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On Ostrowski's Type Inequalities via Convex, s-Convex and Quasi-Convex Stochastic Processes

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Abstract

In this paper we prove classical the Ostrowski's integral inequalities for stochastic processes. Several inequalities for Ostrowski's type via concave, convex, s-convex and quasi-convex stochastic processes are introduced. Some bounds for the difference between the integral mean of a stochastic process X defined on the interval $[a, b]$ and it is value in the midpoint $\frac{a+b}{2}$ are provided. Therefore, the inequalities are related to the left hand side of Hadamard inequality.

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Keywords: Ostrowski's integral inequality, convex stochastic processes, concave stochastic processes, s-convex stochastic processes, quasi-convex stochastic processes, left hand side of Hadamard inequality.

1 Introduction

The research of inequalities on stochastic processes is new. However, the study of convex stochastic processes began in 1974 when Nagy [27], applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation. Soon after, in 1980, K. Nikodem in [28], established some properties of convex stochastic processes and, in [29], introduced properties of quasi-convex stochastic processes. Later, D. Kotrys in 2011 presented in [24] an inequality of Hermite-Hadamard type for Jensen-convex stochastic processes. Nevertheless, in 2014 Set, M. Tomar and S. Maden in [25], presented the s-convex stochastic processes in the second sense and some well-known results concerning s-convex functions are extended to s-convex stochastic processes in the second sense. Also, they investigated a relation between s-convex stochastic processes in the second sense and convex stochastic processes. Recently, in 2015 E. Set, M. Tomar and S. Maden [33], obtained a similar result to the previous one but for s-convex functions. The previous result extended the concept of s-convex functions, which was introduced and improved by H. Hudzik and L. Maligranda [23], and S. S. Dragomir, S. Fitzpatrik [16] in 1998, to s-convex stochastic processes and obtained some results similar to the ones in s-convex functions. Also, during 2015, N. Merentes *et al.*, proved in [6] a generalization for h-convex stochastic processes. In particular, with the function h equals to the identity, a Hermite-Hadamard inequality type for convex stochastic processes were obtained in [6]. Another research in the same year was performed by L. Gonzalez, N. Merentes and M. Valera-López in [22] which establish some estimates of the left and right-hand side of the Hermite-Hadamard inequality for convex stochastic processes with convex or quasi-convex first second derivatives in absolute value establishing for the first time an estimate of error for this kind of inequalities in stochastic processes. Then, J. Materano, N. Merentes and M. Valera-Lopez, [26], tried several type inequalities Simpson giving error limits with Simpson's rule through Peano type and results of the modern theory of inequalities using s-convex and quasi-convex stochastic processes in terms up to second derivative.

The Ostrowski's inequality was performed in 1938. In [30], A. Ostrowski proved the following integral inequality:

Let $f : I \to \mathbf{R}$ be a differentiable mapping on I and let $a,\, b \in I$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $||f'||_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq ||f'||_\infty (b-a)\left[\frac{1}{4} + \frac{\left(t - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \tag{1}$$

for all $t \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Many research has been done about the Ostrowski's inequality. During the years 1997 and 1998, S. S. Dragomir and S. Wang in [17]-[19] extended the inequality (1) for absolutely continuous functions and applied the extended result to numerical quadrature rules and to the estimation of error bounds for special means. The reader can be found other similar results in [8],[11],[12][34].

Further, between 1999 and 2001, S. S. Dragomir in [10]-[12], extended the result (1) to incorporate mappings of bounded variation, Lipschitzian mapping and monotonous mapping, respectively. In these papers, he apply the results in obtaining a Riemann's type quadrature formula for this class of mapping and give some applications for Euler's Beta function.

A extension of the result (1), it has been made considering n-times differentiable mappings on an interior point $t \in (a, b)$, (see [7] and [5]). Aditionally, in [9] was establish some generalizations of Ostrowski inequality for Lipschitz functions and functions of bounded variation. Also, was provide in [20] [31] and [32], generalizations and improvements of a variety of some results for Ostrowski and Simpson inequalities.

Furthermore, in 2001 S.S. Dragomir in [13]-[15] refined the inequality (1) by considering an interval $[a, b]$ with a multiple number of subdivisions.

Recently, several generalizations of the Ostrowski's integral inequality has been the subject of intensive research. In particular, many generalizations, improvements and applications for the Ostrowski's inequality can be found in the literature. In 2009, M. Alomari *et al.* [4], established some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense and, in 2010, M. Alomari and M. Darus, obtained inequalities for differentiable convex and quasi-convex mappings ([2] and [3], respectively), which are connected with Ostrowski's inequality. Additionally, M. Alomari in 2012, [1], obtained a companion inequality of Ostrowski's type using Grss' result and then discussed its applications for a composite quadrature rule and for probability density functions.

The aim of this paper is extend the results by M. Alomari [1], M. Alomari and M. Darus [3]-[4] on Ostrowski's type inequalities for function to concave, convex, s-convex and quasi-convex stochastic processes. The proofs follow from the standard arguments and a Montgomery-type equality.

2 On the Ostrowski's Inequality

We start off proving the Ostrowski's inequality for stochastic processes.

Theorem 2.1 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable on $[a, b]$, where $a, b \in I$ with $a < b$, and X' is bounded, i.e., $||X'||_\infty := \sup |X'(t, \cdot)| < \infty$. If X is concave on I, then the following inequality holds almost everywhere:*

$$\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right| \leq \| X' \|_\infty (b-a) \left[\frac{1}{4} + \frac{\left(t_0 - \frac{a+b}{2} \right)^2}{(b-a)^2} \right], \quad (2)$$

for all $x \in I$

Proof.

Because the process $X(t,\cdot)$ is mean-square differentiable on I and concave on I, then for any t, $y \in I$,

$$\begin{aligned} \frac{X(t_0,\cdot)-X(t,\cdot)}{t_0-t} &\leq X'(t,\cdot), \\ X(t_0,\cdot) &\leq X(t,\cdot)+(t-t_0)X'(t,\cdot), \qquad \textit{(a.e).} \end{aligned}$$

it follows that integrating both sides over $[a,b]$, with respect to u, we get

$$(b-a)X(t_0,\cdot) \leq \int_a^b X(u,\cdot)du + \int_a^b (u-t_0)X'(u,\cdot)du, \qquad \textit{(a.e),}$$

which is equivalent to write

$$X(t_0,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \leq \frac{1}{b-a}\int_a^b (u-t_0)X'(u,\cdot)du, \qquad \textit{(a.e).}$$

Because X' is bounded,

$$\begin{aligned} \left|X(t_0,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \left|\frac{1}{b-a}\int_a^b X'(u,\cdot)(u-t_0)du\right| \\ &\leq \frac{1}{b-a}\int_a^b |X'(u,\cdot)||u-t_0|du \\ &\leq \frac{1}{b-a}\sup_{u\in(a,b)}|X'(u,\cdot)|\int_a^b (u-t_0)dt \\ &\leq \frac{1}{b-a}||X'||_\infty \left[\int_a^{t_0}(t_0-u)du + \int_{t_0}^b (u-t_0)du\right] \\ &\leq \frac{1}{b-a}||X'||_\infty \left[\frac{1}{2}(t_0-a)^2+\frac{1}{2}(b-t_0)^2\right] \\ &= ||X'||_\infty (b-a)\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \qquad \textit{(a.e),} \end{aligned}$$

which completes the proof.

Another Ostrowski's type inequality, which gives the weighted difference between the mean-square integrals of a stochastic process X and its first derivative, is considered bellow:

Theorem 2.2 *Let $X : I \times \Omega \to \mathbf{R}$ be a twice mean-square differentiable stochastic process on I, where $a, b \in I$ with $a < b$. Assume that X and X'*

are concave on (a,b)*. If* $||X'||_\infty = \sup_{t\in(a,b)} |X'(t,\cdot)| < \infty$ *and* $||X''||_\infty = \sup_{t\in(a,b)} |X''(t,\cdot)| < \infty$*, then the following inequality takes place almost everywhere:*

$$\left|X'(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a} - X(t_0,\cdot)\right| \tag{3}$$
$$+||X'||_\infty\left[\frac{(t_0-a)^2+(b-t_0)^2}{2(b-a)}\right] + ||X''||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right],$$

where $t_0, t \in (a,b)$*.*

Proof.

Since X is concave on $[a,b]$ and X' is concave on (a,b), for any $s,u \in (a,b)$

$$X(t_0,\cdot) \leq X(u,\cdot) + (t_0-u)X'(u,\cdot), \qquad (a.e), \tag{4}$$

and

$$X'(t,\cdot) \leq X'(s,\cdot) + (t-s)X''(s,\cdot), \qquad (a.e). \tag{5}$$

Integrating both sides of (4) over $[a,b]$, with respect to u, and (5) over $[a,b]$ with respect to s:

$$(b-a)X(t_0,\cdot) \leq \int_a^b X(u,\cdot)du + \int_a^b (t_0-u)X'(u,\cdot)du, \qquad (a.e) \tag{6}$$

and

$$(b-a)X'(t,\cdot) \leq \int_a^b X'(s,\cdot)ds + \int_a^b (t-s)X''(s,\cdot)ds, \qquad (a.e) \tag{7}$$

Adding (6) and (7), we get

$$\begin{aligned}(b-a)X'(t,\cdot) &- \int_a^b X(u,\cdot)du \\ &\leq X(b,\cdot) - X(a,\cdot) - (b-a)X(t_0,\cdot) \\ &\quad + \int_a^b (t_0-u)X'(u,\cdot)du + \int_a^b (t-s)X''(s,\cdot)ds. \qquad (a.e)\end{aligned} \tag{8}$$

Thus, since $||X'||_\infty = \sup_{t\in(a,b)} |X'(t,\cdot)| < \infty$ and $||X''||_\infty = \sup_{t\in(a,b)} |X''(t,\cdot)| < \infty$, so

$$\begin{aligned}\left|X'(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(t_0,\cdot)\right|\\ &+\frac{1}{b-a}\int_a^b |t_0-u||X'(u,\cdot)|du\\ &+\frac{1}{b-a}\int_a^b |t-s||X''(s,\cdot)|ds\\ &\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(t_0,\cdot)\right|\\ &+\frac{||X'||_\infty}{b-a}\int_a^b |t_0-u|du\\ &+\frac{||X''||_\infty}{b-a}\int_a^b |t-s|ds\\ &\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(t_0,\cdot)\right|\\ &+||X'||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right]\\ &+||X''||_\infty\left[\frac{(t_0-a)^2+(b-t_0)^2}{2(b-a)}\right], \quad (a.e).\end{aligned}$$

for $t, t_0 \in (a,b)$, and proof is completed.

