Mathematica Aeterna, Vol. 6, 2016, no. 2, 135 - 139

Local Extremum Principle for a Class of Elliptic Variational Inequalities

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Abstract

A class of elliptic variational inequalities are considered in this paper. Local extremum principle for weak solutions is obtained using Moser iterative method.

Mathematics Subject Classification: 35J60, 35J67

Keywords: extremum principle, weak solution, variational inequality

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$. And let $W^{1,p}(\Omega)$, $1 < p < \infty$, be the first-order Sobolev space of functions $u \in L^p(\Omega)$ whose distributional gradient ∇u belongs to $L^p(\Omega)$. Suppose that ψ is any functions in Ω with values in $\mathbb{R} \cup \{\pm\infty\}$, and that $\theta \in W^{1,p}(\Omega)$. Let

$$\mathcal{K}^p_{\psi,\theta}(\Omega) = \Big\{ v \in W^{1,p}(\Omega) : v \geq \psi,\ a.e. \text{ and } v - \theta \in W^{1,p}_0(\Omega) \Big\}. \tag{1.1}$$

The function ψ is the obstacle function and θ determines the boundary value.

In this paper, we consider a class of elliptic variational inequalities

$$\begin{cases} u \in \mathcal{K}^p_{\psi,\theta}(\Omega), \\ \displaystyle\int_\Omega \langle A(x, \nabla u), \nabla(v - u)\rangle dx \geq \int_\Omega B(x, \nabla u)(v - u)dx, \quad \forall v \in \mathcal{K}^p_{\psi,\theta}(\Omega), \end{cases} \tag{1.2}$$

where $A(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}^n$, $B(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}$ are Carathéodory functions, for almost all $x\in\Omega$, all $\xi\in\mathbb{R}^n$, satisfying the coercivity and growth conditions:

$$\langle A(x,\xi),\xi\rangle\geq\alpha|\xi|^p;\quad |A(x,\xi)|\leq\beta|\xi|^{p-1};\quad |B(x,\xi)|\leq\gamma|\xi|^{p-1}. \tag{1.3}$$

where α,β,γ are some nonnegative constants, $1<p<n$.

In [1], the extremum principle for very weak solutions of A-harmonic equation is derived by using the stability result of Iwaniec-Hodge decomposition. in this paper we continue to consider the elliptic problems. Local extremum principle for weak solutions of elliptic variational inequalities (1.1) is obtained using Moser iterative method. The main results is in the following.

Theorem 1.1 *Let $u\in W^{1,p}_{loc}(\Omega)$ be the nonnegative weak solution of elliptic variational inequalities (1.1), then*

$$ess\sup_{B_{\frac{R}{2}}} u\leq C\left[1+\left(\fint_{B_R}u^p dx\right)^{\frac{1}{p}}\right], \tag{1.4}$$

where C is only associated with $\alpha,\beta,\gamma,n,p,\mathrm{diam}\Omega$.

2 Proof of Theorem 1.1

Proof Let $u\in W^{1,p}_{loc}(\Omega)$be the nonnegative weak solution of elliptic variational inequalities (1.1). The truncated functions $\xi\in C_0^\infty(B_R)$, $0\leq\xi\leq 1$, $|\nabla\xi|\leq\dfrac{C}{R}$, and $\xi\equiv 1$ in $B_{\frac{R}{2}}$. Let

$$v=u+\xi^p\kappa^t, \tag{2.1}$$

where $\kappa=u+1$, $t\geq 1$ is a constant to be determined. Since

$$v-\theta=(u-\theta)+\xi^p\kappa^t\in W_0^{1,p}(\Omega),\quad v-\psi=(u-\psi)+\xi^p\kappa^t\geq 0, \tag{2.2}$$

then $v\in\mathcal{K}^p_{\psi,\theta}$. We can use (2.1) to (1.1), it yields

$$\begin{aligned} t\int_{B_R}\langle A(x,\nabla u),\kappa^{t-1}\xi^p\nabla\kappa\rangle dx\geq & -p\int_{B_R}\langle A(x,\nabla u),\xi^{p-1}\kappa^t\nabla\xi\rangle dx\\ & +\int_{B_R}B(x,\nabla u)\xi^p\kappa^t dx. \end{aligned} \tag{2.3}$$

