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On inequalities among some cardinal invariants

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Abstract

The strong sequences method was introduced by B. A. Efimov, as a useful method for proving famous theorems in dyadic spaces: Marczewski theorem on cellularity, Shanin theorem on a calibre and Esenin-Volpin theorem. In this paper there will be considered strong sequences on a set with arbitrary relation as generalization of a partially ordered set. In this paper there will be introduced a new cardinal invariant s-length of the strong sequence and investigated relations among s and other well known invariants like: saturation, boundeness, density, calibre.

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1 Introduction

The strong sequences method was introduced by B. A. Efimov, as a useful method for proving famous theorems in dyadic spaces: Marczewski theorem on cellularity, Shanin theorem on a calibre and Esenin-Volpin theorem. Let us remind his main results.

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p : p : T \to \{0,1\}\}.$$

For $s \subset T$, $i : s \to \{0,1\}$ it will be used the following notation

$$H_s^i = \{p \in D^T : p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}}: \alpha \in T\}$ of the Cantor cube as a sequence of so called connected pairs.

A pair (H^i_s, H^i_v) where $card(s) < \omega$ will be called the connected pair if $H^i_s \cap H^i_v \neq \emptyset$

A sequence $(H^{i_\alpha}_{s_\alpha}, H^{i_\alpha}_{v_\alpha})$ consisting of connected pairs is called a strong sequence if $H^{i_\alpha}_{s_\alpha} \cap H^{i_\beta}_{v_\beta} = \emptyset$ whenever $\alpha > \beta$.

and he proved the following

Theorem 1.1 (Efimov) [3] *Let κ be a regular, uncountable cardinal number. In the space D^T there is not a strong sequence*

$$(\{H^i_{\{\alpha\}}: \alpha \in v_\xi\}, \{H^i_{\{\beta\}}: \beta \in w_\xi\}) \quad ; \quad \xi < \kappa$$

such that $|w_\xi| < \kappa$ and $|v_\xi| < \omega$ for each $\xi < \kappa$.

In paper [12] this method was introduced as follows:

Let X be a set, and $B \subset P(X)$ be a family of non-empty subsets of X closed with respect to the finite intersections. Let S be a finite subfamily contained B. A pair (S, H), where $H \subseteq B$, will be called *connected* if $S \cup H$ is centered. A sequence (S_ϕ, H_ϕ) ; $\phi < \alpha$ consisting of connected pairs is called *a strong sequence* if $S_\lambda \cup H_\phi$ is not centered whenever $\lambda > \phi$

and was proved the following

Theorem 1.2 ([12]) *If for $B \subset P(X)$ there exists a strong sequence $(S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$ then the family B contains a subfamily of cardinality λ^+ consisting of pairwise disjoint sets.*

In papers [11] and [12] Turzański investigated implications of this method with well known theorems (i. e. Kurepa theorem [6], Marczewski theorem [7] on cellurality of dyadic spaces, Shanin theorem [10] on a calibre of dyadic spaces, Erdös-Rado theorem and the like.

2 Notation and terminology

In this paper the following notation is used. For given X denote its cardinality by $|X|$. If κ is a cardinal then $[X]^\kappa = \{A \subset X: |A| = \kappa\}$. The smallest cardinal number greater than κ is its successor κ^+. Infinite ordinals are usually denoted by Greek letters. Let us remind that an ordinal number α is the set of all smaller ordinals $\alpha = \{\beta: \beta < \alpha\}$ and we sometimes identify α with the ordered set $(\alpha, \leq)$, defined on α by a natural order. The remaining notations are standard. We will assume AC where will be required.

Let (X, r) be a set with relation r. Let $a, b \in X$. (We sometimes will write X instead of (X, r) in situations when it the relation is obvious).

We say that elements a and b are *comparable* if $(a, b) \in r$ or $(b, a) \in r$.

We say that elements a and b are *compatible* if there exists c such that

$$(a, c) \in r \text{ and } (b, c) \in r.$$

(We say then, that a, b have a *bound*).

We say that $\mathcal{L} \subset X$ is a *chain* if any $a, b \in \mathcal{L}$ are comparable.

