By the definition of operator A, a positive solution of the boundary value problem (1) is equivalent to a nonzero fixed point of A.

In view of the nonnegativeness of $G(t,s)$, $g(t)$ and $f(u)$, it is clear that $Au(t) \geq 0$, $t \in [0,1]$, $A : K \to C^+[0,1]$ is continuous for $u \in K$. Combining with Lemma 3.2, we have

$$\|Au\| = \max_{0\leq t\leq 1} |Au(t)| = \max_{0\leq t\leq 1} \lambda \int_0^1 G(t,s)g(s)f(u(s))\widetilde{d}_q s \leq \lambda \int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s.$$

On the other hand,

$$Au(t) \geq \lambda p(t) \int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s \geq p(t)\|Au\|.$$

Thus, we have $A(K) \subset K$.

Lemma 3.3 *Assume that $(H_1) - (H_3)$ hold, then the operator $A : K \to K$ is completely continuous.*

Proof It is clear that the operator $A : K \to K$ is continuous.

Let $\Omega \subset K$ be bounded, i.e., there exists a positive constant $M_1 > 0$ such that $\|u\| \leq M_1$ for all $u \in \Omega$. Let $L = \max_{\|u\| \leq M_1} \mid f(u(t)) \mid +1$. Then for $u \in \Omega$, from Lemma 3. 2, we have

$$\mid Au(t) \mid \leq \lambda \int_0^1 \mid G(t,s)g(s)f(u(s)) \mid \widetilde{d}_q s \leq \lambda L \int_0^1 M(s)g(s)\widetilde{d}_q s.$$

Hence, $A(\Omega)$ is bounded.

Now we show that A map bounded sets into equicontinuous sets of K. Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, $u \in \Omega$, where Ω is bounded set of K. Then we obtain

$$\begin{aligned}
&\mid Au(t_2) - Au(t_1) \mid \\
= \;& \frac{\lambda}{q+q^{-1}} \mid \int_0^1 t_2^2 g(s)f(u(s))\widetilde{d}_q s - \int_0^{q^2 t_2} (t_2 - q^{-1}s)(t_2 - qs)g(s)f(u(s))\widetilde{d}_q s \\
- \;& \int_0^1 t_1^2 g(s)f(u(s))\widetilde{d}_q s + \int_0^{q^2 t_1} (t_1 - q^{-1}s)(t_1 - qs)g(s)f(u(s))\widetilde{d}_q s \mid \\
\leq \;& \frac{\lambda L}{q+q^{-1}} [\mid t_2^2 - t_1^2 \mid \int_0^1 g(s)\widetilde{d}_q s + \int_{q^2 t_1}^{q^2 t_2} (t_2 - q^{-1}s)(t_2 - qs)g(s)\widetilde{d}_q s \\
+ \;& \int_0^{q^2 t_1} \mid (t_2 - q^{-1}s)(t_2 - qs) - (t_1 - q^{-1}s)(t_1 - qs) \mid g(s)\widetilde{d}_q s] \\
= \;& \frac{\lambda L}{q+q^{-1}} [(t_2^2 - t_1^2) \int_0^1 g(s)\widetilde{d}_q s + (t_2^2 - t_1^2) \int_0^{q^2 t_1} g(s)\widetilde{d}_q s \\
+ \;& (t_2 - t_1) \int_0^{q^2 t_1} s(q+q^{-1})g(s)\widetilde{d}_q s + \int_{q^2 t_1}^{q^2 t_2} (t_2 - q^{-1}s)(t_2 - qs)g(s)\widetilde{d}_q s].
\end{aligned}$$

Obviously the right side of the obove inequality tends to zero independently of $u \in \Omega$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $A : K \to K$ is a completely continuous mapping. The proof is complete.

4 Main results

In this section, we state and prove our main results. To be convenient, we introduce the following notations:

$$F_0 = \lim_{u \to 0^+} \sup \frac{f(u)}{u}; F_\infty = \lim_{u \to +\infty} \sup \frac{f(u)}{u}; f_0 = \lim_{u \to 0^+} \inf \frac{f(u)}{u}; f_\infty = \lim_{u \to +\infty} \inf \frac{f(u)}{u}.$$

$$N_1 = \int_0^1 M(s)g(s)\widetilde{d}_q s; N_2 = \int_0^1 p(s)M(s)g(s)\widetilde{d}_q s.$$

Theorem 4.1 *If there exists $\rho \in (0,1)$ such that $p(\rho)f_\infty N_2 > F_0 N_1$ holds, then for each*

$$\lambda \in ((p(\rho)f_\infty N_2)^{-1}, (F_0 N_1)^{-1}), \tag{13}$$

the boundary value problem (1) has at least one positive solution. Here we impose $(p(\rho)f_\infty N_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0 N_1)^{-1} = +\infty$ if $F_0 = 0$.

