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Structure of 3-Lie algebra J_{27}

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Abstract

In this paper, the 3-Lie algebra J_{27} is constructed by 2-cubic matrices over a field F with $chF = 0$, and the structure of it is studied. It is proved that the 3-Lie algebra J_{27} is solvable but non-nilpotent 3-Lie algebra with two dimensional center, and the concrete expression of all derivations and inner derivations is given.

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1 Introduction

We know that the 3-Lie algebra [1] has wide applications in mathematics and mathematical physics [2, 3]. The realization of 3-Lie algebras is always a hard task in the structural study of 3-Lie algebras. Authors constructed 3-Lie algebras by Lie algebras, associative algebras, pre-Lie algebras and linear functions in [4], and also realized 3-Lie algebras by commutative associative algebras and their derivations and involutions in [5]. In this paper, we continue to construct 3-Lie algebras by 2-cubic matrices [6]. In the following, we suppose that F is a field with characteristic zero, $\langle x_1, \cdots, x_s\rangle$ denotes the subspace generated by vectors $x_1, \cdots, x_s$, and in the multiplication table of a 3-Lie algebra, we omit the zero product of basis vectors.

2 N-cubic matrix

An N-order cubic matrix $A=(a_{ijk})$ (see [6]) over a field F is an ordered object which the elements with 3 indices, and the element in the position (i,j,k) is $(A)_{ijk}=a_{ijk}\in F$, $1\leq i,j,k\leq N$. Denote the set of all cubic matrix over a field F by Ω. Then Ω is an N^3-dimensional vector space over F with

$$A+B=(a_{ijk}+b_{ijk})\in\Omega,\quad \lambda A=(\lambda a_{ijk})\in\Omega,$$

for all $A=(a_{ijk}),B=(b_{ijk})\in\Omega,\ \lambda\in F$, that is,

$$(A+B)_{ijk}=a_{ijk}+b_{ijk},\quad (\lambda A)_{ijk}=\lambda a_{ijk}.$$

Denote E_{ijk} a cubic matrix with the element in the position (i,j,k) is 1 and elsewhere are zero. Then $\{E_{ijk},\ 1\leq i,j,k\leq N\}$ is a basis of Ω, and for every $A=(a_{ijk})\in\Omega$, $A=\sum\limits_{1\leq i,j,k\leq N}a_{ijk}E_{ijk},\ a_{ijk}\in F$.

For all $A=(a_{ijk}),B=(b_{ijk})\in\Omega$, define the multiplication $*_{27}$ in Ω by

$$(A*_{27}B)_{ijk}=\sum_{p,q=1}^{N}a_{qjk}b_{ipk},1\leq i,j,k\leq N,$$

then $(\Omega,*_{27})$ is an associative algebra, and in the basis $\{E_{ijk}|1\leq i,j,k\leq N\}$, we have

$$E_{ijk}*_{27}E_{lmn}=\delta_{kn}E_{ljk},\ 1\leq i,j,k,l,m,n\leq N,$$

where δ_{ij} is 1 in the cases $i=j$, and others are zero, $1\leq i,j\leq N$.

Define linear function $\langle\ \rangle_0:\Omega\to F$ by $\langle A\rangle_0=\sum\limits_{p,q,r=1}^{N}a_{pqr}$, Then we have

$$\langle A*_{27}B\rangle_0=\langle B*_{27}A\rangle_0. \tag{1}$$

So we define the multiplication $[,,]_{27}:\Omega\wedge\Omega\wedge\Omega\to\Omega$ as follows:

$$[A,B,C]_{27}=\langle A\rangle_0(B*_{27}C-C*_{27}B)$$
$$+\langle B\rangle_0(C*_{27}A-A*_{27}C)+\langle C\rangle_0(A*_{27}B-B*_{27}A). \tag{2}$$

3 The structure of J_{27}

First we give the following lemma.