We say that a chain $\mathcal{L} \subset X$ is *a maximal chain* iff for all $x \in X \setminus \mathcal{L}$ there is $(x, a) \notin r$ and $(a, x) \notin r$ for all $a \in \mathcal{L}$.

We say that a set $\mathcal{A} \subset X$ is an *antichain* if any two distinct elements $a, b \in \mathcal{A}$ are incomparable.

We say that an antichain $\mathcal{A} \subset X$ is *a maximal antichain* iff each $x \in X \setminus \mathcal{A}$ is comparable with some $a \in \mathcal{A}$.

If each of two elements in a set $A \subset X$ are compatible, then A is a *directed* set. A set A is κ- *directed* if every subset of A of cardinality less than κ has a bound, i.e. for each $B \subset A$ with $|B| < \kappa$ there exists $a \in A$ such that $(b, a) \in r$ for all $b \in B$.

Now the following definition of strong sequences will be introduced (compare [11])

Definition 2.1 *Let (X, r) be a set with relation r.*
A sequence $(S_\phi, H_\phi); \phi < \alpha$ where $S_\phi, H_\phi \subset X$ and S_ϕ is finite is called a strong sequence if
1^o *$S_\phi \cup H_\phi$ is ω-directed*
2^o *$S_\beta \cup H_\phi$ is not ω-directed for $\beta > \phi$.*

Let us consider the following notation:

$$\hat{s}(X) = sup\{\kappa\colon \text{there exists a strong sequence on } X \text{ of the length } \kappa\}.$$

Let us consider the following definitions of a calibre and a precalibre.

Definition 2.2 *A cardinal κ is a calibre for X if κ is infinite and every set $A \in [X]^\kappa$ has a chain of length κ.*

Definition 2.3 *A cardinal κ is a precalibre for X if κ is infinite and every set $A \in [X]^\kappa$ has an ω -directed subset of cardinality κ.*

Comparing two above definitions and knowing that each chain is an ω - directed set we can conclude that each calibre is a precalibre. Let us notice that the inverse is not true.

Sierpiński poset (see [9], [8]) is an example of a uncountable poset with no uncountable chains nor uncountable antichains. Let us remind it.

Example 2.4 *Let $P = (\boldsymbol{R}, r)$. Let $\leq$ be the natural ordering on $\boldsymbol{R}$ and let $\geq$ be inverse order $\leq^*$. Let $\geq_w$ be any well order on $\boldsymbol{R}$. For arbitrary $x, y \in \boldsymbol{R}$ let set*

$$(x, y) \in r \Leftrightarrow x \geq y \text{ and } x \geq_w y.$$

As $\boldsymbol{R}$ does not contain a copy of ω_1 nor ω_1^* there are no uncountable chains nor uncountable antichains. Let us observe that P is an $\omega-$ directed set. Let us choose arbitrary $x, y \in \boldsymbol{R}$ such that $x \geq_w y$. Then there are less than $2^{\aleph_0}$ many points for which x is their bound. Then there exists $z \in \boldsymbol{R}$ which is a bound of x and y.

3 Main results

Let us start with the theorem which will be crucial for our later investigation.

Theorem 3.1 *Let (X, r) be a set with relation r. Then each regular cardinal number $\kappa > \hat{s}(X)$ is a calibre for X.*

Proof Let us suppose that κ is not a calibre for X. It means that there exists $A \in [X]^\kappa$ of cardinality κ in which each chain has cardinality less than κ.

Let $a_0 \in A$ be an arbitrary element and $A_0 \subset A$ be a maximal chain (with respect to relation r) such that $a_0 \in A_0$. Obviously $|A_0| < \kappa$. Let $(\{a_0\}, A_0)$ be the first pair of a strong sequence.

Let $a_1 \in A \backslash A_0$ be an arbitrary element and $A_1 \subset A \backslash A_0$ be a maximal chain (with respect to relation r) such that $a_1 \in A_1$. Obviously $a_1 \notin A_0$ because A_0 is a maximal chain. Let $(\{a_1\}, A_1)$ is the second pair of the strong sequence.