Proof Let λ satisfy (13) and $\varepsilon > 0$ be such that

$$(p(\rho)(f_\infty - \varepsilon)N_2)^{-1} \le \lambda \le ((F_0 + \varepsilon)N_1)^{-1}. \tag{14}$$

By the definition of F_0, we see that there exists $r_1 > 0$ such that

$$f(u) \le (F_0 + \varepsilon)u, 0 < u \le r_1. \tag{15}$$

So if $u \in K$ with $\|u\| = r_1$, then by (14) and (15), we have

$$\|Au\| \le \lambda \int_0^1 M(s)g(s)(F_0 + \varepsilon)r_1 \widetilde{d}_q s \le \lambda(F_0 + \varepsilon)r_1 N_1 \le r_1 = \|u\|.$$

Hence, if we choose $\Omega_1 = \{u \in C^+[0,1] : \|u\| < r_1\}$, then

$$\|Au\| \le \|u\|, u \in K \cap \partial\Omega_1. \tag{16}$$

Let $r_3 > 0$ be such that

$$f(u) \ge (f_\infty - \varepsilon)u, u \ge r_3. \tag{17}$$

If $u \in K$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, then by (14) and (17), we have

$$\begin{aligned}
\|Au\| &\ge \lambda \int_0^1 G(\rho, s)g(s)f(u(s))\widetilde{d}_q s \ge \lambda \int_0^1 p(\rho)M(s)g(s)(f_\infty - \varepsilon)u(s)\widetilde{d}_q s \\
&\ge \lambda p(\rho)(f_\infty - \varepsilon)\|u\| \int_0^1 p(s)M(s)g(s)\widetilde{d}_q s \\
&= \lambda p(\rho)(f_\infty - \varepsilon)\|u\| N_2 \ge \|u\|.
\end{aligned}$$

Thus, if we set $\Omega_2 = \{u \in C^+[0,1] : \|u\| < r_2\}$, then

$$\|Au\| \ge \|u\|, u \in K \cap \partial\Omega_2. \tag{18}$$

Now, from (16), (18) and Lemma 2.8, we guarantee that A has a fix point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, and clearly u is a positive solution of (1). The proof is complete.

Theorem 4.2 *If there exists $\rho \in (0,1)$ such that $p(\rho)f_0 N_2 > F_\infty N_1$ holds, then for each*

$$\lambda \in ((p(\rho)f_0 N_2)^{-1}, (F_\infty N_1)^{-1}), \tag{19}$$

the boundary value problem (1) has at least one positive solution. Here we impose $(p(\rho)f_0 N_2)^{-1} = 0$ if $f_0 = +\infty$ and $(F_\infty N_1)^{-1} = +\infty$ if $F_\infty = 0$.

Proof Let λ satisfy (19) and $\varepsilon > 0$ be such that

$$(p(\rho)(f_0 - \varepsilon)N_2)^{-1} \leq \lambda \leq ((F_\infty + \varepsilon)N_1)^{-1}. \tag{20}$$

From the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(u) \geq (f_0 - \varepsilon)u, 0 < u \leq r_1.$$

Further, if $u \in K$ with $\|u\| = r_1$, then similar to the second part of Theorem 4.1, we can obtain that $\|Au\| \geq \|u\|$. Thus, if we choose $\Omega_1 = \{u \in C^+[0,1] : \|u\| < r_1\}$, then

$$\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1. \tag{21}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \leq (F_\infty + \varepsilon)u, u \geq R_1. \tag{22}$$

We consider two cases:

Case 1. Suppose f is bounded. Then there exists some $M_2 > 0$, such that

$$f(u) \leq M_2, u \in (0, +\infty).$$

We define $r_3 = \max\{2r_1, \lambda M_2 N_1\}$, and $u \in K$ with $\|u\| = r_3$, then

$$\|Au\| \leq \lambda \int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s \leq \lambda M_2 \int_0^1 M(s)g(s)\widetilde{d}_q s = \lambda M_2 N_1 \leq r_3 = \|u\|.$$

