

Separation by h -convex stochastic processes

Lysis González

Nelson Merentes

Escuela de Matemática, Facultad de Ciencias

Universidad Central de Venezuela

Caracas 1010, Venezuela.

Abstract

A separation theorem by h -convex stochastic processes is presented and a Hyers-Ulam type stability result is obtained as a corollary.

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1 Introduction.

In 1974, B. Nagy [14] considered additive stochastic processes and in 1980 K. Nikodem [16] introduced the notion of convexity on stochastic processes. This line of investigation had been followed by many authors, several types of convexity had been studied and inequalities, as Hermite-Hadamard, Jensen and others, had been proved [2, 9, 10, 12, 15, 16, 23].

Among the problems concerning stochastic processes, separation theorems were considered. A separation theorem is a theorem that gives conditions under which two functions can be separated by other with special characteristics. These theorems are important in mathematics and have interesting applications. Results of this type can be readed in [1, 3, 4, 6, 13, 15, 17, 18, 19, 20, 21, 22, 25].

A characterization of pairs of stochastic processes that can be separated by a quasiconvex one was given by D. Kotrys and K. Nikodem in [15] and analogous separation theorems by convex and strongly convex stochastic processes were proved by L. González, D. Kotrys and K. Nikodem in [7] (submitted for publication).

The aim of this paper is to prove a separation theorem by h -convex stochastic processes. The h -convexity was introduced, first for functions by S.

Varosanec in [24] where generalized convex, s -convex, Godunova-Levin functions and P -functions, then, the notion for stochastic processes were defined and studied by D. Barráez, L. González, N. Merentes and A. Moros in [2].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \rightarrow \mathbb{R}$ is a *random variable* if it is \mathcal{A} -measurable. A function $X : I \times \Omega \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a *stochastic process* if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

If $h : (0, 1) \rightarrow \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, we say that a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is an *h -convex stochastic process* if, for every $t_1, t_2 \in I$, $\lambda \in (0, 1)$, the following inequality is satisfied

$$X(\lambda t_1 + (1 - \lambda)t_2, \cdot) \leq h(\lambda)X(t_1, \cdot) + h(1 - \lambda)X(t_2, \cdot) \quad (a.e)$$

When the inequality holds for h equal to the identity, the stochastic processes is *convex*.

Many properties of h -convex and convex stochastic processes can be found in [2, 11, 16, 23].

Now, we would like to recall the definition of the essential infimum of a family of functions. We will use this notion in the proof of the separation theorem. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and \mathcal{F} be a collection of measurable functions $f : \Omega \rightarrow \mathbb{R}$, considering on \mathbb{R} the Borel σ -algebra. If \mathcal{F} is a countable set, then we may define the pointwise infimum of the functions from \mathcal{F} , which is measurable itself. If \mathcal{F} is uncountable, then the pointwise infimum need not be measurable and in this case, the essential infimum can be used. The *essential infimum* of \mathcal{F} , written as $\text{ess inf } \mathcal{F}$, if it exists, is a measurable function $f : \Omega \rightarrow \mathbb{R}$ satisfying the following two axioms:

- $f \leq g$ almost everywhere, for any $g \in \mathcal{F}$,
- if $h : \Omega \rightarrow \mathbb{R}$ is measurable and $h \leq g$ almost everywhere for every $g \in \mathcal{F}$, then $h \leq f$ almost everywhere.

It can be shown that for a σ -finite measure μ , the essential infimum of \mathcal{F} do exists, whenever \mathcal{F} is a family of measurable functions jointly bounded from below. For more details we refer the reader to [5].

2 Separation by h -convex processes.

In this section we present the main result. First, we recall that a function $h : [0, 1] \rightarrow \mathbb{R}$ is said to be *multiplicative* if $h(xy) = h(x)h(y)$ for all $x, y \in [0, 1]$

Note that if h is non-negative, multiplicative, $h \not\equiv 0$, for every $t \in [0, 1]$, we have

$$h(1) = h\left(t \frac{1}{t}\right) = h(t)h\left(\frac{1}{t}\right)$$

So, $h\left(\frac{1}{t}\right) = \frac{h(1)}{h(t)}$ and considering $t = 1$, we obtain $h(1) = 1$.

