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Matrix-Geometric Method for Queueing Model with State- Dependent Arrival of an Unreliable Server and PH Service

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Abstract

In this paper, we consider a state-dependent queueing system in which the system is subject to random breakdowns. Customer arrive at the system randomly following a Poisson process with state-dependent rates. Service times follows PH distribution and repair times are exponentially distributed. The server may fail to service with probability depending on the number of customer completed since the last repair. The main result of this paper is the matrix-geometric solution of the steady-state queue length from which many performance measurements of this queueing system like the stationary queue length distribution, waiting time distribution and the distribution of regular busy period, system utilization are obtained. Numerical examples are presented for both cases.

Key words: Laplace transform, Markov chain, Matrix-geometric method, Phase-type distribution, Steady-state probability.

1 Introduction

In many congestion problems of the queueing systems, it is a common phenomenon that the server is always available in the service station without any interruption and the service system never fails. But in some real life examples, these assumptions are not realistic. In many practical activities we face the situation where service stations may fail and can be repaired by the repairman available in the system. For example, in the manufacturing system, the machines may break down due to some physical problems. Similarly, many other examples are available in the area of computer communication networks and flexible manufacturing systems where the performance of such systems

may suffer due to the service station breakdowns.
In the resent past, many researchers published a study of the queueing system under random breakdown have been studied a number of different methods including vacation models. We refer the reader to Ke and Wang [5], Jain [2] provided performance modeling for state dependent system incorporating mixed standbys and two models of failures. Single server state dependent finite queueing system was analyzed by Kalyanaraman and Raman [4]. Jain and Bhargava [3] studied an unreliable server M/G/1 queueing mode including the concept of Bernoulli's feedback, repeated attempts and discouragement.
Matrix geometric method is a special approach to provide efficient and stable numerical solution to a continuous time Markov chain where the closed form solutions are complex to be obtained specifically in case of repairs, service and multi-state queueing system. In this chapter to study is devoted to a single unreliable server with state dependent queueing model. Neuts [7] has given the detailed account of matrix geometric solutions for stochastic models. Choi,Kim and Sohraby [1] employed the MGM for the nested QBD chains. An M/M/R queue with vacation was analysed by Lin and Ke [6] with help MGM technique.
The layout of this paper is as follows: In the next part of this paper, we give a assumption of this model. In section 3, we derive the basis properties of PH distribution. In section 4, 5 and 6, we derive the breakdown probability, PH distribution probabilities and conditional probability. The steady state govering equations of the model is constructed in section 7. In section 8, the matrix geometric method for the steady state queue length distribution is obtained. In section 9, we define the steady state probabilities. Some performance measures of the model are discussed in section 10. In section 11 is devoted to the numerical illustration and discussion. Conclusion are given in last of this paper.

2 Model Descriptions

In this paper, We consider an matrix-geometric method for queueing model with state-dependent arrival of an unreliable server and phase service. The basic assumptions are described as follows.

In this paper, we investigation to study of queueing model with state dependent controlled arrivals rates of the customer and phase operational provide by a single unreliable server. An unreliable queueing server model is a more realistic representation of the system which refer to the random breakdown of the server while providing the service. The arrival rate of customers depend upon the state of the server and arrives to the system in accordance with a Poisson process with state-dependent rate $\frac{1}{\lambda_n}$, where n is the number of customers in the system. The maximum number of customers in the system can accommodate is N including the one in service, i.e., λ_n for $n \geq N$. If the system is available when a customer arrives, it is immediately processed for a random amount of time following an exponential distribution. Arrived customers are admitted to service on a first come first serve basis. If the server is busy or in repair, then the arriving customer is put in a buffer, awaiting service. The service system is subject to breakdown in the working or idle state. As soon as server fails, it immediately joins the repair facility, after the server is repaired, it starts to server. The systems operational time, that is the sum of service times since the last repair, follows a phase-type distribution. Once a breakdown occurs it takes a random amount of time for repair and repair times follows exponential distribution with rate $\frac{1}{\beta}$.