Theorem 1[6] *The linear space Ω is a 3-Lie algebra in the multiplication $[,,]_{27}$, which is denoted by J_{27}.*

In the following we suppose $N=2$. For simplifying the multiplication of the 3-Lie algebra J_{27}, we need to find a new basis of Ω. Denote

$e_1=E_{111},e_2=E_{112}-E_{111},\ e_3=E_{111}-E_{121},\ e_4=E_{112}-E_{122}$,

$e_5=E_{211}-E_{111},\ e_6=E_{212}-E_{112},\ e_7=E_{211}-E_{221}-E_{111}+E_{121}$,

$e_8 = E_{212} - E_{222} - E_{112} + E_{122}$.
Then $\{e_1, \cdots, e_8\}$ is a basis of Ω.

Theorem 2 *The multiplication of* 3*-Lie algebra* J_{27} *in the basis* $\{e_1, \cdots, e_8\}$ *is as follows*

$$\begin{cases} [e_1, e_2, e_3] = e_3, \ [e_1, e_2, e_4] = -e_4, \\ [e_1, e_3, e_5] = e_7, \ [e_1, e_2, e_5] = -e_5, \\ [e_1, e_2, e_6] = e_6, \ [e_1, e_4, e_6] = e_8. \end{cases} \quad (3)$$

Proof The result follows from a direction computation according to the multiplication $*_{27}$ and Eq. (1). We omit the computing process.

Theorem 3 *The* 3*-Lie algebra* J_{27} *is solvable but non-nilpotent, and it satisfies that*

1) J_{27} *is an indecomposable* 3*-Lie algebra with two dimensional center* $\langle e_7, e_8\rangle$*, and derived algebra* $J_{27}^1 =< e_3, e_4, e_5, e_6, e_7, e_8 >$.

2) $H =< e_1, e_2, e_7, e_8 >$ *is a Cartan subalgebra of* J_{27}*, and the decomposition of* J_{27} *associate to* H *is*

$$J_{27} = H \dot{+} L_\alpha \dot{+} L_{-\alpha},$$

where $L_\alpha = \{x \in J_{27} \mid ad(h_1, h_2)x = \alpha(h_1, h_2)x, \forall h_1, h_2 \in H\} =< e_3, e_6 >$,
$L_{-\alpha} = \{x \in J_{27} \mid ad(h_1, h_2)x = -\alpha(h_1, h_2)x, \forall h_1, h_2 \in H\} =< e_4, e_5 >$,
and the linear function $\alpha : H \wedge H \to F$ *is defined by* $\alpha(e_1, e_2) = 1$ *and others are zero.*

Proof By the definition of $*_{27}$ and Eq.(3), the derived algebra $J_{27}^1 = [J_{27}, J_{27}, J_{27}] = < e_3, \cdots, e_8 >$, and the center of J_{27} is
$Z(J_{27}) = \{x \in J_{27} | [x, J_{27}, J_{27} = 0]\} =< e_7, e_8 >$.
Since J_{27} can not be decompose into the direct sum of proper ideals, J_{27} is an indecomposable 3-Lie algebra.

From Eq.(3), the derived series $J_{27}^{(1)} = [J_{27}, J_{27}, J_{27}] = J_{27}^1$,
$J_{27}^{(2)} = [J_{27}^{(1)}, J_{27}^{(1)}, J_{27}] = \{e_7, e_8\}$, $J_{27}^{(3)} = [J_{27}^{(2)}, J_{27}^{(2)}, J_{27}] = 0$.
We obtain that J_{27} is solvable. Thanks to the descend center series $J_{27}^{s+1} = [J_{27}^s, J_{27}, J_{27}] = J_{27}^1 \neq 0$ for all $s \geq 1$, the 3-Lie algebra J_{27} is non-nilpotent.

From the multiplication (3), $H = (e_1, e_4, e_5, e_8)$ is a nilpotent subalgebra of J_{27}, and if $[x, H, J_{27}] \subseteq H$ for $x \in J_{27}$, then $x \in H$. Therefore, H is a Cartan subalgebra. Define linear function $\alpha : H \wedge H \to F$ by $\alpha(e_1, e_2) = 1$ and others are zero. Then we have $L_0 = H$, $L_\alpha =< e_3, e_6 >$, $L_{-\alpha} =< e_4, e_5 >$. The proof is completed.

