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3-1 Piecewise NCP Function for Nonlinear Complementary Problem

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Abstract

In this paper, we define a 3-1 piecewise linear NCP function and proposed a modified nonmonotone method for nonlinear complementarity problem. Then, we use a piecewise NCP function to transform the original problem into a semi-smooth equation. This algorithm solves a system of linear equations with the nonsingular coefficient matrix and introduces a nonmonotone linear search. Under some suitable assumptions,we prove the globally convergence of the algorithm.

Mathematics Subject Classification: xxxxx

Keywords: nonlinear complementary problem, piecewise NCP function, nonmonotone, global convergence

1 Introduction

We consider the following nonlinear complementarity problem NCP: find $(x, s) \in R^n \times R^n$, to satisfy :

$$x \geq 0, s = F(x) \geq 0, x^T s = 0, \tag{1}$$

where $F : R^n \to R^n$ is continuously differentiable function. The complementarity problem plays an important role in economics equilibrium, system engineering, optimization and others. In the past few decades, many solving methods has been put forward. The smooth algorithm is a kind of important method. And it has been studied extensively when F is smooth, see for instance [1-3] and references therein. A nonlinear complementarity problem is transformed into a system of nonsmooth equations based on nonlinear complementarity function.

For the case where F is nonsmooth function, [4-6] dealt with the problem (1). They transform (1) into a unconstrained optimization, then solve it by nonsmooth optimization method. In this paper, we try to study nonlinear complementarity with 3-1 piecewise NCP function by nonmonotone line research. We first transform the original problem into a semi-smooth equation. Then propose a Newton method to solve the semi-smooth equation. This method is to solve a system of nonlinear equations:

$$H(x,s) = \begin{pmatrix} s - F(x) \\ \Phi(x,s) \end{pmatrix} = 0, \tag{2}$$

which is equivalent to (1). Among them, $\Phi(x,s)$ is a kind of NCP function satisfies:

$$\Phi(x,s) = \begin{cases} 3x - \dfrac{x^2}{s}, & \text{if } s \geq x > 0, \text{ or } 3s > -x \geq 0 \ , \\ 3s - \dfrac{s^2}{x}, & \text{if } x > s > 0, \text{ or } 3x > -s \geq 0 \ , \\ 9x + 9s, & \text{if } x \leq 0 \text{ and } -x \geq 3s, \text{ or } s \leq -3x \leq 0 \ . \end{cases} \tag{3}$$

$F : R^n \to R^n$ is continuously differentiable P_0-function.

Motivated by the above ideas, we construct a Newton method based on the solution of nonlinear equations obtained by the 3-1 piecewise NCP function. In this algorithm, we only need to solve one nonlinear equations per iteration so that the computational costs are reduced. The method has proved to be implementable and globally convergent without a strict complementarity.

This paper is organized as follows. Section 2 introduces the 3-1 piecewise linear NCP function and the properties of it. And transformed the nonlinear complementary problem into equivalent system of nonlinear equations. Section 3 introduces the algorithm. Section 4 proves the algorithm to be implementable and presents the algorithm's convergence theory.

2 Preliminary Notes

Function $\Phi : R^2 \longrightarrow R$ is called an NCP function when $\Phi(a,b) = 0$ if and only if $a \geq 0, b \geq 0$ and $ab = 0$. The 3-1 piecewise linear NCP function is defined as:

$$\Phi(a,b) = \begin{cases} 3a - \dfrac{a^2}{b}, & \text{if } b \geq a > 0, \text{ or } 3b > -a \geq 0 \ , \\ 3b - \dfrac{b^2}{a}, & \text{if } a > b > 0, \text{ or } 3a > -b \geq 0 \ , \\ 9a + 9b, & \text{if } a \leq 0 \text{ and } -a \geq 3b, \text{ or } b \leq -3a \leq 0 \ . \end{cases} \tag{4}$$

If $(a,b) \neq (0,0)$,then

$$\nabla\Phi(a,b) = \begin{cases} \begin{pmatrix} 3 - \dfrac{2a}{b} \\ \dfrac{a^2}{b^2} \end{pmatrix}, & \text{if } b \geq a > 0, \text{ or } 3b > -a \geq 0 \ , \\ \begin{pmatrix} \dfrac{b^2}{a^2} \\ 3 - \dfrac{2b}{a} \end{pmatrix}, & \text{if } a > b > 0, \text{ or } 3a > -b \geq 0 \ , \\ \begin{pmatrix} 9 \\ 9 \end{pmatrix}, & \text{if } 0 \geq a \text{ and } -a \geq 3b, \text{ or } b \leq -3a \leq 0 \ . \end{cases} \tag{5}$$

So, Φ is continuously differentiable everywhere except at the origin, where it is strongly semismooth.