3 Determination of Breakdown probability

We derive the server breakdown probability. Let $\{X_k, k \geq 1\}$ be a sequence of service times that are independent identically distributed (i.i.d.) random variables following an exponential distribution of rate μ. Without loss of generality, we may assume that the service rate is unit, $\mu = 1$. Denote by Y the service time since the last repair. We assume that Y has a general cumulative distribution function (CDF) F and a probability density function (PDE) $g = \dfrac{F}{dt}$. The Laplace transform (LT) of a function $g(t)$ is denoted by $g^*(s)$.

$$g^*(s) = L\{g\} = \int_0^\infty e^{-sy} g(y)\,\mathrm{d}y \tag{1}$$

The laplace transform of F and its component $\overline{F} = 1 - F$ are given, respectively, by

$$F^*(s) = \frac{1}{s} g^*(s) = \overline{F}^*(s) = \frac{1}{s}[1 - g^*(s)] \tag{2}$$

We restrict ourselves to the server operational time Y having a phase-type distribution with parameter α and T, i.e,.$F(z) = 1 - \alpha e^{Tz}\mathbb{1}$ and $g(z) = \alpha e^{Tz} t'$. Where t' satisfies $T\mathbb{1} + t' = 0$ with $\mathbb{1}$ being defined as a column vector with elements equal to 1. Many nonnegative, continuous distributions can be represented or approximated by a phase-type distribution Neuts [7]. Based on the service times X_k, we define a random variable s_i as the sum of i consecutive service times $s_i = \sum_{k=1}^{i} X_k$ with $s_0 = 0$. It is known that s_i follows a gamma distribution, or Erlang-k distribution with parameters i and μ.

4 Phase-Type Distribution Probability

Theorem 4.1. *If the system service time Y has a phase-type distribution with parameter α and T, the probability q_i that the server will serve at least i customers is given by*

$$q_i = \alpha(i - T/\mu)^{-i}\mathbb{1}, \quad i \geq 0 \tag{3}$$

Immediate results of theorem (4.1) the explicit forms of mixture of exponential (hyper exponential) and Erlang- distribution, we have

❒ For the service time following hyper exponential distribution :

$$q_i = \sum_{j=1}^{m} P_j \left(\frac{\mu}{\mu+\theta}\right)^i \tag{4}$$

❒ For the service time following Erlang-k distribution :

$$q_i = \sum_{j=0}^{k-1} \binom{j+1-1}{j} \frac{\mu^i \theta^j}{(\mu+\theta)^{i+j}}, \quad i > 0 \tag{5}$$

5 Conditional probability

To derive the steady-state balance equations that will be discussed in the next section, we will need the conditional probabilities. That is, given the server has completed a number of services since the last repair, let b_i denote the conditional probability that a breakdown occurs in processing the i^{th} server, given that it has completed $(i-1)$ service after repair. The following theorem will give us the probability of service distribution F with distinct eigenvalues s_j.

Theorem 5.1. *Suppose that the LT$\bar{F}^*(s)$ has m distinct eigenvalues s_j.*

$$\bar{F}^*(s) = \sum_{i=1}^{m} \frac{c_j}{s - s_j} \tag{6}$$

Where c_j are some coefficients. The conditional probability b_j that a breakdown will occur during the next service given that the system has processed $(i-1)$ servics is given as

$$b_j = \frac{\sum_{i=1}^{n} \frac{(-s_j)}{\mu - s_j} \left(\frac{\mu}{\mu - s_j}\right)^{i-l}}{\sum_{i=1}^{n} c_j \left(\frac{\mu}{\mu - s_j}\right)^{i-l}} \tag{7}$$

6 Steady State Analysis

We define the system state by a pair (i, n), where i is the i^{th} service that the system is working on after repair, and n is the number of customers in the system. The states $(0, n)$ represent the system is under repair. We assume the system repair time follow an exponential distribution with rate β. Since the breakdown can occur only when a customer is present in the system, both states $(0, 0)$ and $(1, 0)$ do not exist. That it is impossible that there is no customer in the system when the server is under repair; it is also impossible that the server serves one customer while the system is empty. Now we define the steady-state probability by

$$P_{i,n} = \text{Pro}\left\{\text{state in}(i, n)\right\}, \quad 0 \leq n \leq N, \quad i \geq 0$$