Now we study the inner derivation algebra adJ_{27}. For $e_i, e_j \in \Omega$, denote

$$ad(e_i, e_j)e_k = \sum_{l=1}^{8} a_{kl}^{ij} e_l, \text{ where } a_{kl}^{ij} = -a_{kl}^{ji} \in F.$$

Then the matrix form of $ad(e_i, e_j)$ in the basis $e_1, \cdots, e_8$ is $\sum\limits_{k,l=1}^{8} a_{kl}^{ij} E_{kl}$, where E_{kl} are 8×8-matrix units.

Theorem 4 *Let J_{27} be the 3-Lie algebra in Theorem 1. Then we have*

1) The dimension of inner derivation algebra adJ_{27} is 11, and $\{X_1 = E_{33} - E_{44} - E_{55} + E_{66},\ X_2 = -E_{23} + E_{57},\ X_3 = E_{24} + E_{68},\ X_4 = E_{25} - E_{37},\ X_5 = E_{26} + E_{48},\ X_6 = E_{13}, X_7 = E_{14},\ X_8 = E_{15},\ X_9 = E_{16}, X_{10} = E_{17},\ X_{11} = E_{18}\}$ is a basis of adJ_{27}, the multiplication in it is

$$\begin{cases} [X_1, X_2] = -X_2, [X_1, X_3] = X_3, [X_1, X_4] = X_4, [X_1, X_5] = -X_5 \\ [X_1, X_6] = -X_6, [X_1, X_7] = X_7, [X_1, X_8] = X_8, [X_1, X_9] = -X_9 \\ [X_2, X_8] = -X_{10}, [X_3, X_9] = -X_{11}, [X_4, X_6] = X_{10}, [X_5, X_7] = -X_{11}. \end{cases}$$

2) adJ_{27} is a solvable and indecomposable Lie algebra.

Proof By a direct computation according to Eq.(3) we have $ad(e_1, e_2) = E_{33} - E_{44} - E_{55} + E_{66}$, $ad(e_1, e_3) = -E_{23} + E_{57}$, $ad(e_1, e_4) = E_{24} + E_{68}$, $ad(e_1, e_5) = E_{25} - E_{37}$, $ad(e_1, e_6) = -E_{26} - E_{48}$, $ad(e_2, e_3) = E_{13}$, $ad(e_2, e_4) = -E_{14}$, $ad(e_2, e_5) = -E_{15}$, $ad(e_2, e_6) = E_{16}$, $ad(e_3, e_5) = E_{17}$, $ad(e_4, e_6) = E_{18}$, and others are zero.

Denote $\{X_1 = E_{33} - E_{44} - E_{55} + E_{66},\ X_2 = -E_{23} + E_{57},\ X_3 = E_{24} + E_{68},\ X_4 = E_{25} - E_{37},\ X_5 = E_{26} + E_{48},\ X_6 = E_{13}, X_7 = E_{14},\ X_8 = E_{15},\ X_9 = E_{16}, X_{10} = E_{17},\ X_{11} = E_{18}\}$, then $\{X_1, \cdots, X_{11}\}$ is a basis of adJ_{27}. From $[ad(e_i, e_j), ad(e_k, e_l)]= ad([e_i, e_j, e_k], e_l) + ad(e_k, [e_i, e_j, e_l])$.

From the above discussion, adJ_{27} is an indecomposable solvable Lie algebra. The proof is completed.

Theorem 5 *The dimension of derivation algebra $Der J_{27}$ is 18, and $Der J_{27}$ with a basis $\{X_1, \cdots, X_{18}\}$, where $X_{12} = E_{11} - E_{22} + E_{77} + E_{88}$, $X_{13} = E_{33} + E_{77}$, $X_{14} = E_{44} + E_{88}$, $X_{15} = E_{55} + E_{77}$, $X_{16} = E_{28}$, $X_{17} = E_{12}$, $X_{18} = E_{27}$, and X_i for $1 \le i \le 11$ are defined in Theorem 4. The multiplication in the basis is*

