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Extreme boundary of space semi-additive functionals on finite set

Gayratbay F. Djabbarov

Tashkent State Pedagogical University named after Nizami
str.Yusuf Khos Hojib 103, 100070 Tashkent, Uzbekistan
$gayrat\_77@bk.ru$

Abstract

The present paper is devoted to study of the extreme boundary of the convex compact set of all semi-additive functionals on a finite-point compactum. We shall find some classes of extreme points of the space semi-additive functionals $OS(\mathbf{n})$.

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1. Introduction

The space $P(X)$ of all probability measures on a compactum X is well investigated. In [9], T. Radul introduced the space $O(X)$ of weakly additive order-preserving normalized functionals on a compactum X. Topological and geometric properties of weakly additive order-preserving normalized functionals were studied in [1, 2, 6, 7]. In [5] The space of all of semi-additive positively-homogeneous weakly additive order-preserving normalized functionals was investigated and a general form of semi-additive functionals was given. Also categorical properties of the functor of semi-additive functionals OS have been investigated. Geometrical and topological properties of covariant functors on the category compacts and their continuous mappings had investigated by several authors (see [1, 2, 4, 6, 7, 10]).

It is well-known that the extreme boundary of the space of probability measures on compactum coincides with the set of all Dirac measures on this compactum, and therefore is homeomorphic to the initial compactum. This property plays a crucial rule in investigations of geometric properties of the space of probability measures on compactum. The structure of the extreme boundary of the space of semi-additive functionals on compactum is not yet described. In [8] it was described a general form of extreme points of space semi-additive functionals on three-point space.

The aim of the present paper is to study the extreme boundary of the convex compact set $OS(\mathbf{n})$. We give some classes of extreme points of the space of semi-additive functionals $OS(\mathbf{n})$.

2. Preliminary Notes

Let X be a compact set. Denote by $C(X)$ the algebra of all real-valued continuous functions $f : X \to \mathbb{R}$ with point-wise algebraic operations and sup-norm, i. e., with the norm $\|f\| = \max\{|f(x)| : x \in X\}$. For any $c \in \mathbb{R}$ by c_X we denote the constant function, defined by the formula $c_X(x) = c$, $x \in X$. Let $\varphi,\ \psi\ \in C(X)$. The inequality $\varphi \leq \psi$ means that $\varphi(x) \leq \psi(x)$ for all $x \in X$.

A functional $\nu : C(X) \to \mathbb{R}$ is said to be [9]:

(1) weakly additive if $\nu(\varphi + c_X) = \nu(\varphi) + c\nu(1_X)$ for all $\varphi \in C(X)$ and $c \in \mathbb{R}$;
(2) order-preserving, for any $\varphi,\ \psi \in C(X)$ with $\varphi \leq \psi$ we have $\nu(\varphi) \leq \nu(\psi)$;
(3) normalized if $\nu(1_X) = 1$;
(4) positively homogeneous if $\nu(t\varphi) = t\nu(\varphi)$ for all $\varphi, \in C(X), t \in \mathbb{R},\ t \geq 0$;
(5) semi-additive if $\nu(\varphi + \psi) \leq \nu(\varphi) + \nu(\psi)$ for all $\varphi, \psi \in C(X)$.

For every compactum X we denote

$$V(X) = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi].$$

For every map $f : X \to Y$ by $V(f)$ we denote the map from $V(X)$ to $V(Y)$ defined by

$$V(f)(\nu)(\varphi) = \nu(\varphi \circ f),\ \ \nu \in V(X),\ \varphi \in C(X).$$

For a compactum X we denote by:

– $OS(X)$ the set of all weakly-additive, order-preserving, normalized, positive-homogeneous and semi-additive functionals on $C(X)$;

– $P(X)$ the set of all positive, normalized linear functionals on $C(X)$.