Remark 2.3 *In the inequality (3) one can see that when t, $t_0 \to b^-$:*

$$\begin{aligned}\left|X(b,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(b,\cdot)\right| \\ &+\frac{(b-a)}{2}[||X'||_\infty+||X''||_\infty], \quad (a.e).\end{aligned} \tag{9}$$

Similarly, for $t, t_0 \to a^+$, then we have:

$$\begin{aligned}\left|X'(a,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(a,\cdot)\right| \\ &+\frac{(b-a)}{2}[||X'||_\infty+||X''||_\infty], \quad (a.e)\end{aligned} \tag{10}$$

Also, for $t=t_0=\dfrac{a+b}{2}$,

$$\left|X'\left(\frac{a+b}{2},\cdot\right) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a} - X\left(\frac{a+b}{2},\cdot\right)\right| + \frac{(b-a)}{4}[||X'||_\infty + ||X''||_\infty], \quad (a.e). \tag{11}$$

In the following result we propose an error estimation for the first derivative.

Theorem 2.4 *Considering the assumption in Theorem 2.2. Then the following inequality holds almost everywhere:*

$$\left|X'(t,\cdot) - \frac{X(b,\cdot)-X(a,\cdot)}{b-a}\right| \leq ||X'||_\infty \frac{(b-a)^2}{3} + ||X''||_\infty \left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right], \tag{12}$$

for all $t \in (a,b)$.

Proof.

Integrating the both sides of (8) over $[a,b]$ with respect to t_0

$$\begin{aligned}(b-a)^2 X'(t,\cdot) &- (b-a)\int_a^b X(u,\cdot)du \\ &\leq (b-a)[X(b,\cdot)-X(a,\cdot)] - (b-a)\int_a^b X(t_0,\cdot)dt_0 \\ &+ \int_a^b\int_a^b (t_0-u)X'(u,\cdot)dudt_0 + (b-a)\int_a^b (t-s)X''(s,\cdot)ds, \qquad (a.e),\end{aligned}$$

and writing

$$\begin{aligned}X'(t,\cdot) &- \frac{1}{b-a}\int_a^b X(u,\cdot)du \\ &\leq \frac{X(b,\cdot)-X(a,\cdot)}{b-a} - \frac{1}{b-a}\int_a^b X(t_0,\cdot)dt_0 \\ &- \frac{1}{(b-a)^2}\int_a^b\int_a^b (t_0-u)X'(u,\cdot)dudt_0 \\ &+ \frac{1}{b-a}\int_a^b (t-s)X''(s,\cdot)ds, \qquad (a.e),\end{aligned}$$

but then

$$X'(t,\cdot)-\frac{X(b,\cdot)-X(a,\cdot)}{b-a} \le \frac{1}{b-a}\int_a^b (t-s)X''(s,\cdot)ds - \frac{1}{(b-a)^2}\int_a^b\int_a^b (t_0-u)X'(u,\cdot)dudt_0, \quad (a.e).$$

Since X' and X'' are bounded,

$$\begin{aligned}
&\left|X'(t,\cdot)-\frac{X(b,\cdot)-X(a,\cdot)}{b-a}\right| \\
&\le \frac{1}{b-a}\int_a^b |t-s||X''(s,\cdot)|ds + \frac{1}{(b-a)^2}\int_a^b\int_a^b |t_0-u||X'(u,\cdot)|dudt_0 \\
&\le \frac{||X''||_\infty}{b-a}\int_a^b |t-s|ds + \frac{||X'||_\infty}{(b-a)^2}\int_a^b\int_a^b |t_0-u|dudt_0 \\
&\le ||X''||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right] + ||X'||_\infty\int_a^b \frac{(t_0-a)^2+(b-t_0)^2}{2(b-a)}dt_0 \\
&= ||X''||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right] + ||X'||_\infty\frac{(b-a)^2}{3}, \qquad (a.e),
\end{aligned}$$

for all $t \in (a,b)$.

Remark 2.5 *In the inequality (12), choosing $t = b$,*

$$\left|X'(b,\cdot)-\frac{X(b,\cdot)-X(a,\cdot)}{b-a}\right| \le ||X'||_\infty\frac{(b-a)^2}{3} + ||X''||_\infty\frac{(b-a)}{2}, \qquad (a.e),$$

and choosing $t = \frac{b+a}{2}$,

$$\left|X'\left(\frac{a+b}{2},\cdot\right)-\frac{X(b,\cdot)-X(a,\cdot)}{b-a}\right| \le ||X'||_\infty\frac{(b-a)^2}{3}+||X''||_\infty\frac{(b-a)}{4}, \qquad (a.e).$$

Using the same technique in the proof of Theorem 2.2, we can generalize the inequality (3) for n-times mean-square differentiable stochastic process as follows:

Corollary 2.6 *Let $X : I\times\Omega \to \mathbf{R}$ be $n-$times a mean-square differentiables stochastic process on I, where $a,b \in I$ with $a < b$. Assume that X and $X^{(n-1)}$ are concave stochastic process, $n \ge 2$ on (a,b). If $||X'||_\infty = \sup_{t\in(a,b)}|X'(t,\cdot)| < \infty$ and $||X^{(n)}||_\infty = \sup_{t\in(a,b)}|X^{(n)}(t,\cdot)| < \infty$, then the following inequality shows up almost everywhere:*

$$\left|X^{(n-1)}(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\leq\left|\frac{X^{(n-2)}(b,\cdot)-X^{(n-2)}(a,\cdot)}{b-a}-X(t_0,\cdot)\right|$$
$$+||X'||_\infty\left[\frac{(t_0-a)^2+(b-t_0)^2}{2(b-a)}\right]+||X^{(n)}||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right],$$

for $n\geq 2$ *and* $t,t_0\in(a,b)$.

Proof.

Since X is concave in $[a,b]$ and $X^{(n-1)}$ is concave in (a,b) then for any $s,t\in(a,b)$

$$X(t_0,\cdot)\leq X(u,\cdot)+(t_0-u)X'(u,\cdot)\qquad (a.e), \tag{13}$$

and

$$X^{(n-1)}(t,\cdot)\leq X^{(n-1)}(s,\cdot)+(t-s)X^{(n)}(s,\cdot)\qquad (a.e). \tag{14}$$

Integrating both inequalities (13) and (14) over $[a,b]$ respect with u and s, respectively, we get

$$(b-a)X(t_0,\cdot)\leq\int_a^b X(u,\cdot)du+\int_a^b(t_0-u)X'(u,\cdot)du,\qquad (a.e), \tag{15}$$

and

$$(b-a)X^{(n-1)}(t,\cdot)\leq\int_a^b X^{(n-1)}(s,\cdot)ds+\int_a^b(t-s)X^{(n)}(s,\cdot)ds,\qquad (a.e). \tag{16}$$

Now, we up (15) y (16),

$$(b-a)X^{(n-1)}(t,\cdot)-\int_a^b X(u,\cdot)du\leq X^{(n-2)}(b,\cdot)-X^{(n-2)}(a,\cdot)$$
$$-(b-a)X(t_0,\cdot)+\int_a^b(t_0-u)X'(u,\cdot)du+\int_a^b(t-s)X^{(n)}(s,\cdot)ds,\quad (a.e).$$

Whence,

$$
\begin{aligned}
\left|X^{(n-1)}(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \left|\frac{X^{(n-2)}(b,\cdot)-X^{(n-2)}(a,\cdot)}{b-a}-X(t_0,\cdot)\right| \\
&+\frac{1}{b-a}\int_a^b |t_0-u||X'(u,\cdot)|du \\
&+\frac{1}{b-a}\int_a^b |t-s||X^{(n)}(s,\cdot)|ds \\
&\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(t_0,\cdot)\right| \\
&+\frac{||X'||_\infty}{b-a}\int_a^b |t_0-u|du \\
&+\frac{||X^{(n)}||_\infty}{b-a}\int_a^b |t-s|ds \\
&\leq \left|\frac{X(b,\cdot)-X(a,\cdot)}{b-a}-X(t_0,\cdot)\right| \\
&+||X'||_\infty\left[\frac{(t_0-a)^2+(b-t_0)^2}{2(b-a)}\right] \\
&+||X^{(n)}||_\infty\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right],
\end{aligned}
$$

almost everywhere for t, $t_0 \in (a,b)$.

Next result gives an Ostrowski type inequality involving product of two stochastic processes.