The steady state equations satisfied by the system size probabilities are as follows :

❖ for states system in repair $i = 0$, we have the following balance equations

$$(\lambda_0 + \beta)P_{0,1} = \sum_{i=1}^{\infty} b_i \mu P_{i,1} \tag{8}$$

$$(\lambda_n + \beta)P_{0,1} = \lambda_{n-1}P_{0,n-1} + \sum_{i=1}^{\infty} b_i \mu P_{i,n}, \quad 1 < n \leq N \tag{9}$$

❖ For state $i = 1$, we have

$$(\lambda_1 + \mu)P_{1,1} = \beta P_{0,1} \tag{10}$$

$$(\lambda_n + \mu)P_{1,n} = \beta P_{0,n} + \lambda_{n-1}P_{1,n-1}, \quad 1 < n \leq N \tag{11}$$

❖ Finally the balance equations for state $i > 1$ are

$$\lambda_0 P_{i,0} = \hat{b}_{i-1}\mu P_{i-1,1} \tag{12}$$

$$(\lambda_n + \mu)P_{i,n} = \lambda_{n-1}P_{i,n-1} + \hat{b}_{i-1}\mu P_{i-1,n+1}, \quad 0 < n < N \tag{13}$$

$$\mu P_{i,n} = \lambda_{N-1}P_{i,N-1} \tag{14}$$

The normalization condition is

$$\sum_{i,n} P_{i,n} = 1 \tag{15}$$

7 Matrix Geometry Solution

The theory of matrix-geometry approach was developed Neuts [7] to solve the stationary state probabilities for the vector state Markov Process with repetitive structure, consider the generator matrix Q as shown below :

$$\mathcal{Q} = \begin{bmatrix} \mathcal{B}_0 & \mathcal{C}_0 & 0 & 0 & 0 & \dots \\ b_1\mathcal{B}_1 & \mathcal{A}_1 & \hat{b}_1\mathcal{A}_0 & 0 & 0 & \dots \\ b_2\mathcal{B}_2 & 0 & \mathcal{A}_1 & \hat{b}_2\mathcal{A}_0 & 0 & \dots \\ b_3\mathcal{B}_3 & 0 & 0 & \mathcal{A}_1 & \hat{b}_3\mathcal{A}_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{16}$$

The matrix can be decomposed in to the sub-matrices $\mathcal{B}_0, \mathcal{A}_1, b_n\mathcal{B}_n, \mathcal{C}_0, \hat{b}_n\mathcal{A}_0$ as follows,

$$\mathcal{B}_0 = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & 0 \\ 0 & -\lambda_1-\beta & \lambda_1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\lambda_{N-1}-\beta & \lambda_{N-1} \\ 0 & 0 & \dots & 0 & -\beta \end{bmatrix}$$

$$\mathcal{A}_1 = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & 0 \\ 0 & -\lambda_1-\mu & \lambda_1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -\lambda_{N-1}-\mu & \lambda_{N-1} \\ 0 & 0 & \dots & 0 & -\mu \end{bmatrix}, \quad i>0$$

$$\mathcal{B}_i = b_i \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \mu & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \mu & 0 \\ 0 & 0 & \dots & 0 & \mu \end{bmatrix}, \quad i>0$$

$$\mathcal{C}_0 = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \beta & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \beta & 0 \\ 0 & 0 & \dots & 0 & \beta \end{bmatrix}, \mathcal{A}_0 = \hat{b}_i \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots \\ 0 & \mu & \dots & \dots & \dots \\ 0 & 0 & \dots & & \\ 0 & 0 & \dots & \mu & 0 \end{bmatrix}, \quad i>0$$