Let us suppose that the strong sequence $\{(\{a_\beta\}, A_\beta) \colon \beta < \alpha\}$, where $a_\beta \in A \setminus \bigcup_{\gamma<\beta} A_\gamma$ and $A_\beta \subset \setminus \bigcup_{\gamma<\beta} A_\gamma$ is a maximal chain (with respect to relation r) such that $a_\beta \in A_\beta$, $\beta < \alpha$ has been defined.

Obviously $A \setminus \bigcup_{\beta<\alpha} A_\beta$ is not empty because $|A_\beta| < \kappa$ for all $\beta < \alpha$ and $|\bigcup_{\gamma<\beta} A_\beta| < \kappa$ (because κ is regular). Hence we can choose an arbitrary element $a_\alpha \in A \setminus \bigcup_{\beta<\alpha} A_\beta$ and a maximal chain $A_\alpha \subset A \setminus \bigcup_{\beta<\alpha} A_\beta$ (with respect to relation r) such that $a_\alpha \in A_\alpha$. Let (a_α, A_α) be the next pair of the strong sequence.

According to the construction above we have obtained the strong sequence $\{(\{a_\alpha\}, A_\alpha) : \alpha < \kappa\}$ of the length greater than $\hat{s}(X)$. Contradiction.

Comparing theorem 3.1 and observation before it we immediately obtain that

Corollary 3.2 *Let (X, r) be a set with relation r. If a regular cardinal number κ is not a precalibre for X, then there exists a strong sequence of length κ.*

Proof Let us notice that if κ is not a precalibre, then it is also not a calibre. Our claim immediately follows from proof of theorem 3.1.

In order to obtain some important results let us rewrite definitions of density and boundeness.

A subset $M \subset X$ is *dense* on X if for each $x \in X$ there exists $y \in M$ with $(x, y) \in r$.

We define *density* $d(X)$ as follows:

$$d(X) = \min\{|M| : M \text{ is dense in } X\}.$$

A subset $M \subset X$ is *unbounded* on X if there exists no $x \in X$ such that $(y, x) \in r$ for each $y \in M$.

We define *boundeness* $bd(X)$ as follows:

$$bd(X) = \min\{|M| : M \text{ is an unbounded chain in } X\}.$$

Theorem 3.3 *Let (X, r) be a set with transitive relation r and $d(X) = \kappa$, where κ is a calibre for X. Then X contains an unbounded chain of length κ.*

Proof Let $M = \{x_\alpha : \alpha < \kappa\}$ be a dense set on X. Let $A \subset M$ be a maximal chain. It has length κ. Let us suppose that A has a bound. It means that there exists $p \in A$ such that $(x, p) \in r$ for all $x \in A$. But there exists $x_\xi \in M$ such that $(p, x_\xi) \in r$ because M is dense in X. Contradiction.

Corollary 3.4 *Let (X, r) be a set with transitive relation r with regular density κ and $\kappa > \hat{s}(X)$. Then X contains an unbounded chain of length κ. In other words, if $d(X) > \hat{s}(X)$, then $d(X) = bd(X) > \hat{s}(X)$.*

Proof Applying theorem 3.1 and theorem 3.3 we immediately obtain our claim.

Now we will investigate connections between length of strong sequences and length of antichains.

The minimal cardinal κ such that every antichain on X has length less than κ is a *saturation* of X.
We will use $sat(X)$ to signify the saturation of X.

Let us prove the following theorem (compare [11]).

Theorem 3.5 *If for a set with relation (X, r) there exists a strong sequence $(S_\alpha, H_\alpha); \alpha < (\kappa^\lambda)^+$ such that $|H_\alpha| \leq \kappa^\lambda$ for each $\alpha < (\kappa^\lambda)^+$, then there exists a strong sequence $(S_\alpha, T_\alpha); \alpha < (\lambda)^+$ such that $|T_\alpha| < \omega$ for each $\alpha < (\lambda)^+$,*

Proof Let us take $H_0 \subset X$. Let us notice that if $\alpha > 0$ then the set $S_\alpha \cup H_0$ is not ω-directed. It means that for each $\alpha > 0$ there exists $T \in H_0$ such that $S_\alpha \cup T$ is not ω-directed. Let $B_0 = (\kappa^\lambda)^+ \setminus \{0\}$