Let us consider these sets as subspaces of the space $V(X)$, equipped with the topology of point-wise convergence, in particularly, the base of neighborhoods for the functional $\nu \in \mathcal{F}(X)$, where $\mathcal{F} = OS, P$, is formed by the sets

$$\langle \nu;\ \varphi_1, \varphi_2, \cdots, \varphi_k, \varepsilon\rangle = \{\nu' \in \mathcal{F}(X) : |\nu'(\varphi_i) - \nu(\varphi_i)| < \varepsilon,\ \ i = \overline{1, k}\},$$

where $\varepsilon > 0$, $\varphi_i \in C(X)$, $i = \overline{1, k}, k \in \mathbb{N}$.

For every compactum X the spaces $OS(X)$ and $P(X)$ are convex compact.

Let F be a closed subspace of X. A functional $\nu \in OS(X)$ is said to be supported on F, if $\nu(f) = \nu(g)$ for all $f, g \in C(X)$ with $f|_F = g|_F$. The smallest closed set $F \subset X$ on which the functional μ is supported, is called the support of $\nu \in OS(X)$ and denoted by supp ν, i.e.,

$$\operatorname{supp} \nu = \bigcap \{F : \nu - \text{supported on } F\}.$$

For every convex compactum K we denote by $cc(K)$ the space of all non empty convex compact subsets of K, equipped with the Vietoris topology. Recall [3] that a base of this topology is formed by the sets of the form

$$\langle U_1, \ldots, U_n \rangle = \{A \subseteq cc(K) : A \subseteq U_1 \cup \cdots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for every } i\},$$

where $U_1, \ldots, U_n$ run through the topology of K, $n \in \mathbb{N}$.

For $A \in cc(P(X))$ set

$$\nu_A(\varphi) = \sup\{\mu(\varphi) : \mu \in A\}, \quad \varphi \in C(X). \tag{2.1}$$

Then $\nu_A \in OS(X)$. In [5, Proposition 4.4] it was shown that any functional from $OS(X)$ is represented in the form (2.1), moreover the mapping

$$A \in cc(P(X)) \mapsto \nu_A \in OS(X) \tag{2.2}$$

is an affine homeomorphism between spaces $cc(P(X))$ and $OS(X)$ (see [5, Theorem 1]).

Let X and Y be topological spaces and let $f : X \to Y$ be a continuous mapping. The mapping $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$, where $\mathcal{F} = OS, P$, is defined as the restriction of $V(f)$ on $\mathcal{F}(X)$.

In [5] it was proved that

$$OS(f)(\nu_A) = \nu_{P(f)(A)}. \tag{2.3}$$

Note that for the n-point compact $\mathbf{n} = \{0, 1, ..., n-1\}$, $n \in \mathbb{N}$, the space $C(\mathbf{n})$ is isomorphic to the space $\mathbb{R}^n$, moreover, isomorphism can be defined by

$$f \in C(\mathbf{n}) \to (f(0), f(1), ..., f(n-1)) \in \mathbb{R}^n.$$

In [5] it was shown that the space $OS(\mathbf{2})$ is affine isomorphic to the triangle

$$\triangle = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}, 0 \leq \alpha \leq \beta \leq 1\},$$

moreover, this isomorphism can be defined by a rule

$$(\alpha, \beta) \mapsto \lambda = \alpha\delta_0 + (1-\beta)\delta_1 + (\beta - \alpha)\delta_0 \vee \delta_1,$$

where δ_i is the Dirac functional on i, and a functional $\delta_0 \vee \delta_1 \in OS(\mathbf{2})$ is defined by

$$(\delta_0 \vee \delta_1)(f) = \max\{\delta_0(f), \delta_1(f)\}, \ f \in C(\mathbf{2}).$$