Hence,

$$\|Au\| \leq \|u\|, u \in K_{r_3} = \{u \in K : \|u\| \leq r_3\}. \tag{23}$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, R_1\}$, such that

$$f(u) \leq f(r_4), 0 < u \leq r_4. \tag{24}$$

Let $u \in K$ with $\|u\| = r_4$. Then by (20) and (22), we have

$$\begin{aligned} \|Au\| &\leq \lambda \int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s \leq \lambda \int_0^1 M(s)g(s)(F_\infty + \varepsilon)\|u\|\widetilde{d}_q s \\ &= \lambda(F_\infty + \varepsilon)\|u\|N_1 \leq \|u\|. \end{aligned}$$

Thus, (23) is also true.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in C^+[0,1] : \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|Au\| \leq |u\|, u \in K \cap \partial\Omega_2. \tag{25}$$

Now that we obtain (21) and (25), it follows from Lemma 2.8 that A has a fixed-point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and it is clear u is a positive solution of (1). The proof is complete.

Theorem 4.3 *Suppose there exist* $\rho \in (0,1), r_2 > r_1$ *such that* $p(\rho) > \frac{r_1}{r_2}$, *and* f *satisfy*

$$\max_{0\le u\le r_2} f(u) \le \frac{r_2}{\lambda N_1}, \min_{p(\rho)r_1\le u\le r_1} f(u) \ge \frac{r_1}{\lambda p(\rho) N_2}.$$

Then the boundary value problem (1) has a positive solution $u \in K$ *with* $r_1 \le \|u\| \le r_2$.

Proof choose $\Omega_1 = \{u \in C^+[0,1] : \|u\| < r_1\}$, then for $u \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}\|Au\| &\ge \lambda \int_0^1 G(\rho,s)g(s)f(u(s))\widetilde{d}_q s\\ &\ge \lambda p(\rho)\int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s\\ &\ge \lambda p(\rho)\int_0^1 p(s)M(s)g(s)\min_{p(\rho)r_1\le u\le r_1} f(u(s))\widetilde{d}_q s\\ &\ge \lambda p(\rho)N_2\frac{r_1}{p(\rho)\lambda N_2} = r_1 = \|u\|.\end{aligned}$$

On the other hand, choose $\Omega_2 = \{u \in C^+[0,1] : \|u\| < r_2\}$, then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}\|Au\| &\le \lambda\int_0^1 M(s)g(s)f(u(s))\widetilde{d}_q s \le \lambda\int_0^1 M(s)g(s)\max_{0\le u\le r_2} f(u(s))\widetilde{d}_q s\\ &\le \lambda N_1\frac{r_2}{\lambda N_1} = r_2 = \|u\|.\end{aligned}$$

Thus, by Lemma 2.8, the boundary value problem (1) has a positive solution $u \in K$ with $r_1 \le \|u\| \le r_2$. The proof is complete.

For the reminder of the paper, we will need the following condition.
(H_4) $\sup_{r>0}\min_{u\in(p(\rho)r,r)} f(u) > 0$, where $\rho \in (0,1)$.
Denote

$$\lambda_1 = \sup_{r>0}\frac{r}{N_1\max_{0\le u\le r} f(u)}, \tag{26}$$

$$\lambda_2 = \inf_{r>0}\frac{r}{p(\rho)N_2\min_{p(\rho)r\le u\le r} f(u)}. \tag{27}$$

In view of the continuity of $f(u)$ and (H_4), we have $0 < \lambda_1 \le +\infty$ and $0 \le \lambda_2 < +\infty$.

Theorem 4.4 *Assume* (H_4) *holds. If* $f_0 = +\infty$ *and* $f_\infty = +\infty$, *then the boundary value problem (1) has at least two positive solutions for each* $\lambda \in (0, \lambda_1)$.