Let us consider a function

$$f_0 \colon B_0 \to [H_0]^{<\omega}$$

such that $f_0(\alpha) \in \{T \in [H_0]^{<\omega} \colon S_\alpha \cup T \text{ is not } \omega-\text{directed}\}$ for all $\alpha \in B_0$. Since $|H_0| \leq \kappa^\lambda$, hence the function f_0 determines a partition of B_0 into at most κ^λ elements. But $(\kappa^\lambda)^+$ is regular, hence at least one element of the partition has cardinality $(\kappa^\lambda)^+$. Let

$$P_0 = \{A_0^\xi \subset B_0 \colon |A_0^\xi| = (\kappa^\lambda)^+, f_0|A_0^\xi = \text{ const for } \xi \leq \kappa^\lambda\}.$$

P_0 has the following properties

1) P_0 contains only pairwise disjoint sets
2) $|P_0| \leq \kappa^\lambda$
3) $((\kappa^\lambda)^+ \setminus \bigcup P_0) < (\kappa^\lambda)^+$.

For any $A_0^\xi \in P_0$ let $(S_0, f_0(A_0^\xi))$ be the first pair of strong sequences.

Let $\alpha_0(\xi) = \inf A_0^\xi$ for $\xi < \kappa^\lambda$. For each $\alpha > \alpha_0(\xi)$ the sets $S_\alpha \cup H_{\alpha_0(\xi)}$ are not ω-directed. It means that for each $\alpha > \alpha_0(\xi)$ there exists $T \in [H_{\alpha_0(\xi)}]^{<\omega}$ such that $S_\alpha \cup T$ is not ω-directed. Let $B_{\alpha_0(\xi)} = A_0^\xi \cap \{\alpha < (\kappa^\lambda)^+ \colon \alpha > \alpha_0(\xi)\}$.

Let us consider functions

$$f_{\alpha_0(\xi)} \colon B_{\alpha_0(\xi)} \to [H_{\alpha_0(\xi)}]^{<\omega}$$

such that $f_{\alpha_0(\xi)}(\alpha) \in \{T \in [H_{\alpha_0(\xi)}]^{<\omega} \colon S_\alpha \cup T \text{ are not } \omega-\text{directed}\}$ for $\alpha \in B_{\alpha_0(\xi)}$. Because each function $f_{\alpha_0(\xi)}$ determines a partition of $B_{\alpha_0(\xi)}$ into at most κ^λ elements, hence we can consider a family

$$P_{\alpha_0(\xi)} = \{A_{\alpha_0(\xi)}^\xi \in B_{\alpha_0(\xi)} \colon |A_{\alpha_0(\xi)}^\xi| = (\kappa^\lambda)^+, f_{\alpha_0(\xi)}|A_{\alpha_0(\xi)}^\xi = \text{ const for } \xi \leq \kappa^\lambda\}$$

of the following properties

1) $P_{\alpha_0(\xi)}$ contains only pairwise disjoint sets
2) $|P_{\alpha_0(\xi)}| \leq \kappa^\lambda$
3) $((\kappa^\lambda)^+ \setminus \bigcup P_{\alpha_0(\xi)}) < (\kappa^\lambda)^+$.

For any $A^\xi_{\alpha_0(\xi)} \in P_{\alpha_0(\xi)}$ let $(S_{\alpha_0(\xi)}, f_{\alpha_0(\xi)}(A^\xi_{\alpha_0(\xi)}))$ be the second pair of the strong sequences.

Let us suppose that the following objects have been defined for $\delta < \gamma < \tau < \lambda^+$

sets $B_{\alpha_\gamma(\xi)} = A^\xi_{\alpha_\delta(\xi)} \cap \{\alpha < (\kappa^\lambda)^+ \colon \alpha > \alpha_\gamma(\xi)\}$

functions

$$f_{\alpha_\gamma(\xi)} \colon B_{\alpha_\gamma(\xi)} \to [H_{\alpha_\gamma(\xi)}]^{<\omega}$$

such that $f_{\alpha_\gamma(\xi)}(\alpha) \in \{T \in [H_{\alpha_\gamma(\xi)}]^{<\omega} \colon S_\alpha \cup T \text{ are not } \omega - \text{directed}\}$ for $\alpha \in B_{\alpha_\gamma(\xi)}$,