Theorem 2.7 *Let $X, Y : I \times \Omega \to \mathbf{R}_+$ be two bounded mean-square differentiable stochastic process on I such that X', Y' are mean-square integrables where $a, b \in I$ with $a < b$ whose derivatives $X'Y'$ are bounded. If X is concave and $M = \max_{t\in(a,b)}\{|X(t,\cdot)|, |X'(t,\cdot)|, |Y(t,\cdot)|, |Y'(t,\cdot)|\}$ then:*

$$
\begin{aligned}
&\left|\int_a^b X(t,\cdot)Y(t,\cdot)dt-(b-a)X(u,\cdot)Y(s,\cdot)\right| \\
&\leq M\left[\frac{(u-a)^2+(b-u)^2}{2}+\frac{(s-a)^2+(b-s)}{2}+\frac{b^3-a^3}{3}\right. \\
&\left.+\frac{b^2-a^2}{2}(u+s)+us(b-a)+\left\{\begin{array}{l}\frac{(u-s)^3}{3},\ s<u\\ \frac{(s-u)^3}{3},\ u\leq s\end{array}\right.\right], \quad (a.e),
\end{aligned}
$$

where $t, s, u \in [a,b]$.

Proof.

Since X and Y are concave stochastic process on I, then for any $t, s, u \in (a, b)$,

$$X(t, \cdot) \leq X(u, \cdot) + (t-u)X'(u, \cdot), \qquad \textit{(a.e)},$$

and

$$Y(t, \cdot) \leq Y(s, \cdot) + (t-s)Y'(s, \cdot), \qquad \textit{(a.e)}.$$

Multiplying the above inequalities,

$$\begin{aligned} X(t,\cdot)Y(t,\cdot) &\leq [X(u,\cdot)+(t-u)X'(u,\cdot)]\,[Y(s,\cdot)+(t-s)Y'(s,\cdot)] \\ &= X(u,\cdot)Y(s,\cdot)+(t-s)Y'(s,\cdot)X(u,\cdot) \\ &\quad +(t-u)X'(u,\cdot)Y(s,\cdot)+(t-u)(t-s)X'(u,\cdot)Y'(s,\cdot), \quad \textit{(a.e)}. \end{aligned}$$

Integrating both sides over $[a, b]$, with respect to t, we get

$$\begin{aligned} \int_a^b X(t,\cdot)Y(t,\cdot)dt &\leq (b-a)X(u,\cdot)Y(s,\cdot) + X'(u,\cdot)Y(s,\cdot)\int_a^b (t-u)dt \\ &+X(u,\cdot)Y'(s,\cdot)\int_a^b (t-s)dt + X'(u,\cdot)Y(s,\cdot)\int_a^b (t-u)(t-s)dt, \quad \textit{(a.e)}, \end{aligned}$$

or better

$$\begin{aligned} &\int_a^b X(t,\cdot)Y(t,\cdot)dt - (b-a)X(u,\cdot)Y(s,\cdot) \leq X'(u,\cdot)Y(s,\cdot)\int_a^b (t-u)dt \\ &+X(u,\cdot)Y'(s,\cdot)\int_a^b (t-s)dt + X'(u,\cdot)Y'(s,\cdot)\int_a^b (t-u)(t-s)dt, \quad \textit{(a.e)}. \end{aligned}$$

In this way:

$$\begin{aligned} &\left|\int_a^b X(t,\cdot)Y(t,\cdot)dt - (b-a)X(u,\cdot)Y(s,\cdot)\right| \\ &\quad \leq M\left[\int_a^b |t-u|dt + \int_a^b |t-s|dt + \int_a^b |t-u||t-s|dt\right] \\ &\quad = M\left[\frac{(u-a)^2+(b-u)^2}{2} + \frac{(s-a)^2-(b-s)^2}{2}\right. \\ &\qquad +\frac{u^3-s^3}{3} - 2\left(\begin{cases} 0, \; s \leq u \\ 1, \; u \leq s \end{cases}\right) us^2 + \frac{b^3-a^3}{3} - \frac{b^2-a^2}{2}u \\ &\qquad -\frac{b^2-a^2}{2}s - u^2 s + us^2 + usb - usa + 2\left(\begin{cases} 0, \; s < u \\ 1, \; u \leq s \end{cases}\right) u^2 s \end{aligned}$$

$$
-\frac{2}{3}\left(\left\{\begin{array}{l} 0,\ s<u \\ \\ 1,\ u\leq s \end{array}\right)u^3+\frac{2}{3}\left(\left\{\begin{array}{l} 0,\ s<u \\ \\ 1,\ u\leq s \end{array}\right)s^3\right]
$$

$$
= M\left[\frac{(u-a)^2+(b-u)^2}{2}+\frac{(s-a)^2+(b-s)^2}{2}+\frac{b^3-a^3}{3}\right.
$$

$$
\left.-\frac{b^2-a^2}{2}(u+s)+us(b-a)+\left\{\begin{array}{l} \dfrac{(u-s)^3}{3},\ s<u \\ \\ \dfrac{(s-u)^3}{3},\ u\leq s \end{array}\right.\right],
$$

almost everywhere.

3 Ostrowski's Type via Convex Stochastic Process

We shall introduce some inequalities of Ostrowski's type via convex stochastic process in the second sense. For this, we need to use the following lemma:

Lemma 3.1 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I where $a, b \in I$ with $a < b$. If X' is mean-square integrable, then the following equality is true almost everywhere:*

$$
X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du=(b-a)\int_0^1 p(y)X'(ya+(1-y)b,\cdot)dy,
$$

for each $y \in [0, 1]$, where

$$
p(y)=\left\{\begin{array}{ll} y, & y\in\left[0,\dfrac{b-t}{b-a}\right], \\ \\ y-1, & y\in\left(\dfrac{b-t}{b-a},1\right], \end{array}\right.
$$

for all $t \in [a, b]$.

Proof.

Integrating by parts and considering $\beta = \dfrac{b-t}{b-a}$, we have

$$
\begin{aligned}
I &= \int_0^1 p(y)X'(ya+(1-y)b,\cdot)dy \\
&= \int_0^\beta yX'(ya+(1-y)b,\cdot)dy+\int_\beta^1 (1-y)X'(ya+(1-y)b,\cdot)dy
\end{aligned}
$$

$$
\begin{aligned}
&= y\frac{X(ya+(1-y)b,\cdot)}{b-a}\bigg|_0^{\beta} - \int_0^{\beta}\frac{X(ya+(1-y)b,\cdot)}{a-b}dy \\
&\qquad +(1-y)\frac{X(ya+(1-y)b,\cdot)}{b-a}\bigg|_{\beta}^{1} - \int_{\beta}^{1}\frac{X(ya+(1-y)b,\cdot)}{a-b}dy \\
&= \frac{b-t}{(b-a)^2}X(t,\cdot) - \int_0^{\beta}\frac{X(ya+(1-y)b,\cdot)}{a-b}dy \\
&\qquad +\frac{y-a}{(b-a)^2}X(y,\cdot) - \int_{\beta}^{1}\frac{X(ya+(1-y)b,\cdot)}{a-b}dy \\
&= \frac{1}{b-a}X(t,\cdot) - \int_0^{1}\frac{X(ya+(1-y)b,\cdot)}{a-b}dy \\
&= \frac{1}{b-a}X(t,\cdot) - \int_a^{b} X(u,\cdot)du, \qquad \textit{(a.e)}.
\end{aligned}
$$

Then, multiplying by $(b-a)$ the above integral, gives the desired representation.

We will proceed with the proof of the Ostrowski's inequality when the magnitude of the first derivative is convex.

Theorem 3.2 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I, such that X' is mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|$ is convex on $[a, b]$, then the following inequality holds almost everywhere:*

$$
\begin{aligned}
&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\quad \le \frac{b-a}{6}\left[\left(4\left(\frac{b-t}{b-a}\right)^3 - 3\left(\frac{b-t}{b-a}\right)^2 + 1\right)|X'(a,\cdot)|\right. \\
&\qquad \left. + \left(9\left(\frac{b-t}{b-a}\right)^2 - 4\left(\frac{b-t}{b-a}\right)^3 - 6\left(\frac{b-t}{b-a}\right) + 2\right)|X'(b,\cdot)|\right],
\end{aligned}
$$

for each $t \in [a, b]$.

Proof.

Applying absolute valued to Lemma 3.1 using the triangle inequality, the fact that $|X'|$ is convex and taking $\beta = \dfrac{b-t}{b-a}$, we have:

$$
\begin{aligned}
&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\quad \le (b-a)\int_0^{\beta} y|X'(ya+(1-y)b,\cdot)|dy
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +(b-a)\int_{\beta}^{1}(1-y)|X'(ya+(1-y)b,\cdot)|dy\\
&\leq (b-a)\int_{0}^{\beta} y[y|X'(a,\cdot)|+(1-y)|X'(b,\cdot)|]dy\\
&\qquad +(b-a)\int_{\beta}^{1}(1-y)[y|X'(a,\cdot)|+(1-y)|X'(b,\cdot)|]dy\\
&=(b-a)\Big[|X'(a,\cdot)|\frac{1}{3}\beta^3+|X'(b,\cdot)|\frac{1}{2}\beta^2-|X'(b,\cdot)|\frac{1}{3}\beta^3\\
&\qquad +|X'(a,\cdot)|\frac{1}{6}-|X'(a,\cdot)|\frac{1}{2}\beta^2+|X'(a,\cdot)|\frac{1}{3}\beta^3\\
&\qquad +|X'(b,\cdot)|\frac{1}{3}\left(1-3\beta+3\beta^2-\beta^3\right)\Big]\\
&=(b-a)\Big[\frac{2}{3}\beta^3|X'(a,)|-\frac{1}{2}\beta^2|X(a,\cdot)|+\frac{1}{6}|X'(a,\cdot)|\\
&\quad +\frac{1}{3}|X'(b,\cdot)|-\beta|X'(b,\cdot)|+\frac{3}{2}\beta^2|X'(b,\cdot)|-\frac{2}{3}\beta^3|X'(b,\cdot)|\Big], \qquad (a.e).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\\
&\quad\leq (b-a)\left[\frac{2}{3}\beta^3-\frac{1}{2}\beta^2+\frac{1}{6}\right]|X(a,\cdot)|\\
&\qquad\qquad +(b-a)\left[\frac{1}{3}-\beta+\frac{3}{2}\beta^2-\frac{2}{3}\beta^3\right]|X'(b,\cdot)|, \qquad (a.e).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\\
&\quad\leq \frac{(b-a)}{6}\left[\left(4\left(\frac{b-t}{b-a}\right)^3-3\left(\frac{b-t}{b-a}\right)^2+1\right)|X'(a,\cdot)|\right.\\
&\quad\left.+\left(9\left(\frac{b-t}{b-a}\right)^2-4\left(\frac{b-t}{b-a}\right)^3-6\left(\frac{b-t}{b-a}\right)+2\right)|X'(b,\cdot)|\right], \qquad (a.e),
\end{aligned}
$$

which complete the proof.