Let K be a convex compact subset of a locally convex space E. Recall that the Minkovsky operation is defined as

$$\lambda_1 A_1 + \lambda_2 A_2 = \{\lambda_1 x_1 + \lambda_2 x_2 : x_1 \in A_1, x_2 \in A_2\},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}, A_1, A_2 \in cc(K)$. According to [3] consider the equivalence relation $\sim$ on $cc(E) \times cc(E)$ defined by:

$$(A, B) \sim (C, D) \quad \text{if only if} \quad A + D = B + C.$$

Denote by L the space of the equivalence classes with respect to $\sim$ and let $[A, B]$ be the class containing (A, B). It is well-known that L is a linear space with respect to natural algebraic operations. For a convex neighborhood U of zero put

$$U^* = \{[A, B] : A \subset B + U, B \subset A + U\}.$$

The sets of the form U^* are the base neighborhoods of zero in L. A mapping $\pi : cc(K) \to L$ defined by the rule

$$\pi(A) = [A, \{0\}]$$

is an embedding, moreover

$$\pi(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \pi(A_1) + \lambda_2 \pi(A_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$, $A_1, A_2 \in cc(K)$.

Consider on $C(\mathbf{3})$ functionals of the following form:

$$\nu_1(f) = f(0), \tag{2.4}$$

$$\nu_2(f) = \max\{f(0),\, tf(1) + (1-t)f(2),\, \alpha f(0) + \beta f(1) + \gamma f(2)\}, \tag{2.5}$$

where $0 \leq t \leq 1$, $\alpha + \beta + \gamma = 1$, $\alpha, \beta, \gamma \geq 0$,

$$\nu_3(f) = \max\{\alpha f(0) + (1-\alpha)f(1),\, \beta f(1) + (1-\beta)f(2),\, \gamma f(2) + (1-\gamma)f(0)\}, \tag{2.6}$$

where $0 < \alpha, \beta, \gamma < 1$.

The functionals $\mu, \nu \in OS(X)$ are called *similar*, if there exists a homeomorphism $\Phi : X \to X$ such that $\nu = \mu \circ \Phi$.

The subsets $A, B \subseteq P(X)$ are called *similar*, if there exists a homeomorphism $\tau : X \to X$ such that $A = P(\tau)(B)$.

In [8] it was given a general form of extreme points of space semi-additive functionals on three-point space. Namely, a functional $\nu \in OS(\mathbf{3})$ is an extreme point in $OS(\mathbf{3})$ if and only if ν is similar to a functional of the form (2.4)–(2.6).

3. Main Results

In this paper we will find sufficiently conditions for functional of the form $\nu = \nu_A$, where $A \in cc(P(\mathbf{n}))$, $\dim A = n-1$, to be an extreme point of $OS(\mathbf{n})$.

Let us consider in $P(\mathbf{n})$ subsets of the following forms:

$$A = \text{co}\{K_1, K_2\}, \tag{3.1}$$

where $K_1 \in cc(P(\mathbf{k}))$, $K_2 \in cc(P(\mathbf{n} \setminus \mathbf{k}))$, $k \in \overline{1, n-1}$;

$$A = \text{co}\{\mu_0, K\}, \tag{3.2}$$

where $K \in cc(P(\mathbf{n-1}))$ is a non one-point extreme point, $\mu_0 = (1-\alpha_0)\nu_0 + \alpha_0 \delta_n$, $\nu_0 \in P(\mathbf{n-1})$, $0 < \alpha_0 < 1$,

$$A = \text{co}\{\delta_n, \mu_0, K\}, \tag{3.3}$$

where $K \in cc(P(\mathbf{n-1}))$ is a subset with dimension $\leq n-2$, μ_0 is a point which does not lie on the hyperplane generated by δ_n and K,

$$A = \text{co}\{\mu_0, \ldots, \mu_{n-1}\}, \tag{3.4}$$

where

$$\mu_i = \sum_{j \neq i} \alpha_{ij} \delta_j, \tag{3.5}$$

$\alpha_{ij} > 0$ for all $i \neq j$ and $\sum\limits_{j\neq i} \alpha_{ij} = 1$ for all $i \in \mathbf{n}$.

The following theorem is the main result of this paper.

Theorem 3.1. *Let A be a subset in $P(\mathbf{n})$ one of the forms* (3.1)–(3.4)*. Then ν_A is an extreme point in $OS(\mathbf{n})$.*

The proof is separated to several Lemmata.