Detailed Property and application of piecewise NCP function see[7].

It is easy to check the following Proposition

Proposition 2.1 For the function $\Phi(a,b)$ the following holds.

1. $\Phi(a,b) = 0 \Longleftrightarrow a \geq 0, b \geq 0$ and $ab = 0$;
2. the square of Φ is continuously differentiable;
3. Φ is twice continuously differentiable everywhere except at the origin, but it is strongly semismooth at the origin and is a pseudo-smooth NCP function.

Construct function: $H : R^{2n} \longrightarrow R^{2n}$

$$H(x,s) = \begin{pmatrix} s - F(x) \\ \Phi(x,s) \end{pmatrix}. \tag{6}$$

For the update of s, we require it infinitely close to $F(x)$. So we order $s - F(x) = 0$. Therefore, contacting the first line of Proposition 2.1, we know that nonlinear complementarity problem (1) is equivalent to solving the minimization problem:

$$\min \ \Psi(x,s)$$

$$\Psi(x,s) := \|H(x,s)\|$$

3 Algorithm

If (x,s)=(0,0), let$\xi_i^k = 1$, $\eta_i^k = 1$
Otherwise we denote (ξ_i^k,η_i^k)=$\nabla\Phi(x,s)$
Clearly $\xi_i^k > 0$ and $\eta_i^k > 0$, let

$$V^k = \begin{pmatrix} -F'(x^k) & I \\ diag(\xi_i^k) & diag(\eta_i^k) \end{pmatrix} \tag{7}$$

where I is identity matrix of $n \times n$, $diag(\xi_i^k)$ or $diag(\eta_i^k)$ denotes the diagonal matrix whose i diagonal element is ξ_i^k or η_i^k respectively.

We now present the algorithm combining a Newton method with the nonmonotone line search, the following algorithm is obtained d and λ by calculating system of nonlinear equations, which from the Hessian of H. In order to solve: min $\Psi(x, s)$, we adopt the nonmonotone line search based on [8], so that the trial step is more flexible.

Algorithm 3.1

Step0 Initialization:

Given initial point $x^0 \in R^n$, $s^0 \in R^n$, $\tau \in (0, 1)$, $0 < \theta < 1$.

Step1 Calculation of the search direction:

Calculate d^k and λ^k by solving the following linear system in (d, λ):

$$V^k \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} F(x^k) - s^k \\ -\Phi(x^k, s^k) \end{pmatrix} \tag{8}$$

Step2 Nonmonotone line search.

Step2.1 if

$\Psi(x^k + d^k, s^k + \lambda^k) \leq \theta \Psi(x^k, s^k)$

and the next inequation (9) holds. then let $x^{k+1} = x^k + d^k$, $s^{k+1} = s^k + \lambda^k$. Go to step 3.

Step2.2 Nonmonotone line search.

Let $x^{k+1} = x^k + \alpha_k d^k$, $s^{k+1} = s^k + \alpha_k \lambda^k$, where $\alpha_k = \tau^j$ $(0 < \tau < 1)$ and j is the smallest non-negative integer satisfying :

$$\|\Phi(x^{k+1}, s^{k+1})\| \leq \max\{\frac{r_k + 1}{2}, \theta\} \cdot p_{max}^k \tag{9}$$

where $r_k = \dfrac{\max\limits_{0 \leq r \leq m(k+1)-1} \|\Phi^{k+1-r}\|}{\max\limits_{0 \leq r \leq m(k)-1} \|\Phi^{k-r}\|}$,$p_{max}^k = \max\limits_{0 \leq r \leq m(k)-1} \|\Phi^{k-r}\|$,$m(0) = 0$, $0 \leq m(k) \leq min\{m(k-1)+1, M\}$, M is a positive constant.

If x^{k+1}, s^{k+1} satisfy $\Psi(x^{k+1}, s^{k+1}) = 0$ then stop.

Step3 Update: Let k=k+1 and go to Step 1.

4 Main Results

In this section, we discuss the global convergence property of algorithm with the nonmonotone line search. In order to achieve the convergence of the algorithm, we give some Assumptions as follows:

Assumption 4.1

$F : R^n \to R^n$ is continuously differentiable P_0-function, so that $F'(x)$ is positive semidefinite.