By the condition (1.2),

$$t\int_{B_R}\langle A(x,\nabla u),\kappa^{t-1}\xi^p\nabla\kappa\rangle dx\geq t\int_{B_R}\alpha|\nabla\kappa|^p\kappa^{t-1}\xi^p dx, \tag{2.4}$$

$$\left|-p\int_{B_R}\langle A(x,\nabla u),\xi^{p-1}\kappa^t\nabla\xi\rangle dx\right| \le p\beta\int_{B_R}|\nabla\kappa|^{p-1}\xi^{p-1}\kappa^t|\nabla\xi|dx, \tag{2.5}$$

$$\left|\int_{B_R}B(x,\nabla u)\xi^p\kappa^t dx\right| \le \gamma\int_{B_R}|\nabla\kappa|^{p-1}\xi^p\kappa^t dx. \tag{2.6}$$

Combined (2.5)-(2.7) with (2.4), and noticing that $t\ge 1$, we have

$$\begin{aligned}&\int_{B_R}\alpha\kappa^{t-1}\xi^p|\nabla\kappa|^p dx\\ \le\ & p\beta\int_{B_R}\xi^{p-1}\kappa^t|\nabla\kappa|^{p-1}|\nabla\xi|dx+\gamma\int_{B_R}\xi^p\kappa^t|\nabla\kappa|^{p-1}dx\\ =\ & p\beta I_1+\gamma I_2.\end{aligned} \tag{2.7}$$

By Young's inequality,

$$I_1\le\varepsilon\int_{B_R}\xi^p\kappa^{t-1}|\nabla\kappa|^p dx+c(\varepsilon)\int_{B_R}|\nabla\xi|^p\kappa^{t+p-1}dx, \tag{2.8}$$

$$I_2\le\varepsilon\int_{B_R}\xi^p\kappa^{t-1}|\nabla\kappa|^p dx+c(\varepsilon)\int_{B_R}\xi^p\kappa^{t+p-1}dx. \tag{2.9}$$

Combined (2.9)-(2.10) with (2.8), and let ε small enough to satisfy $\alpha-p\beta\varepsilon-\gamma\varepsilon>0$, it yields

$$\int_{B_R}\xi^p\kappa^{t-1}|\nabla\kappa|^p dx\le C\left[\int_{B_R}|\nabla\xi|^p\kappa^{t+p-1}dx+\int_{B_R}\xi^p\kappa^{t+p-1}dx\right], \tag{2.10}$$

where C is associated with $\alpha,\beta,\gamma,p,\varepsilon$, and is independent of t. Let

$$W=\kappa^{\frac{t+p-1}{p}}, \tag{2.11}$$

then

$$|\nabla W|=\frac{t+p-1}{p}\kappa^{\frac{t-1}{p}}|\nabla\kappa|,\quad \kappa^{t-1}|\nabla\kappa|^p=|\nabla W|^p\left(\frac{p}{t+p-1}\right)^p. \tag{2.12}$$

Submit (2.13) into (2.12) , yields

$$\left(\frac{p}{t+p-1}\right)^p\int_{B_R}\xi^p|\nabla W|^p dx\le C\left[\int_{B_R}|\nabla\xi|^pW^p dx+\int_\Omega\xi^pW^p dx\right]. \tag{2.13}$$

Setting $s=t+p-1$, then by (2.12),

$$\int_{B_R}\xi^p|\nabla W|^p dx\le Cs^p\left[\int_{B_R}|\nabla\xi|^pW^p dx+\int_{B_R}\xi^pW^p dx\right]. \tag{2.14}$$

Since

$$\begin{aligned}\int_{B_R} |\nabla(\xi W)|^p dx &= \int_{B_R} |W\nabla\xi + \xi\nabla W|^p \, dx \\ &\leq 2^{p-1}\int_{B_R} |\nabla\xi|^p W^p dx + 2^{p-1}\int_{B_R} \xi^p |\nabla W|^p dx. \quad (2.15)\end{aligned}$$

Combined (2.16) with (2.17) yields

$$\int_{B_R} |\nabla(\xi W)|^p dx \leq Cs^p \left[\int_{B_R} |\nabla\xi|^p W^p dx + \int_{\Omega} \xi^p W^p dx\right]. \quad (2.16)$$