Let $\mathcal{X}$ be a steady-state probability vector of $\mathcal{Q}$ partitioned of $\mathcal{X} = [\mathcal{X}_1, \mathcal{X}_2, \dots \mathcal{X}_i, \dots]$, where $\mathcal{X}$ satisfies

$$\mathcal{X}\mathcal{Q} = 0, \ \mathcal{X}\mathbb{1} = 1$$

$\mathbb{1}$ is the column vector of appropriate dimention with all elements equal to 1. Then equations (8) to (15) can be written in matrix forms

$$\mathcal{X}_0\mathcal{B}_0 + \sum_{i=1}^{\infty} \mathcal{X}_i\mathcal{B}_n = 0 \tag{17}$$

$$\mathcal{X}_{i-1}\mathcal{A}_{i-1} + \mathcal{X}_i\mathcal{A}_i = 0, \quad i > 0 \tag{18}$$

8 Solution of the Steady State Probabilities

The matrix $\mathcal{A}$ is nonsingular. Thus its inverse exists. If we let $H_0 = -\mathcal{B}_{01}\mathcal{A}_1^{-1}$ and $H = -\mathcal{A}_0\mathcal{A}_1^{-1}$, it is possible to express $\mathcal{X}_i$ in terms of $\mathcal{X}_0$, Repeatedly using equation (18), we obtain the following relation for $\mathcal{X}_i$.

Theorem 8.1. *The steady state $\mathcal{X}_i$ has a matrix product form in terms of $\mathcal{X}_0$ and H*

$$\mathcal{X}_1 = \mathcal{X}_0 H_0 \tag{19}$$

$$\mathcal{X}_i = \mathcal{X}_0 \prod_{k=1}^{i-1} \hat{b_k} H_0 \mathcal{R}^{i-1}, \, i > 1 \tag{20}$$

Theorem 8.2. *For the service time follows an Erlank-k distribution, we have*

$$\sum_{i=1}^{\infty} q_{i-1}\mathcal{R}_{i-1} = I + \sum_{j=1}^{k} c_j \frac{\mu\mathcal{R}}{\theta} \left(\frac{\theta}{\mu+\theta}\right)^j \left(I - \frac{\mu}{\mu+\theta}\mathcal{R}\right)^{-j} \tag{21}$$

Theorem 8.3. *Using vector $\mathcal{X} = [\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]$, we have the marginal probability of queue length that is given in terms of $\mathcal{X}_0$*

$$\mathcal{X} = \mathcal{X}_0 \left[I + H_0 \sum_{j=1}^{m} c_j L_j \right]$$

9 Performance of the System

In this section, some useful performance measure of the proposed model in terms of the steady-state probabilities explicity. Some of the system performance induces are as follows,

❖ Expected mean queue length :

$$E(N) = \sum_{i=0}^{N} \mathcal{X}_i \tag{22}$$

❖ Probability of system being idle :

$$P_{\mathrm{ID}} = \mathcal{X}_0 \tag{23}$$

- Probability of system being breakdown :

$$P_{BD} = \mathcal{X}_0 \mathbb{1} \tag{24}$$

- Probability system being busy :

$$P_{\text{BS}} = \mathcal{X}_i - \mathcal{X}_0 - \mathcal{X}_0 \mathbb{1} \tag{25}$$

At steady-state, the above probabilities can be interpreted as the long-run percentage of time. A computation procedure is presented below to summarize the computation steady-state probability $\mathcal{X}$.

10 Numerical Results and Discussion

In this section, we present the numerical illustration to evaluate by using the results derived in the previous section. For numerical illustration, lets us consider the values $\lambda_n = 1, \mu = 1, \theta = 0.1, \beta = 1$ and N = 5,10,15,20,25,30,35. Using these vaues in the equations (22) to (25), we compute the tables (1 and 2 respectively. Corresponding graphs have been show in figures (1 and 2) respectively.

- From table (1) and figure (1) we can observed that the expected queue length increases when the system capcity increases.

- From table (2) and figure (2) we can observed that the expected system breakdown time increases when the system capcity increases.