families

$$P_{\alpha_\gamma(\xi)} = \{A^\xi_{\alpha_\gamma(\xi)} \in B_{\alpha_\gamma(\xi)} \colon |A^\xi_{\alpha_\gamma(\xi)}| = (\kappa^\lambda)^+, f_{\alpha_\gamma(\xi)} | A^\xi_{\alpha_\gamma(\xi)} = \text{ const for } \xi \leq \kappa^\lambda\}$$

of the following properties

1) $P_{\alpha_\gamma(\xi)}$ contains only pairwise disjoint sets
2) $|P_{\alpha_\gamma(\xi)}| \leq \kappa^\lambda$
3) $((\kappa^\lambda)^+ \setminus \bigcup P_{\alpha_\gamma(\xi)}) < (\kappa^\lambda)^+$,

the strong sequences

$$\{(S_{\alpha_\gamma(\xi)}, f_{\alpha_\gamma(\xi)}(A^\xi_{\alpha_{\gamma(\xi)}})) \colon \alpha_\gamma(\xi) < \tau, \xi \leq \kappa^\lambda\}.$$

Let $B_{\alpha_\beta(\xi)} = A^\xi_{\alpha_\gamma(\xi)} \cap \{\alpha < (\kappa^\lambda)^+ \colon \alpha > \alpha_\beta(\xi)\}$. Let us consider a function

$$f_{\alpha_\beta(\xi)} \colon B_{\alpha_\beta(\xi)} \to [H_{\alpha_\beta(\xi)}]^{<\omega}$$

such that $f_{\alpha_\beta(\xi)}(\alpha) \in \{T \in [H_{\alpha_\beta(\xi)}]^{<\omega} \colon S_\alpha \cup T \text{ are not } \omega - \text{directed}\}$ for $\alpha \in B_{\alpha_\beta(\xi)}$,

the families

$$P_{\alpha_\beta(\xi)} = \{A^\xi_{\alpha_\beta(\xi)} \in B_{\alpha_\beta(\xi)} \colon |A^\xi_{\alpha_\beta(\xi)}| = (\kappa^\lambda)^+, f_{\alpha_\beta(\xi)} | A^\xi_{\alpha_\beta(\xi)} = \text{ const for } \xi \leq \kappa^\lambda\}$$

of the following properties

1) $P_{\alpha_\beta(\xi)}$ contains only pairwise disjoint sets
2) $|P_{\alpha_\beta(\xi)}| \leq \kappa^\lambda$
3) $((\kappa^\lambda)^+ \setminus \bigcup P_{\alpha_\beta(\xi)}) < (\kappa^\lambda)^+$.

For each set $A^\xi_{\alpha_\beta(\xi)} \in P_{\alpha_\beta(\xi)}$ let us consider

$$\{(S_{\alpha_\beta(\xi)}, f_{\alpha_\beta(\xi)}(A^\xi_{\alpha_{\beta(\xi)}})) \colon \alpha_\beta(\xi) < \tau, \xi \leq \kappa^\lambda\}$$

the next pair of the strong sequence.

Let us notice that some of defined sequences may be shorter than λ^+. In order to find the proper one on each step we can consider

$$\eta_\beta = \sup\{\alpha_\beta : \alpha_\beta = \inf A^\xi_{\alpha_\beta(\xi)}, A^\xi_{\alpha_\beta(\xi)} \in P_{\alpha_\beta(\xi)}\}$$

Such element exists because $|P_{\alpha_\beta(\xi)}| \leq \kappa^\lambda$. According to our choice it is obvious that the strong sequence $(S_{\eta_\beta}, f_{\eta_\beta}(A_{\eta_\beta}))_{\eta_\beta<\lambda^+}$ exists.