Then setting $p^* = \dfrac{np}{n-p}$, by the Sobolev imbedding theorem,

$$\left[\int_{B_R} (\xi W)^{p^*} dx\right]^{\frac{p}{p^*}} \leq \int_{B_R} |\nabla(\xi W)|^p dx. \quad (2.17)$$

Since $W = \kappa^{\frac{t+p-1}{p}} = \kappa^{\frac{s}{p}}$, the above inequality becomes

$$\left[\int_{B_R} \xi^{p^*} \kappa^{\frac{ns}{n-p}} dx\right]^{\frac{n-p}{n}} \leq Cs^p \left[\int_{B_R} |\nabla\xi|^p \kappa^s dx + \int_{B_R} \xi^p \kappa^s dx\right]. \quad (2.18)$$

Noticing that $0 \leq \xi \leq 1$, $\xi \in C_0^\infty(B_R)$, $|\nabla\xi| \leq \dfrac{C}{R}$, $\xi \equiv 1$ in $B_{\frac{R}{2}}$, we have

$$\left[\int_{B_{\frac{R}{2}}} \kappa^{\frac{ns}{n-p}} dx\right]^{\frac{n-p}{n}} \leq Cs^p \left[\frac{C}{R^p}\int_{B_R} \kappa^s dx + \int_{B_R} \kappa^s dx\right]. \quad (2.19)$$

By the above inequality, we can choosing C large enough or R small enough to get

$$\left[\int_{B_{\frac{R}{2}}} \kappa^{\frac{ns}{n-p}} dx\right]^{\frac{n-p}{n}} \leq \frac{Cs^p}{R^p}\int_{B_R} \kappa^s dx. \quad (2.20)$$

Let $S_m = p\left(\dfrac{n}{n-p}\right)^m$, $m = 0, 1, 2, \ldots.$ Using S_m instead of s in (2.22), and setting S_m times square on both sides, we have

$$\|\kappa\|_{L^{S_{m+1}}(B_{\frac{R}{2}})} \leq \left(\frac{C}{R^p}\right)^{\frac{1}{S_m}} \cdot S_m^{\frac{p}{S_m}} \cdot \|\kappa\|_{L^{S_m}(B_R)}. \quad (2.21)$$

After iteration the above inequality yields

$$\|\kappa\|_{L^{S_{m+1}}(B_{\frac{R}{2}})} \leq \left(\frac{C}{R^p}\right)^{\sum_{m=0}^{\infty}\frac{1}{S_m}} \cdot \prod_{m=0}^{\infty} S_m^{\frac{p}{S_m}} \cdot \|\kappa\|_{L^p(B_R)}. \tag{2.22}$$

Since $\sum_{m=0}^{\infty} \frac{1}{S_m} = \frac{n}{p^2}$, and it is easy to verify the convergence of the series $\prod_{m=0}^{\infty} S_m^{\frac{p}{S_m}}$, then by (2.24) we have

$$\|\kappa\|_{L^{S_{m+1}}(B_{\frac{R}{2}})} \leq \frac{C}{R^{\frac{n}{p}}}\|\kappa\|_{L^p(B_R)} \leq C\left(\fint_{B_R} \kappa^p dx\right)^{\frac{1}{p}}. \tag{2.23}$$

Noticing that $S_m \to \infty$ when $m \to \infty$. Let $m \to \infty$, (2.24) yields

$$\|\kappa\|_{L^\infty(B_{\frac{R}{2}})} \leq C\left(\fint_{B_R} \kappa^p dx\right)^{\frac{1}{p}}. \tag{2.24}$$

Noticing that $\kappa = u + 1$ and u is nonnegative, then

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}} u = \|u\|_{L^\infty(B_{\frac{R}{2}})} \leq \|\kappa\|_{L^\infty(B_{\frac{R}{2}})}. \tag{2.25}$$

$$\left(\fint_{B_R} \kappa^p dx\right)^{\frac{1}{p}} = \left(\fint_{B_R} (u+1)^p dx\right)^{\frac{1}{p}} \leq C\left[\left(\fint_{B_R} u^p dx\right)^{\frac{1}{p}} + 1\right]. \tag{2.26}$$

Combined the above two inequalities into (2.26), we have the desired result (1.4). The proof is completed.

□

ACKNOWLEDGEMENTS. The research is supported by Scientific Research Foundation of Hebei United University(z201219) and National Science Foundation of Hebei Province (A2013209278).

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Received: January 3, 2016