We resume this property in the following

Remark 1. *If h is multiplicative then it is non-negative and either $h \equiv 0$ or $h(1) = 1$.*

Our main result is the following sandwich theorem type, which allows us to separate stochastic processes by an h -convex one.

Theorem 2. *Let $h : (0, 1) \rightarrow \mathbb{R}$ be a positive multiplicative function and $X, Y : I \times \Omega \rightarrow \mathbb{R}$ be positive stochastic processes. There exists an h -convex stochastic process $Z : I \times \Omega \rightarrow \mathbb{R}$ such that*

$$X(t, \cdot) \leq Z(t, \cdot) \leq Y(t, \cdot) \quad (a.e)$$

for all $t \in I$, if and only if

$$X\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^n h(\lambda_i) Y(t_i, \cdot) \quad (a.e) \quad (1)$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ and $\lambda_1, \dots, \lambda_n \in I$ with $\lambda_1 + \dots + \lambda_n = 1$.

Proof. Fix $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$, $\lambda_1, \dots, \lambda_n \in [0, 1]$ with $\lambda_1 + \dots + \lambda_n = 1$. By the Jensen inequality for h -convex stochastic processes ([2]), we have

$$X\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq Z\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^n h(\lambda_i) Z(t_i, \cdot) \leq \sum_{i=1}^n h(\lambda_i) Y(t_i, \cdot) \quad (a.e)$$

Reciprocally, for every $t \in I$ we define the stochastic process $Z : I \times \Omega \rightarrow \mathbb{R}$ by

$$Z(t, \cdot) = \text{ess inf} \left\{ \sum_{i=1}^n h(\lambda_i) Y(t_i, \cdot) : n \in \mathbb{N}, t_1, \dots, t_n \in I, \lambda_1, \dots, \lambda_n \in [0, 1] \right. \\ \left. \text{such that } \lambda_1 + \dots + \lambda_n = 1 \text{ and } t = \lambda_1 t_1 + \dots + \lambda_n t_n \right\}.$$

Using the hypothesis and definition of essential infimum we have

$$X(t, \cdot) \leq Z(t, \cdot) \quad (a.e), \quad t \in I$$

Considering $n = 1$, $\lambda_1 = 1$, $t_1 = t$, also the following inequality holds

$$Z(t, \cdot) \leq Y(t, \cdot) \quad (a.e), \quad t \in I$$

Now, we will prove that Z is an h -convex stochastic process. For every $t_1, t_2 \in I$ and $\lambda \in [0, 1]$ consider $u_1, \dots, u_n \in I$, $v_1, \dots, v_m \in I$, $\alpha_1, \dots, \alpha_n \in [0, 1]$,

$\beta_1, \dots, \beta_m \in [0, 1]$ such that $\alpha_1 + \dots + \alpha_n = 1$, $\beta_1 + \dots + \beta_m = 1$ and $t_1 = \sum_{i=1}^n \alpha_i u_i$, $t_2 = \sum_{i=1}^m \beta_i v_i$.

Then,

$$\lambda t_1 + (1 - \lambda)t_2 = \sum_{i=1}^n \lambda \alpha_i u_i + \sum_{i=1}^m (1 - \lambda) \beta_i v_i = \sum_{i=1}^{n+m} \gamma_i w_i$$

where,

$$\gamma_i = \begin{cases} \lambda \alpha_i & \text{if } i = 1, \dots, n \\ (1 - \lambda) \beta_{i-n} & \text{if } i = n + 1, \dots, n + m \end{cases}$$

$$w_i = \begin{cases} u_i & \text{if } i = 1, \dots, n \\ v_{i-n} & \text{if } i = n + 1, \dots, n + m \end{cases}$$

Note that for every i , $\gamma_i \in [0, 1]$ and $\sum_{i=1}^{n+m} \gamma_i = 1$, hence, $\lambda t_1 + (1 - \lambda)t_2$ is a convex combination of $u_1, \dots, u_n; v_1, \dots, v_m$.