Table 1: System capacity Vs Expected mean queue length

N	Exponential	Hyper Exponential	Erlang-2
	E(N)	E(N)	E(N)
5	2.0027	2.5642	1.9002
10	2.1134	7.0367	5.4321
15	10.4351	11.3367	9.1003
20	14.4655	15.3479	12.5013
25	18.1003	19.2100	16.1034
30	21.0950	22.3436	19.1263
35	24.0691	25.3940	22.1500

Table 2: System capacity Vs Expected of system beakdown

N	Exponential	Hyper Exponential	Erlang-2
	E(W)	E(W)	E(W)
5	0.1068	0.0601	0.1133
10	0.0554	0.0296	0.0611
15	0.0259	0.0112	0.0300
20	0.0113	0.0035	0.0151
25	0.0041	0.0015	0.0078
30	0.0005	0.0012	0.0040
35	0.0003	0.0005	0.0031

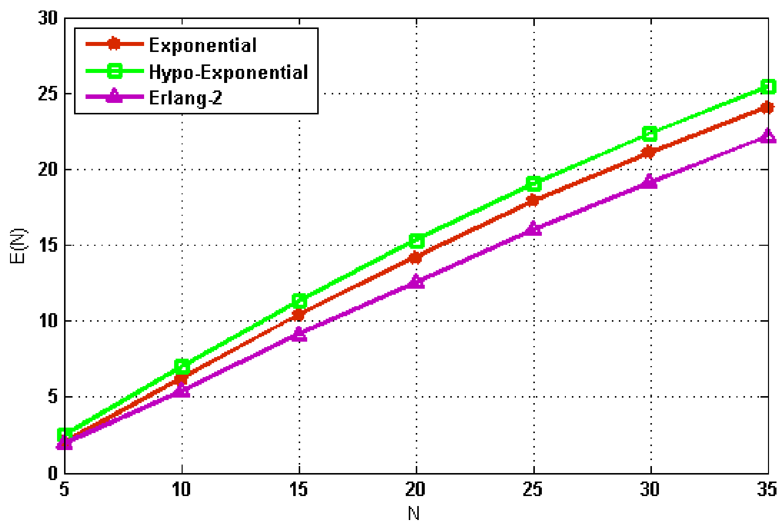

Figure 1: N Vs Expected queue length.

11 Conclusion

In the present parer, we have studied the single server model for unreliable server queue. For real life situations, where the arrival of customer or job depends on the server status, such performance indices established may be very helpful in the designing and development of many system in the field of computer system, manufacturing system and telecommunication networks, etc.

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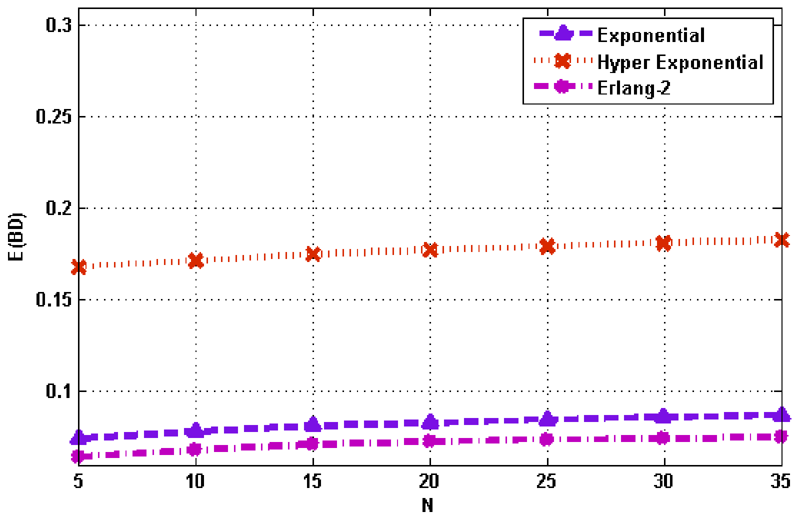

Figure 2: N Vs Expected system breakdown.

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