Proof Define

$$a(r) = \frac{r}{N_1 \max_{0 \le u \le r} f(u)}.$$

By the continuity of $f(u)$, $f_0 = +\infty$ and $f_\infty = +\infty$, we have that $a(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r\to 0} a(r) = \lim_{r\to+\infty} a(r) = 0.$$

By (26), there exists $r_0 \in (0, +\infty)$, such that $a(r_0) = \sup_{r>0} a(r) = \lambda_1$, then for $\lambda \in (0, \lambda_1)$, there exist constants $m_1, m_2 (0 < m_1 < r_0 < m_2 < +\infty)$ with

$$a(m_1) = a(m_2) = \lambda.$$

Thus,

$$f(u) \le \frac{m_1}{\lambda N_1}, u \in [0, m_1], \tag{28}$$

$$f(u) \le \frac{m_2}{\lambda N_1}, u \in [0, m_2]. \tag{29}$$

On the other hand, applying the conditions $f_0 = +\infty$ and $f_\infty = +\infty$, there exist constants $n_1, n_2 (0 < n_1 < m_1 < r_0 < m_2 < n_2 < +\infty)$ with

$$\frac{f(u)}{u} \ge \frac{1}{p^2(\rho)\lambda N_2}, u \in (0, n_1) \bigcup (p(\rho)n_2, +\infty), \tag{30}$$

then

$$\min_{p(\rho)n_1 \le u \le n_1} f(u) \ge \frac{n_1}{\lambda p(\rho) N_2}, \tag{31}$$

$$\min_{p(\rho)n_2 \le u \le n_2} f(u) \ge \frac{n_2}{\lambda p(\rho) N_2}. \tag{32}$$

By (28) and(31), (29) and (32), combining with Theorem 4.3, we can complete the proof.

Corollary 4.5 *Assume (H_4) holds. If $f_0 = +\infty$ or $f_\infty = +\infty$, then the boundary value problem (1) has at least one positive solution for each $\lambda \in (0, \lambda_1)$.*

Theorem 4.6 *Assume (H_4) holds. If $f_0 = 0$ and $f_\infty = 0$, then for each $\lambda \in (\lambda_2, +\infty)$ the boundary value problem (1) has at least two positive solutions.*

Proof Define

$$b(r) = \frac{r}{p(\rho) N_2 \min_{p(\rho) r \le u \le r} f(u)}.$$

By the continuity of $f(u)$, $f_0 = 0$ and $f_\infty = 0$, we easily see that $b(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r\to 0} b(r) = \lim_{r\to+\infty} b(r) = +\infty.$$

By (27), there exists $r_0 \in (0, +\infty)$, such that $b(r_0) = \inf_{r>0} b(r) = \lambda_2$. For $\lambda \in (\lambda_2, +\infty)$, there exist constants $n_3, n_4 (0 < n_3 < r_0 < n_4 < +\infty)$ with

$$b(n_3) = b(n_4) = \lambda.$$

Therefore,

$$f(u) \geq \frac{n_3}{\lambda p(\rho) N_2}, u \in [p(\rho) n_3, n_3],$$

$$f(u) \geq \frac{n_4}{\lambda p(\rho) N_2}, u \in [p(\rho) n_4, n_4].$$

On the other hand, using $f_0 = 0$, we know that there exists a constants $m_3 (0 < m_3 < n_3)$ with

$$\frac{f(u)}{u} \leq \frac{1}{\lambda N_1}, u \in (0, m_3),$$

$$\max_{0 \leq u \leq m_3} f(u) \leq \frac{m_3}{\lambda N_1}. \tag{33}$$

In view of $f_\infty = 0$, there exists a constant $m_4 \in (n_4, +\infty)$, such that

$$\frac{f(u)}{u} \leq \frac{1}{\lambda N_1}, u \in (m_4, +\infty).$$

Let $M_3 = \max_{0 \leq u \leq m_4} f(u)$ and $m_4 \geq \lambda N_1 M_3$. It is easily seen that

$$\max_{0 \leq u \leq m_4} f(u) \leq \frac{m_4}{\lambda N_1}. \tag{34}$$

By (33), (34) and Theorem 4.3, we can complete the proof.

Corollary 4.7 *Assume (H_4) holds. If $f_0 = 0$ or $f_\infty = 0$, then for each $\lambda \in (\lambda_2, +\infty)$ the boundary value problem (1) has at least one positive solution.*

By the above theorems, we can obtain the following results.

Corollary 4.8 *Assume (H_4) holds. If $f_0 = +\infty, f_\infty = l$, or if $f_0 = l, f_\infty = +\infty$, then for any $\lambda \in (0, (lN_1)^{-1})$ the boundary value problem (1) has at least one positive solution.*

Corollary 4.9 *Assume (H_4) holds. If $f_0 = 0, f_\infty = l$, or if $f_0 = l, f_\infty = 0$, then for any $\lambda \in ((p(\rho) l N_2)^{-1}, +\infty)$ the boundary value problem (1) has at least one positive solution.*

5 Nonexistence

In this section, we give some sufficient conditions for the nonexistence of positive solution to the problem (1).