One can deduce a Ostrowski's type inequality for stochastic processes whose derivative are bounded, as follows:

Corollary 3.3 *In Theorem (3.2), and considering additionally that, if* $|X'(t,\cdot)| \leq$

M, for $M > 0$, then inequality

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \le M(b-a)\left[\left(\frac{b-t}{b-a}\right)^2 - \left(\frac{b-t}{b-a}\right) + \frac{1}{2}\right],$$

holds almost everywhere. The constant $\frac{1}{2}$ is the best possible in the sense that cannot be replaced by a smaller one.

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 3.4 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable on $[a,b]$, where $a, b \in I$ with $a < b$. If $|X'|^{p/p-1}$ is convex on $[a,b]$, the following inequality takes place almost everywhere:*

$$\begin{aligned}\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\\ \le \frac{2^{-1/q}}{[(b-a)(p+1)]^{1/p}} &\Big[(b-t)^{\frac{p+1}{p}}(|X'(t,\cdot)|^q + |X(b,\cdot)|^q)^{1/q}\\ &+ (t-a)^{\frac{p+1}{p}}(|X'(a,\cdot)|^q + |X'(t,\cdot)|^q)^{1/q}\Big],\end{aligned}$$

for each $t \in [a,b]$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof.

Suppose $p > 1$. From Lemma 3.1, using the Hölder inequality and considering $\beta = \dfrac{b-t}{b-a}$, we have:

$$\begin{aligned}&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\\ &\le (b-a)\int_0^\beta y|X'(ya+(1-y)b,\cdot)|dy\\ &\quad +(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy\\ &\le (b-a)\left(\int_0^\beta y^p dy\right)^{1/p}\left(\int_0^\beta |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}\\ &\quad +(b-a)\left(\int_\beta^1 (1-y)^p dy\right)^{1/p}\left(\int_\beta^1 |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}, \quad (a.e).\end{aligned}$$

Since $|X'|$ is convex, by Hermite-Hadamard inequality,

$$\int_0^{\beta} |X'(ya+(1-y)b,\cdot)|dy \le \frac{|X'(b,\cdot)|+|X'(t,\cdot)|}{2}, \qquad (a.e),$$

and

$$\int_{\beta}^{1} |X'(ya+(1-y)b,\cdot)|dy \le \frac{|X'(b,\cdot)|+|X'(t,\cdot)|}{2}, \qquad (a.e).$$

Therefore, if $\gamma = \dfrac{t-a}{b-a}$

$$\begin{aligned}
&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\quad \le (b-a)\left(\frac{1}{p+1}\beta^{p+1}\right)^{1/p}\left(\frac{|X'(b,\cdot)|^q+|X'(t,\cdot)|^q}{2}\right)^{1/q} \\
&\qquad + (b-a)\left(\frac{1}{p+1}\gamma^{p+1}\right)^{1/p}\left(\frac{|X'(t,\cdot)|^q+|X'(a,\cdot)|^q}{2}\right)^{1/q} \\
&\quad = \left(\frac{(b-a)^p}{p+1}\beta^{p+1}\right)^{1/p}\left(\frac{|X'(b,\cdot)|^q+|X'(t,\cdot)|^q}{2}\right)^{1/q} \\
&\qquad + \left(\frac{(b-a)^p}{p+1}\gamma^{p+1}\right)^{1/p}\left(\frac{|X'(t,\cdot)|^q+|X'(a,\cdot)|^q}{2}\right)^{1/q} \\
&\quad = \frac{1}{2^{1/q}}\left[\left(\frac{1}{p+1}\frac{(b-t)^{p+1}}{(b-a)}\right)^{1/p}(|X'(b,\cdot)|^q+|X'(t,\cdot)|^q)^{1/q}\right. \\
&\qquad \left. + \left(\frac{1}{p+1}\frac{(t-a)^{p+1}}{(b-a)}\right)^{1/p}(|X'(t,\cdot)|^q+|X'(a,\cdot)|^q)^{1/q}\right] \\
&\quad = \frac{2^{-1/q}}{[(p+q)(b-a)]^{1/p}}\left[(b-t)^{\frac{p+1}{p}}(|X'(b,\cdot)|^q+|X'(t,\cdot)|^q)^{1/q}\right. \\
&\qquad \left. +(t-a)^{\frac{p+1}{p}}(|X'(a,\cdot)|^q+|X'(t,\cdot)|^q)^{1/q}\right], \qquad (a.e),
\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Corollary 3.5 *In Theorem (3.4), if additionally* $|X'(t,\cdot)| \le M$, *for some* $M>0$, *then inequality*

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \le M\frac{(b-t)^{\frac{p+1}{p}}+(t-a)^{\frac{p+1}{p}}}{(p+1)^{1/p}(b-a)^{1/p}}.$$

holds almost everywhere, where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Corollary 3.6 *In Theorem (3.4), choose* $t = \frac{a+b}{2}$*, then*

$$\begin{aligned}
&\left| X\left(\frac{a+b}{2}, \cdot\right) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right| \\
&\quad \leq \frac{(b-a)}{4(p+1)^{1/p}} \left[\left(\left| X'\left(\frac{a+b}{2}, \cdot\right)\right|^q + |X'(b,\cdot)|^q \right)^{1/q} \right. \\
&\qquad \left. + \left(|X'(a,\cdot)|^q + \left| X'\left(\frac{a+b}{2}, \cdot\right)\right|^q \right)^{1/q} \right],
\end{aligned}$$

holds almost everywhere.

Theorem 3.7 *Let* $f : I \times \Omega \to \mathbf{R}$ *be a mean-square differentiable stochastic process on* I *such that* X' *is mean-square integrable on* $[a, b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|X'|^{p/(p-1)}$ *is concave on* $[a, b]$*, then the following inequality is true almost everywhere:*

$$\begin{aligned}
\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right| \;\leq\; & \frac{(b-a)}{(p+1)^{1/p}} \left[\left(\frac{b-t}{b-a}\right)^{(p+1)/p} \left| X'\left(\frac{t+b}{2}, \cdot\right)\right| \right. \qquad (17) \\
& \left. + \left(\frac{t-a}{b-a}\right)^{(p+1)/p} \left| X'\left(\frac{a+t}{2}, \cdot\right)\right| \right],
\end{aligned}$$

for each $t \in [a, b]$*, where* $p > 1$*.*

Proof.

Suppose that $p > 1$. As the Theorem (3.4),

$$\begin{aligned}
&\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right| \\
&\quad \leq (b-a)\int_0^\beta y|X'(ya+(1-y)b,\cdot)|dy \\
&\quad\; +(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy \\
&\quad \leq (b-a)\left(\int_0^\beta y^p dy\right)^{1/p} \left(\int_0^\beta |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q} \\
&\quad\; +(b-a)\left(\int_\beta^1 (1-y)^p dy\right)^{1/p} \left(\int_\beta^1 |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q},
\end{aligned}$$

almost everywhere. Since $|X'|^q$ is concave on $[a, b]$, by Hermite-Hadamard's inequality, we get

$$\int_0^\beta |X'(ya+(1-y)b,\cdot)|^q dy \leq \left| X'\left(\frac{b+t}{2}, \cdot\right)\right|^q,$$

and

$$\int_{\beta}^{1} |X'(ya+(1-y)b,\cdot)|^q dy \leq \left|X'\left(\frac{a+t}{2},\cdot\right)^q\right|.$$

Therefore,

$$\begin{aligned}
&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\qquad \leq (b-a)\left(\frac{1}{p+1}\beta^{p+1}\right)^{1/p}\left(\left|X'\left(\frac{t+b}{2},\cdot\right)\right|^q\right)^{1/q} \\
&\qquad\quad +(b-a)\left(\frac{1}{p+1}\gamma^{p+1}\right)^{1/p}\left(\left|X'\left(\frac{a+t}{2},\cdot\right)\right|^q\right)^{1/q} \\
&\qquad = \frac{(b-a)}{(p+1)^{1/p}}\left(\frac{b-t}{b-a}\right)^{(p+1)/p}\left|X'\left(\frac{t+b}{2},\cdot\right)\right| \\
&\qquad\quad +\frac{(b-a)}{(p+1)^{1/p}}\gamma^{(p+1)/p}\left|X'\left(\frac{a+t}{2},\cdot\right)\right| \\
&\qquad = \frac{(b-a)}{(p+1)^{1/p}}\left[\beta^{(p+1)/p}\left|X'\left(\frac{t+b}{2},\cdot\right)\right|\right. \\
&\qquad\qquad \left.+\gamma^{(p+1)/p}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|\right].
\end{aligned}$$

almost everywhere, and proof is completed.

Corollary 3.8 *In Theorem (3.7), choose* $t=\frac{a+b}{2}$*, then*

$$\begin{aligned}
&\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\qquad \leq \frac{(b-a)}{(2^{p+1}(p+1))^{1/p}}\left[\left|X'\left(\frac{a+3b}{4},\cdot\right)\right|+\left|X'\left(\frac{3a+b}{4},\cdot\right)\right|\right], \quad (a.e).
\end{aligned}$$

for each $y\in[a,b]$*, where* $p>1$*.*

The following result refines the above inequality (17).