$$\begin{array}{llll}
[X_1, X_2] = -X_2, [X_1, X_3] = X_3, [X_2, X_8] = -X_{10}, & [X_{18}, X_{17}] = -X_{10}, \\
[X_1, X_6] = -X_6, [X_1, X_7] = X_7, [X_4, X_6] = X_{10}, & [X_{18}, X_{12}] = 2X_{18}, \\
[X_1, X_5] = X_5, [X_1, X_4] = X_4, [X_3, X_9] = -X_{11}, & [X_1, X_8] = X_8, \\
[X_2, X_{12}] = X_2, [X_3, X_{12}] = X_3, [X_5, X_{12}] = X_5, & [X_6, X_{12}] = -X_6, \\
[X_2, X_{13}] = X_2, [X_4, X_{15}] = X_4, [X_8, X_{12}] = -X_8, & [X_9, X_{12}] = -X_9, \\
[X_4, X_{12}] = X_4, [X_3, X_{14}] = X_3, [X_{16}, X_{12}] = 2X_{16}, & [X_1, X_9] = -X_9, \\
[X_6, X_{13}] = X_6, [X_{10}, X_{13}] = X_{10}, [X_{18}, X_{13}] = X_{18}, & [X_3, X_{17}] = -X_7, \\
[X_7, X_{14}] = X_7, [X_{11}, X_{14}] = X_{11}, [X_{16}, X_{14}] = X_{16}, & [X_5, X_7] = -X_{11}, \\
[X_8, X_{15}] = X_8, [X_{10}, X_{15}] = X_{10}, [X_{18}, X_{15}] = X_{18}, & \\
[X_2, X_{17}] = X_6, [X_4, X_{17}] = -X_8, [X_5, X_{17}] = -X_9, & \\
[X_{17}, X_{12}] = -2X_{17}, [X_7, X_{12}] = -X_7, [X_{16}, X_{17}] = -X_{11}. &
\end{array}$$

Proof The result follows from a direct computation.

Theorem 6 *The subalgebra $H =< X_1, X_{12}, X_{13}, X_{14}, X_{15} >$ is a Cartan subalgebra of $Der J_{27}$, and the decomposition of $Der J_{27}$ associate to H is*

$$Der J_{27} = H \dot{+} Der^1 J_{27} = H \dot{+} L_{\alpha_1} \dot{+} L_{\alpha_2} \dot{+} L_{\alpha_3} \dot{+} L_{\alpha_4} \dot{+} L_{\alpha_5} \dot{+} \sum_{i=1}^{8} L_{\beta_i},$$

where $\alpha_i, \beta_j \in H^*$, and the form of vectors of α_i, β_j, $1 \le i \le 5$, $1 \le j \le 8$

under the basis $X_1, X_{12}, X_{13}, X_{14}, X_{15}$ are as follows

$$\alpha_1 = (-1,-1,-1,0,0),\ \alpha_2 = (1,-1,0,-1,0),\ \alpha_3 = (1,-1,0,0,-1),$$

$$\alpha_4 = (-1,-1,0,0,0),\ \alpha_5 = (-1,1,-1,0,0),$$

$$\beta_1 = -\alpha_1+\alpha_2+\alpha_5,\ \beta_2 = -\alpha_1+\alpha_3+\alpha_5,\ \beta_3 = -\alpha_1+\alpha_4+\alpha_5, \beta_4 = \alpha_3+\alpha_5,$$

$$\beta_5 = -\alpha_1+\alpha_2+\alpha_4+\alpha_5,\ \beta_6 = -\alpha_1+\alpha_5,\ \beta_7 = \alpha_1+\alpha_3, \beta_8 = \alpha_2+\alpha_4.$$

The corresponding root subspace is $L_{\alpha_1} =< X_2 >$, $L_{\alpha_2} =< X_3 >$, $L_{\alpha_3} =< X_4 >$, $L_{\alpha_4} =< X_5 >$, $L_{\alpha_5} =< X_6 >$, $L_{\beta_1} =< X_7 >$, $L_{\beta_2} =< X_8 >$, $L_{\beta_3} =< X_9 >$, $L_{\beta_4} =< X_{10} >$, $L_{\beta_5} =< X_{11} >$, $L_{\beta_6} =< X_{17} >$, $L_{\beta_7} =< X_{18} >$, $L_{\beta_8} =< X_{16} >$.

Proof The result follows from Theorem 5.

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