Lemma 4.1 If $\Phi^k \neq 0$ then given any $\varepsilon > 0$ there is a $\bar{t} > 0$, such that for any $0 < t \leq \bar{t}$ and all k,

$\|\Phi^k\|^2 - \|\Phi(x^k + td^k, s^k + t\lambda^k)\|^2 \geq (2-\varepsilon)t\|\Phi^k\|^2$

Proof:If $\Phi^k \neq 0$ implies

$$diag(\xi^k) \cdot d^k + diag(\eta^k) \cdot \lambda^k = -\Phi(x^k, s^k) \tag{10}$$

We define that if $(x, s) \neq (0, 0)$, then

$(\overline{\xi}_i^k, \overline{\eta}_i^k) = (\xi_i^k, \eta_i^k)$

otherwise

$\overline{\xi}_i^k d^k + \overline{\eta}_i^k \lambda^k = \phi_i'((x^k, s^k), (d^k, \lambda^k))$

where $\phi_i'((x^k, s^k), (d^k, \lambda^k))$ is the direction derivative of $\phi_i(x, s)$ at (x^k, s^k) in the direction (d^k, λ^k). Let $diag(\overline{\xi}^k)$ or $diag(\overline{\eta}^k)$ denote the diagonal matrix whose ith diagonal element is $\overline{\xi}_i^k$ or $\overline{\eta}_i^k$, respectively.

Clearly, for all i,

$$\phi_i(x^k + td^k, s^k + t\lambda^k) - \phi_i^k - t(\overline{\xi}_i^k d^k + \overline{\eta}_i^k \lambda^k) = \circ(t) \tag{11}$$

It follows by the definition of above,we have

$$\|\Phi^k + t(diag(\overline{\xi}^k)d^k + diag(\overline{\eta}^k)\lambda^k)\|^2 = (1-2t)\|\Phi^k\|^2 + t\|diag(\overline{\xi}^k)d^k + diag(\overline{\eta}^k)\lambda^k\|^2 \tag{12}$$

It follows from (11) and (12) that, given any $\varepsilon > 0$, there is a $\bar{t} > 0$, such that for any $0 < t \leq \bar{t}$,

$\|\Phi^k\|^2 - \|\Phi(x^k + td^k, s^k + t\lambda^k)\|^2 \geq (2-\varepsilon)t\|\Phi^k\|^2$

Hence, this lemma holds.

From lemma 4.1 we know that if $\Phi^k \neq 0$, then (d^k, λ^k) is the decreasing direction of $\|\Phi^k\|$.

Lemma 4.2 For all k, there is an $\alpha_{min} > 0$ such that $\alpha_k \geq \alpha_{min} > 0$.

Proof: Assume $\Phi^k \neq 0$ for sufficiently large k, it follows by Lemma 4.1 that, for all k, $\Phi^k \neq 0$ and any $\alpha \leq min\{\frac{1-\theta}{2-\varepsilon}, \bar{t}\}$

$\|\Phi(x^k + \alpha d^k, s^k + \alpha\lambda^k)\|^2 \leq [1-(2-\varepsilon)\alpha]\|\Phi^k\|^2 \leq \theta^2\|\Phi^k\|^2 \leq \theta^2 \max_{0\leq r\leq m(k)-1} \|\Phi^{k-r}\|^2 \leq$ $[\max\{\frac{r_k+1}{2}, \theta\}]^2 \max_{0\leq r\leq m(k)-1} \|\Phi^{k-r}\|^2$

So, There must be α to meet (9).

Lemma 4.3 If $H(x^k, s^k) \neq 0$ then V^k is nonsingular.

Proof: Assume $H(x^k, s^k) \neq 0$. If $V^k(u, v) = 0$ for some $(u, v) \in R^{2n}$, where $u = (u_1 \cdots u_n)^\top$, $v = (v_1 \cdots v_n)^\top$, then

$$-F'(x^k)u + Iv = 0 \tag{13}$$

$$diag(\xi^k)u + diag(\eta^k)v = 0 \tag{14}$$

From the definitions of ξ_i^k and η_i^k ,we know that $\xi_i^k > 0$ and $\eta_i^k > 0$ for all i. So, $diag(\eta^k)$ is nonsingular. We have

$$v = -(diag(\eta^k))^{-1} diag(\xi^k) u \tag{15}$$

Puting (15) into (13), and multiplying by $u^\top$,we have
$-u^\top F'(x^k)u - u^\top(diag(\eta^k))^{-1}diag(\xi^k)u = 0$
The fact that $F(x)$ is the P_0 function, so all the principal minor determinant of $F'(x)$ is non-negative, that is to say, $F'(x)$ is positive semidefinite. And matrix $diag(\eta^k))^{-1}diag(\xi^k)$ is positive definite. showing $u = 0$. It follows from (15), $v = 0$.Hence, V^k is nonsingular.

Lemma 4.4 If V^* is an accumulation matrix of $\{V^k\}$, then V^* is nonsingular.