Theorem 3.9 *Let* $X: I\times\Omega\to\mathbf{R}$ *a mean-square differentiable stochastic process on* I *such that* X' *is mean-square integrable, where* $a,b\in I$ *with* $a<b$*. If* $|X'|^{p/(p-1)}$ *is concave on* $[a,b]$*, then:*

$$\begin{aligned}
&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \qquad (18)\\
&\quad \leq \frac{(b-t)^2}{(b-a)(p+1)^{1/p}}\left|X'\left(\frac{t+b}{2},\cdot\right)\right|+\frac{(t-a)^2}{(b-a)(p+1)^{1/p}}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|, \quad (a.e).
\end{aligned}$$

for each $t\in[a,b]$*, where* $p>1$*.*

Proof.

Suppose that $p > 1$. From Lemma 3.1 and using the Hölder inequality,

$$\begin{aligned}
&\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right| \\
&\quad \leq (b-a)\int_a^{\beta} t_0 |X'(ya+(1-y)b,\cdot)|dy \\
&\qquad\qquad +(b-a)\int_{\beta}^{1} |y-1||X'(ya+(1-y)b,\cdot)|dy \\
&\quad \leq \left(\int_0^{\beta} y^p dy\right)^{1/p}\left(\int_0^{\beta} |X'(ya+(1-y)b,\cdot)|^q\right)^{1/q} \\
&\qquad +(b-a)\left(\int_{\beta}^{1}(1-y)^p\right)^{1/p}\left(\int_{\beta}^{1}|X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}, \quad \textit{(a.e).}
\end{aligned}$$

Since $|X'|^q$ is concave on $[a,b]$, we can use the Jense's integral inequality to obtain

$$\begin{aligned}
\int_0^{\beta} |X'(ya+(1-y)b,\cdot)|^q dy &= \int_0^{\beta} 1|X'(ya+(1-y)b,\cdot)|^q dy \\
&\leq \left(\int_0^{\beta} vdy\right)\left|X'\left(\frac{1}{\int_0^{\beta} vdy}\int_0^{\beta}(ya+(1-y)b)dy,\cdot\right)\right|^q \\
&= \frac{b-t}{b-a}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|^q, \qquad \textit{(a.e).}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\beta}^{1} |X'(ya+(1-y)b,\cdot)|^q dy &= \int_{\beta}^{1} 1|X'(ya+(1-y)b,\cdot)|^q dy \\
&\leq \left(\int_{\beta}^{1} vdy\right)\left|X'\left(\frac{1}{\int_{\beta}^{1} vdy}\int_{\beta}^{1}(ya+(1-y)b)dy,\cdot\right)\right|^q \\
&= \frac{t-a}{b-a}\left|X'\left(\frac{a+t}{2},\cdot\right)\right|^q \qquad \textit{(a.e).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left| X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \right| &\leq \frac{(b-t)^2}{(b-a)(p+1)^{1/p}}\left|X'\left(\frac{a+b}{2},\cdot\right)\right| \\
&\quad + \frac{(t-a)^2}{(b-a)(p+1)^{1/p}}\left|X'\left(\frac{a+b}{2},\cdot\right)\right|, \qquad \textit{(a.e).}
\end{aligned}$$

Corollary 3.10 *In Theorem (3.9), choose* $t = \frac{a+b}{2}$*, then*

$$\left|X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{b-a}\int_a^b X(u,\cdot)\right| \\ \leq \frac{(b-a)}{4(p+1)^{1/p}}\left[\left|X'\left(\frac{a+3b}{4},\cdot\right)\right| + \left|X'\left(\frac{3a+b}{2},\cdot\right)\right|\right], \qquad (19)$$

holds almost everywhere for each $t \in [a,b]$*, where* $p > 1$*.*

A different approach for powers of the absolute value of the first derivative leads to the following result:

Theorem 3.11 *Let* $X : I \times \Omega \to \mathbf{R}$ *be a mean-square differentiable stochastic process on* I *such that* X' *is mean-square integrable, where* $a, b \in I$ *with* $a < b$*. If* $|X'|^q$ *is convex on* $[a,b], q \geq 1$*, and* $|X'(t)| \leq M, t \in [a,b]$*, then the following inequality holds almost everywhere:*

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ \leq (b-a)\left(\frac{1}{2}\frac{(b-t)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2}\frac{(b-t)^2}{(b-a)^2} - \frac{1}{3}\frac{(b-t)^3}{(b-a)^3}\right)|X'(b,\cdot)|^q\right. \\ \left. + \frac{1}{3}\frac{(b-t)^3}{(b-a)^3}|X'(a,\cdot)|^q\right\}^{1/q} + (b-a)\left(\frac{1}{2} - \frac{(b-t)}{(b-a)} + \frac{1}{2}\frac{(b-t)^2}{(b-a)^2}\right)^{1-\frac{1}{q}} \\ \times\left\{\left[\frac{1}{3}\left(1 - \frac{(b-t)^3}{(b-a)^3}\right) - \left(1 - \frac{(b-t)^2}{(b-a)^2}\right) + \left(1 - \frac{(b-x)}{(b-a)}\right)\right]|X'(b,\cdot)|^q\right. \\ \left. + \left[\frac{1}{2}\left(1 - \frac{(b-t)^2}{(b-a)^2}\right) - \frac{1}{3}\left(1 - \frac{(b-t)^3}{(b-a)^3}\right)\right]|X'(a,\cdot)|^q\right\}^{\frac{1}{q}},$$

for each $t \in [a,b]$*.*

Proof.

Suppose that $q \geq 1$. From Lemma 3.1 and using the power mean inequality, we have

$$\left|X(t,\cdot)\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ \leq (b-a)\int_0^\beta y|X'(ya+(1-y)b,\cdot)|dy \\ +(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy$$

$$\leq (b-a)\left(\int_0^\beta ydy\right)^{1-\frac{1}{q}}\left(\int_0^\beta y|X'(ya+(1-y)b,\cdot)|^q dy\right)^{\frac{1}{q}}$$
$$+(b-a)\left(\int_\beta^1 (1-y)dy\right)^{1-\frac{1}{q}}\left(\int_\beta^1 (1-y)|X'(ya+(1-y)b,\cdot)|^q dy\right)^{\frac{1}{q}}, \quad (a.e).$$

Since $|X'|^q$ is convex,

$$\int_0^\beta y|X'(ya+(1-y)b,\cdot)|^q dy$$
$$\leq \int_0^\beta y[y|X'(a,\cdot)|^q+(1-y)|X'(b,\cdot)|^q]dy$$
$$=\left(\frac{1}{2}\frac{(b-t)^2}{(b-a)^2}-\frac{1}{3}\frac{(b-t)^3}{(b-a)^3}\right)|X'(b,\cdot)|^q+\frac{1}{3}\frac{(b-t)^3}{(b-a)^3}|X'(a,\cdot)|^q, \quad (a.e),$$

and

$$\int_\beta^1 (1-y)|X'(ya+(1-y)b,\cdot)|^q dy$$
$$\leq \int_\beta^1 (1-y)[y|X'(a,\cdot)|^q+(1-y)|X'(b,\cdot)|^q]dy$$
$$=\left[\frac{1}{3}\left(1-\frac{(b-t)^3}{(b-a)^3}\right)-\left(1-\frac{(b-t)^2}{(b-a)^2}\right)+\left(1-\frac{(b-t)}{(b-a)}\right)\right]|X'(b,\cdot)|^q$$
$$+\left[\frac{1}{2}\left(1-\frac{(b-t)^2}{(b-a)^2}\right)-\frac{1}{3}\left(1-\frac{(b-t)^3}{(b-a)^3}\right)\right]|X'(a,\cdot)|^q, \quad (a.e).$$

Therefore,

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|$$
$$\leq (b-a)\left(\frac{1}{2}\frac{(b-t)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{2}\frac{(b-t)^2}{(b-a)^2}-\frac{1}{3}\frac{(b-t)^3}{(b-a)^3}\right)|X'(b,\cdot)|^q\right.$$
$$\left.+\frac{1}{3}\frac{(b-t)^3}{(b-a)^3}|X'(a,\cdot)|^q\right\}^{\frac{1}{q}}+(b-a)\left(\frac{1}{2}-\frac{(b-t)}{(b-a)}+\frac{1}{2}\frac{(b-t)^2}{(b-a)^2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{1}{3}\left(1-\frac{(b-t)^3}{(b-a)^3}\right)-\left(1-\frac{(b-t)^2}{(b-a)^2}\right)+\left(1-\frac{(b-t)}{(b-a)}\right)\right]|X'(b,\cdot)|^q\right.$$
$$\left.+\left[\frac{1}{2}\left(1-\frac{(b-x)^2}{(b-a)^2}\right)-\frac{1}{3}\left(1-\frac{(b-t)^3}{(b-a)^3}\right)\right]|X'(a,\cdot)|^q\right\}^{\frac{1}{q}}, \quad (a.e).$$

Corollary 3.12 *In Theorem (3.11), choose* $t=\frac{a+b}{2}$*, then*

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \tag{20}$$

$$\leq (b-a)\frac{8^{\frac{1}{q}}}{192}\left[(|X'(a,\cdot)|^q+2|X'(b,\cdot)|^q)^{\frac{1}{q}}+(2|X'(a,\cdot)|^q+|X'(b,\cdot)|^q)^{\frac{1}{q}}\right], \quad (a.e).$$

For instance, if $q=1$, then (20) becomes

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\leq \frac{(b-a)}{8}(|X'(a,\cdot)|+|X'(b,\cdot)|), \quad (a.e).$$

In the following inequality, we may refine the result (17) and (18).