Lemma 3.2. *Let A be a subset in $P(\mathbf{n})$ of the form* (3.1)*. Then A is an extreme point in $cc(P(\mathbf{n}))$.*

Proof. Let A be a subset in $P(\mathbf{n})$ of the form (3.1) and let $A = (B + C)/2$, where $B, C \in cc(P(\mathbf{n}))$. Put

$$B_1 = B \cap P(\mathbf{k}),\ B_2 = B \cap P(\mathbf{n} \setminus \mathbf{k}),$$

$$C_1 = C \cap P(\mathbf{k}),\ C_2 = C \cap P(\mathbf{n} \setminus \mathbf{k}).$$

Since $P(\mathbf{k})$ and $P(\mathbf{n} \setminus \mathbf{k})$ both are faces in $P(\mathbf{n})$, it follows that $K_i = (B_i + C_i)/2$, $i = 1, 2$.

Let $\lambda \in B_1$, $\nu \in C_2$. Since $(\lambda + \nu)/2 \in A$, there exist $\mu_1 \in K_1, \mu_2 \in K_2$ such that $(\lambda + \nu)/2 = t_1\mu_1 + t_2\mu_2$, where $t_1, t_2 \geq 0, t_1 + t_2 = 1$. Take a characteristic function $\chi_{\mathbf{k}}$ of the set $\mathbf{k}$, i.e. $\chi_{\mathbf{k}}(i) = 1$ for $i \in \mathbf{k}$ and $\chi_{\mathbf{k}}(i) = 0$ for $i \notin \mathbf{k}$. From

$$(\lambda(\chi_{\mathbf{k}}) + \nu(\chi_{\mathbf{k}}))/2 = t_1\mu_1(\chi_{\mathbf{k}}) + t_2\mu_2(\chi_{\mathbf{k}}),$$

it follows that $1/2 = t_1$, and therefore $t_1 = t_2 = 1/2$. Thus $\lambda + \nu = \mu_1 + \mu_2$.

Now take an arbitrary function $f : \mathbf{n} \to \mathbb{R}$ such that $f|_{\mathbf{n}\setminus\mathbf{k}} \equiv 0$. Since $\nu(f) = \mu_2(f) = 0$, from $\lambda+\nu = \mu_1+\mu_2$ we have that $\lambda(f) = \mu_1(f)$, and therefore $\lambda = \mu_1$ and $\nu = \lambda_2$. Thus $\lambda \in K_1$, $\nu \in K_2$. This means that $B_1 \subseteq K_1$, $C_2 \subseteq K_2$.

By a similar way we obtain that $B_2 \subseteq K_2$, $C_1 \subseteq K_1$. Thus

$$B \subseteq A \text{ and } C \subseteq A.$$

Let $\mu \in A$ be an arbitrary extreme point in A. Since $A = (B + C)/2$, there exist $\lambda \in B, \nu \in C$ such that $\mu = (\lambda + \nu)/2$. Since μ is an extreme point in A, it follows that $\lambda = \nu = \mu$. So, $\mu \in B$ and $\mu \in C$. Since μ be an arbitrary extreme point, it follows that

$$A \subseteq B \text{ and } A \subseteq C.$$

Thus $B = C = A$. The proof is complete. □

Lemma 3.3. *Let A be a subset in $P(\mathbf{n})$ of the form* (3.2)*. Then A is an extreme point in $cc(P(\mathbf{n}))$.*

Proof. Let A be a subset in $P(\mathbf{n})$ of the form (3.2), i.e.

$$A = \mathrm{co}\{\mu_0, K\},$$

where $K \in cc(P(\mathbf{n-1}))$ is a non one-point extreme point, $\mu_0 = (1-\alpha_0)\nu_0+\alpha_0\delta_n$, $\nu_0 \in P(\mathbf{n-1})$, $0 < \alpha_0 < 1$.

Let us take $B, C \in cc(P(\mathbf{n}))$ such that $A = (B + C)/2$. Set

$$B_0 = B \cap P(\mathbf{n-1}),\ C_0 = C \cap P(\mathbf{n-1}).$$

Taking into account that $P(\mathbf{n-1})$ is a face in $P(\mathbf{n})$, we obtain that $K = (B_0 + C_0)/2$. Since K is an extreme point in $cc(P(\mathbf{n-1}))$, we obtain that $K = B_0 = C_0$.