Theorem 5.1 *Assume (H_4) holds. If $F_0 < +\infty$ and $F_\infty < +\infty$, then there exists a constant $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the boundary value problem (1) has no positive solution.*

Proof Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers l_1, l_2, r_1 and r_2, such that $r_1 < r_2$ and

$$f(u) \le l_1 u, u \in [0, r_1],$$

$$f(u) \le l_2 u, u \in [r_2, +\infty).$$

Let $M_4 = \max\{l_1, l_2, \max_{r_1 \le u \le r_2}\{\frac{f(u)}{u}\}\}$. Then we have

$$f(u) \le M_4 u, u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (M_4 N_1)^{-1}$. Since $Av(t) = v(t)$, for $t \in [0, 1]$,

$$\begin{aligned} \|v\| &= \|Av\| \le \lambda \int_0^1 M(s)g(s)f(v(s))\widetilde{d}_q s \\ &\le \lambda M_4 \|v\| \int_0^1 M(s)g(s)\widetilde{d}_q s \\ &< \lambda_0 M_4 \|v\| \int_0^1 M(s)g(s)\widetilde{d}_q s = \|v\|, \end{aligned}$$

which is a contradiction. Therefore, (1) has no positive solution. The proof is complete.

Theorem 5.2 *Assume (H_4) holds. If $f_0 > 0$ and $f_\infty > 0$, then there exists a constant $\lambda_0 > 0$ such that for all $\lambda \in (\lambda_0, +\infty)$, the boundary value problem (1) has no positive solution.*

Proof By $f_0 > 0$ and $f_\infty > 0$, we know that there exist positive numbers l_3, l_4, r_3 and r_4, such that $r_3 < r_4$ and

$$f(u) \ge l_3 u, u \in [0, r_3],$$

$$f(u) \ge l_4 u, u \in [r_4, +\infty).$$

Let $M_5 = \min\{l_3, l_4, \min_{r_3 \le u \le r_4}\{\frac{f(u)}{u}\}\} > 0$. Then we get

$$f(u) \ge M_5 u, u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1). We will show that this leads to a contradiction for $\lambda > \lambda_0 := (p(\rho)M_5N_2)^{-1}$. Since $Av(t) = v(t)$, for $t \in [0,1]$,

$$\begin{aligned} \|v\| &= \|Av\| \geq \lambda \int_0^1 G(\rho,s)g(s)f(v(s))\widetilde{d}_q s \\ &\geq p(\rho)\lambda \int_0^1 M(s)g(s)f(v(s))\widetilde{d}_q s \\ &\geq p(\rho)\lambda M_5\|v\| \int_0^1 p(s)M(s)g(s)\widetilde{d}_q s \\ &> p(\rho)\lambda M_5\|v\|N_2 = \|v\|, \end{aligned}$$

which is a contradiction. Thus, (1) has no positive solution. The proof is complete.

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References

[1] A. Strominger, Information in black hole radiation, Phys. Rev. Lett. 71(1993),3743-3746.

[2] D. Youm, q-deformed conformal quantum mechanics, Phys. Rev. D 62(2000).

[3] A. Lavagno, P.N. Swamy, q-deformed structures and nonextensive statistics: a comparative study, Phys. A 305(1-2)(2002),310-315.

[4] G. Boole, Calculus of Finite Differences, Chelsea Publishing Company, New York, 1957.

[5] T. Ernst, The different tongues of q-calculus, Proc. Est. Acad. Sci. 57(2)(2008),8199.

[6] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2002, Universitext.

[7] R. Koekoek, P.A. Lesky, R.F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, in: Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[8] A. Lavagno, G. Gervino, Quantum mechanics in q-deformed calculus, J. Phys.: Conf. Ser. 174(2009).

[9] Artur M.C. Brito da Cruz, Natália Martins, The q-symmetric variational calculus, Computers and Mathematics with Applications, 64(2012),2241-2250.

[10] Li Wang, Chengmin Hou, Application of differential transformation method to nonlinear q-symmetric damped systems, Mathematica Aeterna, 5(3)(2015),503-514.

[11] Yige Zhao, Shurong Sun, Zhenlai Han, Qiuping Li, Positive solutions to boundary value problems of nonlinear fractional differential equations, Abstract and Applied Analysis, 2011(2011), 16 pages.