Theorem 3.13 *Let $f: I\times\Omega\to\mathbf{R}$ be a mean-square differentiable stochastic process, on I such that X' is mean-square integrable, where $a,b\in I$ with $a<b$. If $|X'|^q$ is concave on $[a,b]$, $q\geq 1$, then the following inequality holds almost everywhere:*

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|$$
$$\leq 2^{-1/q}(b-a)\left[\left(\frac{b-t}{b-a}\right)^2\left|X'\left(\frac{b+2t}{3},\cdot\right)\right|+\left(\frac{t-a}{b-a}\right)^2\left|X'\left(\frac{a+2t}{3},\cdot\right)\right|\right], \quad (a.e),$$

for each $y\in[a,b]$.

Proof.

First, we note that by concavity of $|X'|^q$ and the power-mean inequality,

$$|X'(\alpha t+(1-\alpha)t_0,\cdot)|^q\geq \alpha|X'(t,\cdot)|^q+(1-\alpha)|X'(t_0,\cdot)|^q, \quad (a.e)$$

Hence,

$$|X'(\alpha t+(1-\alpha)t_0,\cdot)|\geq \alpha|X'(t,\cdot)|+(1-\alpha)|X'(t_0,\cdot)|, \quad (a.e)$$

so, $|X'|$ is also concave,

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|$$
$$\leq (b-a)\int_0^\beta y|X'(ya+(1-y)b,\cdot)|dy$$
$$+(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy$$
$$\leq (b-a)\left(\int_0^\beta ydy\right)^{1-1/q}\left(\int_0^\beta y|X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}$$
$$+(b-a)\left(\int_\beta^1 (1-y)dy\right)^{1-1/q}\left(\int_\beta^1 (1-y)|X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}, \quad (a.e).$$

Accordingly, by Lemma 3.1, the Jensen integral inequality and taken $\beta = \dfrac{b-t}{b-a}$ and $\gamma = \dfrac{t-a}{b-a}$, we have

$$\begin{aligned}\int_0^\beta y|X'(ya+(1-y)b,\cdot)|^q dy &\le \left(\int_0^\beta ydt\right)\left|X'\left(\frac{\int_0^\beta y(ya+(1-y)b)dy}{\int_0^\beta ydy},\cdot\right)\right|^q \\ &= \frac{1}{2}\left(\frac{b-t}{b-a}\right)^2\left|X'\left(\frac{2t+b}{3},\cdot\right)\right|^q, \qquad (a.e),\end{aligned}$$

and

$$\begin{aligned}\int_\beta^1 (1-y)|X'(ya+(1-y)b,\cdot)|^q dt &= \int_0^\gamma v|X'((1-v)a+vb,\cdot)|^q dv \\ &\le \left(\int_0^\gamma vdv\right)\left|X'\left(\frac{\int_0^\gamma (v(1-v)a+vb,\cdot)dv}{\int_0^\gamma vdv},\cdot\right)\right|^q \\ &= \frac{1}{2}\left(\frac{t-a}{b-a}\right)^2\left|X'\left(\frac{a+2t}{3},\cdot\right)\right|^q, \qquad (a.e).\end{aligned}$$

So,

$$\begin{aligned}&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ &\le 2^{-1/q}(b-a)\left[\left(\frac{b-t}{b-a}\right)^2\left|X'\left(\frac{b+2t}{2},\cdot\right)\right|+\left(\frac{t-a}{b-a}\right)^2\left|X'\left(\frac{a+2t}{3},\cdot\right)\right|\right], \quad (a.e).\end{aligned}$$

3.1 Results and discussion

The results presented in this section represent refinements of error estimate in an Ostrowski type integral inequality when the first derivative is convex, concave or bounded. Also, the corresponding version for power of the first derivative in absolute value is incorporated. For this type of estimations the Hölder inequality is implemented and these estimates are enhanced applied the well known power mean inequality. All inequalities built in this section represent an improvement of the original Ostrowski inequality (1) using the above conditions.

The results presented in the above section are a generalization of the theorems established in [2].

4 Ostrowski's Type Inequalities for s-Convex Stochastic Process

In this section, we consider some inequalities of Ostrowski's type for s-convex (s-concave) stochastic process. We start with the following result:

Lemma 4.1 *Let $X : [a,b]\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I where $a,b \in I$ with $a<b$. If X' is mean-square integrable the following equality holds almost everywhere:*

$$\begin{aligned} &X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du \qquad (21)\\ &= \frac{(t-a)^2}{b-a}\int_0^1 yX'(yt+(1-y)a,\cdot)dy - \frac{(b-t)^2}{b-a}\int_0^1 yX'(yt+(1-y)b,\cdot)dy, \end{aligned}$$

for each $y \in [a,b]$.

Proof.

We perform a change of variable, $u = yt+(1-y)a$ and $w = yt+(1-y)b$. Then integrating by parts:

$$\begin{aligned} &\frac{(t-a)^2}{b-a}\int_0^1 yX'(yt+(1-y)a,\cdot)dy - \frac{(b-t)^2}{b-a}\int_0^1 yX'(yt+(1-y)b,\cdot)dy \\ &= \frac{(t-a)^2}{b-a}\int_a^t \frac{(u-a)}{(t-a)}X'(u,\cdot)\frac{du}{(t-a)} - \frac{(b-t)^2}{b-a}\int_t^b \frac{(b-w)}{(b-t)}X'(w,\cdot)\frac{dw}{(b-t)} \\ &= \frac{1}{b-a}\int_a^t (u-a)X'(u,\cdot)du - \frac{1}{b-a}\int_t^b (b-w)X'(w,\cdot)dw \\ &= \frac{1}{b-a}\left[(t-a)X(t,\cdot) - \int_a^t X(u,\cdot)du\right] + \frac{1}{b-a}\left[(b-t)X(t,\cdot) - \int_t^b X(w,\cdot)dw\right] \\ &= \frac{1}{b-a}\left[(t-a)X(t,\cdot) - \int_a^t X(u,\cdot)du + (b-t)X(t,\cdot) - \int_t^b X(w,\cdot)dw\right] \\ &= \frac{1}{b-a}\left[X(t,\cdot)(b-a) - \int_a^b X(u,\cdot)du\right] \\ &= X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du, \quad \textit{(a.e).} \end{aligned}$$

so proof is completed.

Theorem 4.2 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable where $a,b \in I$ with $a<b$.*

If $|X'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $|X'(t,\cdot)| \leq M, x \in [a,b]$, then the following inequality holds:

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{M}{b-a}\left[\frac{(t-a)^2+(b-t)^2}{s+1}\right], \quad (a.e),$$

takes place $t \in [a,b]$.

Proof.

By Lemma 4.1 and since $|X'|$ is s-convex, we have

$$\begin{aligned}
&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\leq \frac{(t-a)^2}{b-a}\int_0^1 y|X'(yt+(1-y)a,\cdot)|dy \\
&\qquad +\frac{(b-t)^2}{b-a}\int_0^1 y|X'(yt+(1-y)b,\cdot)|dy \\
&\leq \frac{(t-a)^2}{b-a}\int_0^1 t[y^s|X'(t,\cdot)|+(1-y)^s|X'(a,\cdot)|]dy \\
&\qquad +\frac{(b-t)^2}{b-a}\int_0^1 y[y^s|X'(t,\cdot)|+(1-y)^s|X'(b,\cdot)|]dy \\
&= \frac{(t-a)^2}{b-a}|X'(t,\cdot)|\int_0^1 y^{s+1}dy + |X'(a,\cdot)|\int_0^1 y(1-y)^s dy \\
&\qquad +\frac{(b-t)^2}{b-a}|X'(t,\cdot)|\int_0^1 y^{s+1} + |X'(b,\cdot)|\int_0^1 y(1-y)^s dy \\
&= \frac{(t-a)^2}{b-a}\left(|X'(t,\cdot)|\frac{1}{s+2} + |X'(a,\cdot)|\frac{1}{(s+1)(s+2)}\right) \\
&\qquad +\frac{(b-t)^2}{b-a}\left(|X'(t,\cdot)|\frac{1}{s+2} + |X'(b,\cdot)|\frac{1}{(s+1)(s+2)}\right) \\
&\leq \frac{M}{b-a}\left[\frac{(t-a)^2+(b-t)^2}{s+1}\right].
\end{aligned}$$

almost everywhere.

Theorem 4.3 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is a mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|^q$ is s-convex in the second sense on $[a,b]$, for some fixed $s \in (0,1], p, q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and $|X'(t,\cdot)| \leq M, t \in [a,b]$, then the following inequality holds almost everywhere:*

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(t-a)^2+(b-t)^2}{b-a}\right],$$

for each $t \in [a, b]$.

Proof.