Proof: It is clear that V^k is nonsingular for all $k = 0, 1, 2, \ldots$. Since ξ_i^k and η_i^k are bounded without loss of generality, Let $\xi_i^k \longrightarrow \xi_i^*$, $\eta_i^k \longrightarrow \eta_i^*$ and let $x^k \longrightarrow x^*$, then

$$V^k \longrightarrow V^* = \begin{pmatrix} -F'(x^*) & I \\ diag(\xi_i^*) & diag(\eta_i^*) \end{pmatrix} \tag{16}$$

Let $(u, v) \in R^{2n}$ be the solution of $V^*(u, v) = 0$.

$$-F'(x^*)u + Iv = 0 \tag{17}$$

$$diag(\xi^*)u + diag(\eta^*)v = 0 \tag{18}$$

in the next section, V^* is proven to be nonsingular, which is equivalent to showing that $(u, v) = (0, 0)$.

First, consider such an $j \in J$ for which $\xi_j^* = 0$. From the definition of the 3-1 piecewise NCP function, it is only possible in the second area and $x > s > 0$ or $3x > -s \geq 0$
$\xi_j^k = (\frac{s}{x})^2 \longrightarrow 0$
hence
$\eta_j^k = (3 - \frac{2s}{x}) \longrightarrow 3 \neq 0$
then for such an $j \in J$, we deduce that the matrix $diag(\eta_j^*)$ is nonsingular, and $v_j = 0, j \in J$ by (18).

For $j \notin J$ such that $\xi_j^* \neq 0$, substituting (18) into (17) and multiplying (17) by $v_j^\top$, then
$v_j^\top \cdot F'(x^*) \cdot \sum_{j:\xi_j^* \neq 0} \frac{\eta_j^*}{\xi_j^*} \cdot v_j + v_j^\top I v_j = 0$
$F'(x^*)$ is positive semidefinite together with the $\xi^* > 0$,$\eta^* \geq 0$ implies $v_j = 0$, $j \notin J$.
This proves $(u, v) = (0, 0)$ and hence V^* is nonsingular.

Lemma 4.5 $\Phi(x^k, s^k) \longrightarrow 0, k \longrightarrow \infty$.

Proof: In view of convenience, let $\|\Phi^{l(k)}\| = \max_{0\le r\le m(k)-1} \|\Phi^{k-r}\|$

where $k - m(k) + 1 \le l(k) \le k$

If for all sufficiently large k, (9) holds, since $m(k+1) \le m(k)+1$, then

$$\begin{aligned} p^k_{max} &= \max_{0\le r\le m(k)-1} \|\Phi^{k-r}\| = \|\Phi^{l(k)}\| \\ &\le \max_{0\le r\le m(k-1)} \|\Phi^{k-r}\| \\ &= \max\{\|\Phi^{l(k-1)}\|, \|\Phi^k\|\} \\ &= \|\Phi^{l(k-1)}\| \end{aligned}$$

So, $p^k_{max} \le \theta\|\Phi^{l(k-1)}\| = \theta p^{k-1}_{max} < p^{k-1}_{max}$. Then, p^k_{max} is monotone decreasing. Consider the nonnegativity of the p^k_{max}, $p^k_{max} \longrightarrow 0, k \longrightarrow \infty$.

Therefore $\|\Phi^{k+1}\| \le p^k_{max} \longrightarrow 0$ holds by the Algorithm 3.1.

That is, $\lim_{k\longrightarrow\infty} \|\Phi^k\| = 0$

Lemma 4.6$d^k \longrightarrow 0, \lambda^k \longrightarrow 0, H^k \longrightarrow 0.$

Proof: Suppose the contrary that exists $\varepsilon_1 > 0, \varepsilon_2 > 0$ for a subsequence (x^k, s^k) such that $\|d^k\| \ge \varepsilon_1 > 0$, $\|\lambda^k\| \ge \varepsilon_2 > 0$. If $\Phi^k \ne 0$, then (d^k, λ^k) is the decreasing direction of $\|\Phi^k\|$ by lemma 4.1, which contradict $\lim_{k\longrightarrow\infty} \|\Phi^k\| = 0$. Hence, $d^k \longrightarrow 0, \lambda^k \longrightarrow 0$. V^* is nonsingular from lemma 4.4 together with

$$V^* \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F(x^*) - s^* \\ 0 \end{pmatrix} \tag{19}$$

it is seen that $F(x^*) - s^* = 0$ and $\Phi(x^*, s^*) = 0$ namely, $\Psi(x^*, s^*) = 0$, so $(x^k, s^k) \longrightarrow (x^*, s^*)$ is the solving of NCP.

Theorem 4.1 Algorithm 3.1 is implemented to generate a sequence (x^k, s^k) and (x^*, s^*) be an accumulation point of (x^k, s^k). Then (x^*, s^*) is the solution of NCP.

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