Corollary 3.6 *If for a set (X, r) with relation r there exists a strong sequence $(S_\alpha, H_\alpha); \alpha < (\kappa^\lambda)^+$ such that $|H_\alpha| \leq \kappa$ for each $\alpha < (\kappa^\lambda)^+$, then there exists A of cardinality greater than λ which consists of pairwise incomparable elements.*

Proof Let $(S_\alpha, H_\alpha)_{\alpha<(\kappa^\lambda)^+}$ be a strong sequence. According to theorem 3.5 there exists a strong sequence $(S_\alpha, T_\alpha)_{\alpha<\lambda^+}$ such that $|T_\alpha| < \omega$. According to definition of a strong sequence for all $\alpha < \lambda^+$ the set $S_\alpha \cup T_\alpha$ is ω-directed. For each $\alpha < \lambda^+$ let us consider the sets

$$A_\alpha = \{a \in X : (b, a) \in r \text{ for all } b \in S_\alpha \cup T_\alpha\}.$$

Now let us take all sequences of the form $(a_\alpha)_{\alpha<\lambda^+}$ such that $a_\alpha \in A_\alpha, \alpha < \lambda^+$. According to definition of strong sequence at least one of such sequences contains only pairwise incomparable elements. The elements of such sequence form required set A.

As a corollary we obtain

Corollary 3.7 *Let (X, r) be a set with relation. Let κ be a cardinal number. Then for each $A \subset X$ such that $|A| \geq (\kappa^{sat(X)})^+$ there exists an ω- directed set $B \subset A$ such that $|B| > sat(X)$.*

Proof Let $A \subset X$ be a set of required cardinality. Let us suppose that each ω- directed subset of A has cardinality not grater than $sat(X)$.

Let us choose an arbitrary element $x_0 \in A$. Let $B_0 \subset A$ be a maximal ω - directed set (with respect to relation r) such that $x_0 \in B_0$. Let $(\{x_0\}, B_0)$ be the first pair of a strong sequence.

Let $x_1 \in A \setminus B_0$ be an arbitrary element. Let $B_1 \subset A \setminus B_0$ be a maximal ω - directed set such that $x_1 \in B_1$. Let $(\{x_1\}, B_1)$ be the second pair of the strong sequence.

Let us suppose that the strong sequence $\{(\{x_\beta\}, B_\beta) : \beta < \alpha\}$ where $x_\beta \in A \setminus \bigcup_{\gamma<\beta} B_\gamma$ is an arbitrary element and $B_\beta \subset A \setminus \bigcup_{\gamma<\beta} B_\gamma$ is a maximal ω - directed set such that $x_\beta \in B_\beta$.

According to our assumption $|B_\beta| \leq sat(X)$ for $\beta < \alpha$, hence $|\bigcup_{\beta<\alpha} B_\beta| \leq sat(X)$. Hence there exists an element $x_\alpha \in A \setminus \bigcup_{\beta<\alpha} B_\beta$. Let $B_\alpha \subset A \setminus \bigcup_{\beta<\alpha} B_\beta$ be a maximal ω - directed set such that $x_\alpha \in B_\alpha$. Let $(\{x_\alpha\}, B_\alpha)$ be the next pair of the strong sequence.

Applying previous theorem for $\lambda = sat(X)$ we have obtained an antichain of cardinality greater than $sat(X)$. Contradiction.

The next theorem is one of the main results of this paper.

Theorem 3.8 *Let (X, r) be a set with relation. If $sat(X)$ is regular then $sat(X) \leq \hat{s}(X)$.*

Proof Let us suppose not, i.e. $sat(X) > \hat{s}(X)$. According to our assumption $sat(X)$ is regular and according to theorem 3.1 $sat(X)$ is a calibre for X. Contradiction.

Applying corollary 3.4 and theorem 3.8 we almost immediately obtain the following corollary

Corollary 3.9 *Let (X, r) be aset with relation r. Let $d(X)$ and $sat(X)$ be regular cardinals. If $d(X) > \hat{s}(X)$, then $sat(X) \leq \hat{s}(X) < bd(X) = d(X)$.*

4 Applications

A. Preordered sets

Let $(X, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). For symplifying notation we will use X instead of $(X, \leq)$ Let $bd(X)$ and $d(X)$ mean boundeness and density respectively.

The following theorem is true

Theorem 4.1 *Let $(X, \leq)$ be a preordered set without maximal elements. Then $bd(X)$ is regular and $bd(X) \leq cf(d(X)) \leq d(X)$. Moreover*
1) if $d(X)$ is a calibre then $b(X) = d(X)$;
2) if $d(X) > \hat{s}(X)$, then $sat(X) \leq \hat{s}(X) < bd(X) = d(X)$.