Suppose $p > 1$ from Lemma 4.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\leq \frac{(t-a)^2}{b-a}\int_0^1 t|X'(yt+(1-y)a,\cdot)|dy + \frac{(b-t)^2}{b-a}\int_0^1 t|X'(yt+(1-y)b,\cdot)|dy \\
&\leq \frac{(t-a)^2}{b-a}\left(\int_0^1 y^p dy\right)^{\frac{1}{p}}\left(\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy\right)^{1/q} \\
&\qquad + \frac{(b-t)^2}{b-a}\left(\int_0^1 y^p dy\right)^{\frac{1}{p}}\left(\int_0^1 |X'(yt+(1-y)b,\cdot)|^q dy\right)^{1/q}, \quad (a.e).
\end{aligned}$$

Since $|X'|^q$ is s-convex in the second sense and $|X'(t,\cdot)| \leq M$, hence

$$\begin{aligned}
\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy &\leq \int_0^1 [y^s|X'(t,\cdot)|^q + (1-y)^s|X'(a,\cdot)|^q]dy \\
&= |X'(t,\cdot)|^q\int_0^1 y^s dy + |X'(a,\cdot)|^q\int_0^1 (1-y)^s dy \\
&= \frac{|X'(t,\cdot)|^q + |X'(a,\cdot)|^q}{s+1} \\
&\leq \frac{2M^q}{s+1}, \quad (a.e).
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 |X'(yt+(1-y)b,\cdot)|^q dy &\leq \int_0^1 [y^s|X'(t,\cdot)|^q + (1-y)^s|X'(b,\cdot)|^q]dt \\
&= |X'(t,\cdot)|^q\int_0^1 y^s dy + |X'(b,\cdot)|^q\int_0^1 (1-y)^s dy \\
&= \frac{|X'(t,\cdot)|^q + |X'(b,\cdot)|^q}{s+1} \\
&\leq \frac{2M^q}{s+1}, \quad (a.e).
\end{aligned}$$

Therefore,

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)\right| \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(t-a)^2+(b-t)^2}{b-a}\right], \quad (a.e),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which is required.

The previous theorem can be formulated in case that X is convex as follows:

Corollary 4.4 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|^{p/(p-1)}$ is convex on $[a, b], p > 1$, and $|X'(y, \cdot)| \leq M, y \in [a, b]$, then:*

$$\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right| \leq \frac{M}{b-a} \left[\frac{(t-a)^2 + (b-t)^2}{(1+p)^{\frac{1}{p}}} \right], \quad (a.e),$$

for each $t \in [a, b]$.

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 4.5 *Let $X : I \times \Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|^q$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, and $|X'(t, \cdot)| \leq M, t \in [a, b]$, then the following inequality holds almost everywhere:*

$$\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right| \leq M \left(\frac{2}{s+1} \right)^{1/q} \left[\frac{(t-a)^2 + (b-t)^2}{2(b-a)} \right], \quad (22)$$

for each $t \in [a, b]$.

Proof.

Suppose that $q \leq 1$. By the Lemma 4.1 and using the power mean inequality:

$$\begin{aligned}
&\left| X(t, \cdot) - \frac{1}{b-a} \int_a^b X(u, \cdot) du \right| \\
&\leq \frac{(x-a)^2}{b-a} \int_0^1 y|X'(yt + (1-y)a, \cdot)| dy + \frac{(b-x)^2}{b-a} \int_0^1 y|X'(yt + (1-y)b, \cdot)| dy \\
&\leq \frac{(x-a)^2}{b-a} \left(\int_0^1 y dy \right)^{1-1/q} \left(\int_0^1 y|X'(yt + (1+y)a, \cdot)|^q dy \right)^{1/q} \\
&\qquad + \frac{(b-x)^2}{b-a} \left(\int_0^1 y dy \right)^{1-1/q} \left(\int_0^1 y|X'(yt + (1+y)a, \cdot)|^q dy \right)^{1/q},
\end{aligned}$$

almost everywhere. Since $|X'|^q$ is s-convex, we have

$$\begin{aligned}
&\int_0^1 y|X'(yt+(1+y)a,\cdot)|^q dy \\
&\quad\leq \int_0^1 [y^{s+1}|X'(yt+(1+y)a,\cdot)|^q + y(1-y)^s|X'(yt+(1+y)a,\cdot)|^q]dy \\
&\quad= \frac{|X'(t,\cdot)|^q+(s+1)|X'(a,\cdot)|^q}{(s+1)(s+2)} \leq \frac{M^q}{s+1},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 y|X'(yt+(1+y)b,\cdot)|^q dy &\leq \int_0^1 [y^{s+1}|X'(t,\cdot)|^q + y(1-y)^s|X'(a,\cdot)|^q]dy \\
&= \frac{|X'(t,\cdot)|^q+(s+1)|X'(a,\cdot)|^q}{(s+1)(s+2)} \leq \frac{M^q}{s+1}, \quad (a.e).
\end{aligned}$$

Therefore,

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq M\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(t-a)^2+(b-t)^2}{2(b-a)}\right] \qquad (a.e.),$$

which is the desired result.

A midpoint type inequality for stochastic processes whose derivatives in absolute value are s-convex in the second sense may be obtained from the previous results as follows:

Corollary 4.6 *If in (22) we choose* $t=\frac{a+b}{2}$*, then:*

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{M(b-a)}{4}\left(\frac{2}{s+1}\right)^{1/q}, \quad q\geq 1, \quad (a.e),$$

where $s\in(0,1]$ *and* $|X'|^q$ *is* s*-convex in the second sense on* $[a,b], q\geq 1$.

Now, we obtain a Ostrowski's type inequality for the following result holds for s-concave mapping.

Theorem 4.7 *Let* $X: I\times\Omega\to\mathbf{R}$ *be a mean-square differentiable stochastic process on* I *such that* X' *is a mean-square integrable where* $a,b\in I$ *with* $a<b$*. If* $|X'|^q$ *is* s*-concave on* $[a,b], p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$*, then the following inequality holds almost everywhere:*

$$\begin{aligned}
&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \qquad (23)\\
&\quad\leq \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(t-a)^2\left|X'\left(\frac{t+a}{2},\cdot\right)\right|+(b-t)^2\left|X'\left(\frac{b+t}{2},\cdot\right)\right|\right],
\end{aligned}$$

for each $t\in[a,b]$.

Proof.

Suppose that $q > 1$. From Lemma 4.1 and using the Hlder inequality,

$$\begin{aligned} &\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ &\leq \frac{(x-a)^2}{b-a}\int_0^1 y|X'(yt+(1-y)a,\cdot)|dy \\ &\quad + \frac{(b-x)^2}{b-a}\int_0^1 y|X'(yt+(1-y)b,\cdot)|dy \\ &\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy\right)^{1/q} \\ &\quad + \frac{(b-x)^2}{b-a}\left(\int_0^1 y^p dy\right)^{1/p}\left(\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy\right)^{1/q}, \end{aligned}$$

almost everywhere.

But since $|X'|^q$ is concave, using the inequality (21), we have

$$\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy \leq 2^{s-1}\left|X'\left(\frac{x+a}{2},\cdot\right)\right|^q, \quad (a.e),$$

and

$$\int_0^1 |X'(yt+(1-y)a,\cdot)|^q dy \leq 2^{s-1}\left|X'\left(\frac{b+x}{2},\cdot\right)\right|^q, \quad (a.e).$$

Combining the above numbered inequalities,

$$\begin{aligned} &\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ &\leq \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(t-a)^2\left|X'\left(\frac{t+a}{2},\cdot\right)\right| + (b-t)^2\left|X'\left(\frac{b+t}{2},\cdot\right)\right|\right], \quad (a.e). \end{aligned}$$

Therefore, we can deduce the following midpoint type inequality for stochastic processes whose derivative in absolute value are s-concave in the second sense:

Corollary 4.8 *If in (23) we choose $s = 1$ and $t = \frac{a+b}{2}$, then:*

$$\begin{aligned} &\left|X\left(\frac{a+b}{2},\cdot\right) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\ &\qquad \leq \frac{b-a}{4(1+p)^{1/p}}\left[\left|X'\left(\frac{3a+b}{4}\right)\right| + \left|X'\left(\frac{a+3b}{4}\right)\right|\right] \quad (a.e), \end{aligned}$$

where $|X'|^q$ is concave on $[a,b], p > 1$.

4.1 Results and discussion

In this section, the theorems and corollaries established represent different refinements of the weighted difference in absolute value between the mean integrals of the stochastic process X and its first derivative, considering the s-convexity condition on first derivative of the process in absolute value. In addition, the corresponding version for powers of the first derivative in absolute value is incorporated implementing the Hölder inequality and the well known power mean inequality. The inequalities in this section are a betterment of the error estimation on the orginal Ostriwski inequality (1) using the aforementioned conditions.
The result developed in the before section is a generalization of the research done in [4].

5 Ostrowski's Type Inequalities for Quasi-Convex Stochastic Process

In the following, some Ostrowski type inequalities for absolutely continuous stochastic process whose first derivative satisfies certain convexity assumptions are considered.

Theorem 5.1 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process on I such that X' is mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|$ is quasi-convex on $[a, b]$, then the following inequality holds almost everywhere:*

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-t)^2}{2(b-a)}\max\{|X'(t,\cdot)|, |X'(b,\cdot)|\} + \frac{(t-a)^2}{2(b-a)}\max\{|X'(t,\cdot)|, |X'(a,\cdot)|\},$$

for each $t \in [a, b]$

Proof.