Because of the definition of Z , the multiplicity and non-negativity of h , the following inequality holds almost everywhere for every $n \in \mathbb{N}$, $u_1, \dots, u_n \in I$, $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \dots + \lambda_n = 1$, $t_1 = \lambda_1 u_1 + \dots + \lambda_n u_n$ and $m \in \mathbb{N}$, $\beta_1, \dots, \beta_m \in [0, 1]$ such that $\beta_1 + \dots + \beta_m = 1$, $t_2 = \beta_1 v_1 + \dots + \beta_m v_m$,

$$Z(\lambda t_1 + (1 - \lambda)t_2, \cdot) \leq h(\lambda) \sum_{i=1}^n h(\alpha_i) Y(u_i, \cdot) + h(1 - \lambda) \sum_{i=1}^m h(\beta_i) Y(v_i, \cdot) \quad (2)$$

Therefore, taking the essential infimum in the first term of the right hand side of (2), for all m we have

$$Z(\lambda t_1 + (1 - \lambda)t_2, \cdot) \leq h(\lambda) Z(t_1, \cdot) + h(1 - \lambda) \sum_{i=1}^m h(\beta_i) Y(v_i, \cdot) \quad (a.e.)$$

Hence, for all m

$$Z(\lambda t_1 + (1 - \lambda)t_2, \cdot) - h(\lambda) Z(t_1, \cdot) \leq h(1 - \lambda) \sum_{i=1}^m h(\beta_i) Y(v_i, \cdot) \quad (a.e.)$$

Using the second axiom of the definition of essential infimum,

$$Z(\lambda t_1 + (1 - \lambda)t_2, \cdot) - h(\lambda)Z(t_1, \cdot) \leq h(1 - \lambda)Z(t_2, \cdot) \quad (a.e)$$

wich implies that

$$Z(\lambda t_1 + (1 - \lambda)t_2, \cdot) \leq h(\lambda)Z(t_1, \cdot) + h(1 - \lambda)Z(t_2, \cdot) \quad (a.e)$$

□

As a corollary of the above separation theorem, the following Hyers-Ulam-type stability result for h -convex stochastic process. The classical Hyers-Ulam theorem can be readed in [8].

Let be $\epsilon > 0$. We say that the stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is $\epsilon - h$ -convex if

$$X\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^n h(\lambda_i)X(t_i, \cdot) + \epsilon \quad (a.e)$$

for every $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$, $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \dots + \lambda_n = 1$.

Corollary 3. *Let $h : (0, 1) \rightarrow \mathbb{R}$ be a positive function and $\epsilon > 0$. If a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is $\epsilon - h$ -convex, then exists an h -convex stochastic process Z such that $X(t, \cdot) - \epsilon \leq Z(t, \cdot) \leq X(t, \cdot)$ (a.e) for all $t \in I$.*

Proof. Let us define the stochastic process $Y(t, \cdot) = X(t, \cdot) - \epsilon$, $t \in I$.

By the $\epsilon - h$ -convexity of X , for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$, $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \dots + \lambda_n = 1$, we have

$$X\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^n h(\lambda_i)X(t_i, \cdot) + \epsilon \quad (a.e.)$$

that is,

$$Y\left(\sum_{i=1}^n \lambda_i t_i, \cdot\right) \leq \sum_{i=1}^n h(\lambda_i)X(t_i, \cdot) \quad (a.e.)$$

Using the theorem (2), exists an h -convex stochastic process $Z : I \times \Omega \rightarrow \mathbb{R}$ such that for every $t \in I$,

$$Y(t, \cdot) \leq Z(t, \cdot) \leq X(t, \cdot) \quad (a.e.)$$

□

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