Take $\lambda_0 \in B, \nu_0 \in C$ such that $\mu_0 = (\lambda_0 + \nu_0)/2$. Let μ be an arbitrary extreme point of K. Since $(\lambda_0 + \mu)/2, (\nu_0 + \mu)/2 \in A$, there exist $\mu_1, \mu_2 \in K$ such that

$$(\lambda_0 + \mu)/2 = t_1\mu_1 + (1 - t_1)\mu_0 \text{ and } (\nu_0 + \mu)/2 = t_2\mu_2 + (1 - t_2)\mu_0,$$

where $0 \le t_1 \le 1,\ 0 \le t_2 \le 1$. Summing the last two equalities we get

$$(\lambda_0 + \nu_0)/2 + \mu = t_1\mu_1 + t_2\mu_2 + (2 - t_1 - t_2)\mu_0,$$

i.e.

$$\mu = t_1\mu_1 + t_2\mu_2 + (1 - t_1 - t_2)\mu_0.$$

Take a function $f : \mathbf{n} \to \mathbb{R}$ defined by $f(n) = 1$ and $f|_{\mathbf{n-1}} \equiv 0$. From

$$\mu(f) = t_1\mu_1(f) + t_2\mu_2(f) + (1 - t_1 - t_2)\mu_0(f)$$

it follows that $(1-t_1-t_2)\alpha_0 = 0$. Thus $t_1+t_2 = 1$. So, $\mu = t_1\mu_1+t_2\mu_2$. Since μ is an extreme point, we obtain that $\mu = \mu_1 = \mu_2$. Thus $(\lambda_0+\mu)/2 = t_1\mu+(1-t_1)\mu_0$ and $(\nu_0 + \mu)/2 = t_2\mu + (1 - t_2)\mu_0$. This means that λ_0 and ν_0 both lie on the line passing through μ_0 and μ. Since μ be an arbitrary and K is a non one-point, it follows that $\lambda_0 = \nu_0$. Thus $A = B = C$. The proof is complete. □

The proof of the following Lemma is similar to the proof of Lemma 3.3.

Lemma 3.4. *Let A be a subset in $P(\mathbf{n})$ of the form (3.3). Then A is an extreme point in $cc(P(\mathbf{n}))$.*

Let A be a subset in $P(\mathbf{n})$ of the form (3.4). It is clear that $A = \mathrm{co}\{\mu_0, \ldots, \mu_{n-1}\}$ is a $(n-1)$-dimensional simplex with vertices $\mu_0, \ldots, \mu_{n-1}$. In particular, if $\alpha_{ij} = \frac{1}{n}$ for all $i \ne j$, it follows that A is a simplex with vertices on the barycenters of $(n-1)$-dimensional faces in $P(\mathbf{n})$. In this case A coincides with a set of all points of the form: $\mu_i = \sum\limits_{i=0}^{n-1} t_i\delta_i$ with $0 \le t_i \le 1/n$ for all $i \in \mathbf{n}$ and $\sum\limits_{i=0}^{n-1} t_i = 1$.

Lemma 3.5. *Let A be a subset in $P(\mathbf{n})$ of the form (3.4). Then A is an extreme point in $cc(P(\mathbf{n}))$.*

Proof. Let us consider the following two cases.

Case 1. $\alpha_{ij} = \frac{1}{n}$ for all $i \ne j$. Let $A = (B + C)/2$, where B, C are convex subsets in $P(\mathbf{n})$. Then there exist $\lambda_i \in B, \nu_i \in C$ such that $\mu_i = (\lambda_i + \nu_i)/2$ for all $i \in \mathbf{n}$. We put

$$\lambda_i = \sum_{j \ne i} \left(\frac{1}{n} + t_{ij}\right)\delta_j,$$

$$\nu_i = \sum_{j \ne i} \left(\frac{1}{n} - t_{ij}\right)\delta_j,$$

where $|t_{ij}| \le 1/n$ for all $i \ne j$ and $\sum\limits_{j \ne i} t_{ij} = 0$ for all $i \in \mathbf{n}$.