By Lemma 3.1, since $|X'|$ is quasi-convex and considering $\beta = \dfrac{b-t}{b-a}$, we have

$$\left|X(t,\cdot) - \frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq (b-a)\int_0^\beta |y||X'(ya + (1-y)b,\cdot)|dy$$

$$
\begin{aligned}
&\quad +(b-a)\int_{\beta}^{1}|y-1||X'(ya+(1-y)b,\cdot)|dy \\
&\le (b-a)\int_{0}^{\beta} y\max\{|X'(a,\cdot)|,|X'(b,\cdot)|\}dy \\
&\quad +(b-a)\int_{\beta}^{1}(1-y)\max\{|X'(a,\cdot)|,|X'(b,\cdot)|\}dy \\
&= \frac{1}{2}\frac{(b-t)^2}{(b-a)}\max\{|X'(a,\cdot)|,|X'(b,\cdot)|\} \\
&\quad +\frac{1}{2}\frac{(t-a)^2}{(b-a)}\max\{|X'(a,\cdot)|,|X'(b,\cdot)|\}, \quad (a.e).
\end{aligned}
$$

Corollary 5.2 *In Theorem (5.1) taking additionally that, if X' is bounded on $[a,b]$, i.e., there exist $M>0$ such that $|X'(t,\cdot)|\le M, t\in[a,b]$, then inequality (2) holds almost everywhere.*

Corollary 5.3 *In Theorem (5.1) assuming additionally that, if*

1. *X' is increasing, then*

$$
\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)dy\right|\le\frac{(b-t)^2}{2(b-a)}|X'(b,\cdot)|+\frac{(t-a)^2}{2(b-a)}|X'(t,\cdot)|, \quad (a.e). \tag{24}
$$

2. *X' is decreasing, then*

$$
\left|X(t\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\le\frac{(b-t)^2}{2(b-a)}|X'(t,\cdot)|+\frac{(t-a)^2}{2(b-a)}|X'(a,\cdot)|, \quad (a.e). \tag{25}
$$

Corollary 5.4 *In Theorem (5.1), choose $t=\frac{a+b}{2}$, then*

$$
\begin{aligned}
&\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \\
&\quad\le\frac{(b-a)}{8}\left[\max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|,|X'(b,\cdot)|\right\}\right. \\
&\qquad\left.+\max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|,|X'(a,\cdot)|\right\}\right], \quad (a.e).
\end{aligned}
$$

Therefore,

1. *If $|X'|$ is increasing, then*

$$
\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)\right|\le\frac{b-a}{8}\left[|X'(b,\cdot)|+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|\right], \quad (a.e). \tag{26}
$$

2. If $|X'|$ is decreasing, then

$$\left|X\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \le \frac{b-a}{8}\left[|X'(a,\cdot)|+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|\right], \quad (a.e). \tag{27}$$

The corresponding version for powers via quasi-convex stochastic processes is incorporated in the following result:

Theorem 5.5 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean-square differentiable stochastic process, on I such that X' is mean-square integrable where $a, b \in I$ with $a < b$. If $|X'|$ is quasi-convex on $[a, b]$, then:*

$$\begin{aligned}&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)\right|\\ &\quad\le\left(\frac{(b-t)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left[\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\}\right]^{1/q}\\ &\quad\quad+\left(\frac{(t-a)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}}\left[\max\{|X'(t,\cdot)|^q,|X'(a,\cdot)|^q\}\right]^{1/q}, \quad (a.e),\end{aligned}$$

for each $t \in [a, b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof.

Suppose that $p > 1$. From Lemma 3.1, using the Hölder inequality and $\beta=\frac{b-t}{b-a}$,

$$\begin{aligned}&\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\\ &\le (b-a)\int_0^\beta |y||X'(ya+(1-y)b,\cdot)|dy\\ &\quad+(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy\\ &\le (b-a)\left(\int_0^\beta |y|^p dy\right)^{1/p}\left(\int_0^\beta |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}\\ &\quad+(b-a)\left(\int_\beta^1 (1-y)^p dy\right)^{1/p}\left(\int_\beta^1 |X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}\\ &=\frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left(\frac{(b-t)^{p+1}}{(b-a)^p(b-a)}\right)^{1/p}\left[\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\}\right]^{1/q}\end{aligned}$$

$$+(b-a)\left(\frac{1}{p+1}\gamma^{p+1}\right)^{1/p}\left[\max\{|X'(t,\cdot)|^q,|X'(a,\cdot)|^q\}\right]^{1/q}$$

$$=\frac{(b-t)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}}\left[\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\}\right]^{1/q}$$

$$+\frac{(t-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)^{\frac{1}{p}}}\left[\max\{|X'(t,\cdot)|^q,|X'(a,\cdot)|^q\}\right]^{1/q}, \quad (a.e).$$

This completes this proof.

Corollary 5.6 *In Theorem (5.5), additionally, if*

1. *If $|X'|$ is increasing, then*

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{1}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[(b-t)^{\frac{p+1}{p}}|X'(b,\cdot)|+(t-a)^{\frac{p+1}{p}}|X'(t,\cdot)|\right], \quad (a.e).$$

2. *If $|X'|$ is decreasing, then*

$$\left|X'(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{1}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[(b-a)^{\frac{p+1}{p}}|X'(t,\cdot)|+(t-a)^{\frac{p+1}{p}}|X(a,\cdot)|\right], \quad (a.e).$$

Corollary 5.7 *In Theorem (5.5), choose $t=\frac{a+b}{2}$, then we have*

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)\right| \leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[\max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q,|X'(b,\cdot)|^q\right\}^{1/q} + \max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q,|X'(a,\cdot)|^q\right\}^{1/q}\right], \quad (a.e).$$

Therefore we have

1. *If $|X'|$ is increasing, then*

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[|X'(b,\cdot)|+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|\right], \quad (a.e). \tag{28}$$

2. If $|X'|$ is decreasing, then

$$\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \tag{29}$$
$$\leq \frac{(b-a)}{2^{1/p}(p+1)^{1/p}}\left[|X'(a,\cdot)|+\left|X'\left(\frac{a+b}{2},\cdot\right)\right|\right], \quad (a.e).$$

Theorem 5.8 *Let $X : I\times\Omega \to \mathbf{R}$ be a mean square differentiable stochastic process on I such that X' is integrable, where $a, b \in I$ with $a < b$. If $|X'|^q$ is quasi-convex on $[a, b], q \geq 1$, and $|X'(x, \cdot)| \leq M, x \in [a, b]$, then the following inequality hold almost everywhere:*

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| \leq \frac{(t-a)^2}{2(b-a)}(\max\{|X'(t,\cdot)|^q,|X'(a,\cdot)|^q\})^{1/q}$$
$$+\frac{(b-t)^2}{2(b-a)}(\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\})^{1/q}.$$

for each $t \in [a, b]$.

Proof.

Suppose that $q \geq 1$. from Lemma 3.1, using the power mean inequality and $\beta = \dfrac{b-t}{b-a}$, we have:

$$\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|$$
$$\leq (b-a)\int_0^\beta y|X'(ya+(1-y)b,\cdot)|dy$$
$$+(b-a)\int_\beta^1 |y-1||X'(ya+(1-y)b,\cdot)|dy$$
$$\leq (b-a)\left(\int_0^\beta ydy\right)^{1-1/q}\left(\int_0^\beta y|X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}$$
$$+(b-a)\left(\int_\beta^1 ydy\right)^{1-1/q}\left(\int_\beta^1 y|X'(ya+(1-y)b,\cdot)|^q dy\right)^{1/q}, \quad (a.e).$$

Since $|X'|^q$ is quasi-convex,

$$\int_0^\beta y|X(ya+(1-y)b,\cdot)|^q dy \leq \int_0^\beta y\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\}dy$$
$$= \frac{(b-t)^2}{2(b-a)^2}\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\}, \quad (a.e).$$

and

$$\begin{aligned}\int_0^{\beta}(1-y)|X(ya+(1-y)b,\cdot)|^q dy &\leq \int_0^{\beta}(1-y)\max\{|X'(a,\cdot)|^q,|X'(t,\cdot)|^q\}dy\\ &= \frac{(t-a)^2}{2(b-a)^2}\max\{|X'(a,\cdot)|^q,|X'(t,\cdot)|^q\}, \quad (a.e).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|X(t,\cdot)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right| &\leq \frac{(t-a)^2}{2(b-a)}(\max\{|X'(t,\cdot)|^q,|X'(a,\cdot)|^q\})^{1/q}\\ &\quad+\frac{(b-t)^2}{2(b-a)}(\max\{|X'(t,\cdot)|^q,|X'(b,\cdot)|^q\})^{1/q}, \quad (a.e),\end{aligned}$$

which is the desired result.

Corollary 5.9 *In Theorem (5.8), and*

1. $|X'|$ *is increasing, then (24) holds almost everywhere.*
2. $|X'|$ *is decreasing, then (25) holds almost everywhere.*

Corollary 5.10 *In Theorem 5.8, choose* $t=\frac{a+b}{2}$*, then*

$$\begin{aligned}&\left|X\left(\frac{a+b}{2},\cdot\right)-\frac{1}{b-a}\int_a^b X(u,\cdot)du\right|\\ &\quad\leq\frac{(b-a)}{8}\left[\left(\max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q,|X'(b,\cdot)|^q\right\}\right)^{1/q}\right.\\ &\qquad\left.+\left(\max\left\{\left|X'\left(\frac{a+b}{2},\cdot\right)\right|^q,|X'(a,\cdot)|^q\right\}\right)^{1/q}\right], \quad (a.e).\end{aligned}$$

Therefore,

1. *If* $|X'|$ *is increasing, then (26) holds, almost everywhere.*
2. *If* $|X'|$ *is decreasing, then (27) holds, almost everywhere.*

5.1 Results and discussion

The terms set out in the theorems and corollaries of this section, represent a refined estimate of the error on the original Ostrowski's inequality (1). The results presented represent a different and novel estimated when quasi-convexity or bounded condition on the first derivative of the process is established. Further, the corresponding version for powers of the first derivative in absolute value where it is implemented the Hölder inequality. This result is enhanced when the well known power mean inequality are implemented.
The estimates presented generalizes the results given in [3].

6 Conclusion

In this work we establish, for the first time, an Ostrowski's integral inequality to stochastic processes. Also, different Ostrowski's type inequalities are given through this paper, giving also an estimate of the weighted difference between the mean-square integrals of a stochastic process X and its first derivative. The error estimated involving the first, second and n-th derivatives are obtained.

Other refinements are established when the magnitude of the first derivative is convex, concave, s-convex or quasi-convex, and their corresponding version for powers of the first derivative in the absolute value. Additionally, we deduce an Ostrowski's type inequality for stochastic processes whose derivatives are bounded.

Is important stand out that all the results obtained in this paper are the counterpart for stochastic processes of theorems previously established for functions and contributes to new error estimation applied to numerical analysis of stochastic processes.

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