Proof The first part of the theorem follows from [2], p. 195, the second one - from theorem 3.3 and the third one - from corollary 3.9.

Now let us consider a set $F = (\omega^\omega, \leq^*)$ of all functions from ω to ω, where

$$\alpha \leq^* \beta = \{n \in \omega : \neg(\alpha(n) \leq \beta(n))\} \text{ is a finite set.}$$

Let us denote

$$bd(F) = bd\{\omega^\omega, \leq^*\} \text{ and } d(F) = d(\omega^\omega, \leq^*).$$

According to [2], p. 196

$$\aleph_0 < bd(F) \leq cf(d(F)) \leq d(F) \leq 2^{\aleph_0}.$$

The following corollary is true

Corollary 4.2 *Let $F = (\omega^\omega, \leq^*)$. Let $sat(F)$ be regular and $d(F) > \hat{s}(F)$. Then*

$$\aleph_0 < sat(F) \leq \hat{s}(F) \leq 2^{\aleph_0}.$$

Proof The inequalities follows from previous remark and corollary 3.9.

B. Partially ordered sets

Let P be a partially ordered set. Let us notice that all results presented above become true. Moreover using [4] (pp. 157-158) one can formulate the following lemma

Lemma 4.3 *Let P be a partially ordered set. Then $sat(P)$ is a regular uncountable cardinal.*

Let us notice that $sat(P) < \kappa$ for κ being a calibre for P. Using lemma 4.6 and corollaries 3.9 and 4.4 we quickly obtain the following

Corollary 4.4 *Let P be a partially ordered set and $\hat{s}(P)$ be an uncountable cardinal number. Then $sat(P) \leq \hat{s}(P)$. Moreover*
1) if $\kappa > \hat{s}(P)$ then $sat(P) < \kappa$;
2) if $d(P) > \hat{s}(P)$, then $sat(P) \leq \hat{s}(P) < bd(P) = d(P)$.

Let us consider the following example.

Example 4.5 *Let $(P, \leq_{lex})$ be a set with the lexicographical order. Then $sat(P) = \hat{s}(P)$.*

C. Families of sets

Let $\mathcal{P}(X)$ be a family of all sets ordered by inclusion. Then instead of $sat(X)$ we will consider cellularity (we use standard notation for cellularity $c(X)$). Then according to above considerations we obtain

Corollary 4.6 *Let $\mathcal{P}(X)$ be a family of sets. Then $c(X) \leq \hat{s}(X)$.*

The following question is whether the inequality from corollary 4.6 can be substituted by $<$.

Corollary 4.7 *Let X be a regular, first countable and ccc space with $d(X) = \aleph_2$. Then $c(X) < \hat{s}(X)$.*

Proof Let $D \subset X$ be a dense set. Let $x_0 \in D$ be an arbitrary element and let B_{x_0} be a base in the point x_0. Let $U_0 \in B_{x_0}$ be an arbitrary open set such that $x_0 \in U_0$. Let $(\{U_0\}, B_{x_0})$ be the first pair of a strong sequence.

Suppose that the strong sequence $(\{U_\beta\}, B_{x_\beta})_{\beta<\alpha}$ for $\alpha < \aleph_2$ has been defined.

Let us take the set $\{x_\beta \colon \beta < \alpha\}$. Obviously the set $D \setminus cl\{x_\beta \colon \beta < \alpha\}$ is nonempty and we can choose a point $x_\alpha \in D \setminus cl\{x_\beta \colon \beta < \alpha\}$. Since regularity of X we can find a set $U \supset cl\{x_\beta \colon \beta < \alpha\}$ and a neighbourhood V of x_α such that $U \cap V = \emptyset$. Hence for all $\beta < \alpha$ there exists $U_\beta \in B_{x_\beta}$ such that $U_\beta \cap V_\alpha = \emptyset$ (where V_α is a neigbourhood of x_α). Let $(\{x_\alpha\}, B_{x_\alpha})$ be the next pair of the strong sequence. Hence we obtain a strong sequence of length $\aleph_2$.

Example 4.8 *Let us notice that an example of the space with the properties required in corollary 4.7 one can find in [1]. (See also [11], p. 41).*

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