Since $(\lambda_k + \nu_p)/2 \in A$, $k, p \in \mathbf{n}$, it follows that

$$\frac{1}{n} + t_{kj} - t_{pj} \le \frac{1}{n},$$

i.e. $t_{kj} - t_{pj} \le 0$. Interchanging k and p we get $t_{pj} - t_{kj} \le 0$. Thus $t_{kj} = t_{pj}$ for all k, p, j with $k \ne j$, $p \ne j$. Denote $t_j = t_{ij}$. Then $\sum\limits_{j \ne i} t_j = 0$. Since

$$0 = \sum_{j \ne i} t_j = \sum_{j \ne k} t_j + t_i - t_k = t_i - t_k,$$

it follows that $t_1 = \ldots = t_n$. Thus $t_i = 0$ for all i. This means that $\lambda_i = \nu_i = \mu_i$ for all i. Thus $A = B = C$.

Case 2. Let $\mu_0, \ldots, \mu_{n-1}$ be arbitrary points of the form (3.5).

Let $A = (B + C)/2$, where B, C are convex subsets in $P(\mathbf{n})$. Then there exist $\lambda_i \in B, \nu_i \in C$ such that $\mu_i = (\lambda_i + \nu_i)/2$ for all $i \in \mathbf{n}$. Let

$$\lambda_i = \sum_{j \ne i} (\alpha_{ij} + t_{ij})\, \delta_j,$$

$$\nu_i = \sum_{j \ne i} (\alpha_{ij} - t_{ij})\, \delta_j,$$

where $|t_{ij}| \le \alpha_{ij}$ for all $i \ne j$ and $\sum\limits_{j \ne i} t_{ij} = 0$ for all $i \in \mathbf{n}$.

Set

$$\lambda'_i = \sum_{j \ne i} (\alpha_{ij} + \varepsilon t_{ij})\, \delta_j,$$

$$\nu'_i = \sum_{j \ne i} (\alpha_{ij} - \varepsilon t_{ij})\, \delta_j,$$

where $0 < \varepsilon < 1$. Since $\mu_i = (\lambda'_i + \nu'_i)/2$ for all $i \in \mathbf{n}$, it follows that

$$A \subseteq (\mathrm{co}\{\lambda'_i\} + \mathrm{co}\{\nu'_i\})/2.$$

Taking into account

$$\frac{\lambda'_k + \nu'_p}{2} = \frac{t_1\lambda_k + (1 - t_1)\lambda_p + t_2\nu_k + (1 - t_2)\nu_p}{2},$$

where $t_1 = (1 + \varepsilon)/2$, $t_2 = (1 - \varepsilon)/2$, we obtain that

$$A \supseteq (\mathrm{co}\{\lambda'_i\} + \mathrm{co}\{\nu'_i\})/2.$$

Thus

$$A = (\mathrm{co}\{\lambda'_i\} + \mathrm{co}\{\nu'_i\})/2. \tag{3.6}$$

So, if necessary, replacing t_{ij} with εt_{ij}, we can assume that $|t_{ij}| \le \min\{\alpha_{ij}, 1/n\}$.

Set

$$\mu'_i = \sum_{j \ne i} \frac{1}{n} \delta_j,$$

$$\lambda_i' = \sum_{j \neq i} \left(\frac{1}{n} + t_{ij} \right) \delta_j,$$

$$\nu_i' = \sum_{j \neq i} \left(\frac{1}{n} - t_{ij} \right) \delta_j.$$

Since co$\{\mu_i\}$ and co$\{\mu_i'\}$ both are simplex with the same dimensions, taking into account (3.6) we obtain that

$$\text{co}\{\mu_i'\} = (\text{co}\{\lambda_i'\} + \text{co}\{\nu_i'\})/2.$$

By case 1 it follows that $t_{ij} = 0$. Thus $B = C = A$. The proof is complete. $\square$

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