Mathematica Aeterna, Vol. 6, 2016, no. 2, 255 - 259

Removable Singularities for a class of Elliptic Variational Inequalities

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Abstract

Removable Singularities for weak solutions of A class of elliptic variational inequalities is obtained in this paper.

Mathematics Subject Classification: 35J20, 35A20

Keywords: Removable Singularities, weak solution, variational inequality

1 Introduction

It is well known that variational inequalities are systematically used in the theory of many practical problems. In this paper, we will consider a class of elliptic variational inequalities, we are committed to the removable singularities for weak solutions.

Let Ω be a bounded domain of $\mathbb{R}^n$. let B_R be a cube with radius R. $B_{\rho R}$ is the ball with the same center and radius ρR. $|E|$ denotes the Lebesgue measure of a set$E \subseteq \mathbb{R}^n$. And let $W^{1,p}(\Omega)$, $1 < p < \infty$, be the first-order Sobolev space of functions $u \in L^p(\Omega)$ whose distributional gradient ∇u belongs to $L^p(\Omega)$. Suppose that ψ_1, ψ_2 are any functions in Ω with values in $\mathbb{R} \cup \{\pm\infty\}$, and that $\theta \in W^{1,p}(\Omega)$. Let

$$\mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega) = \Big\{ v \in W^{1,p}(\Omega) : \psi_1 \le v \le \psi_2,\ a.e. \text{ and } v - \theta \in W^{1,p}_0(\Omega) \Big\}. \quad (1.1)$$

The functions ψ_1, ψ_2 are the obstacle functions and θ determines the boundary value.

In this paper, we consider a class of elliptic variational inequalities

$$\begin{cases} u \in \mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega), \\ \int_\Omega \langle A(x,\nabla u), \nabla(v-u)\rangle dx \geq 0, \end{cases} \tag{1.2}$$

where $A(x,\xi):\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is any Carathéodory function, for almost all $x\in\Omega$, all $\xi\in\mathbb{R}^n$, satisfying the coercivity and growth conditions:

$$\langle A(x,\xi),\xi\rangle \geq \alpha|\xi|^p; \quad |A(x,\xi)| \leq \beta|\xi|^{p-1}, \tag{1.3}$$

where α,β are some nonnegative constants, $1<p<n$.

Definition 1.1 [1−2] *A compact set $E\subset\mathbf{R}^n$ is said to have zero r-capacity for $1<p\leq n$, if for some bounded domain Ω containing E there exists a sequence $\{\varphi_k(x)\}$, $k=1,2,...$, of functions $\varphi_k(x)\in C_0^\infty(\Omega)$ such that*

(*1*) $0\leq\varphi_k(x)\leq 1$;
(*2*) *each $\varphi_k(x)$equals to 1 on its own neighborhood of E,*
(*3*) $\lim\limits_{k\to\infty}\|\nabla\varphi_k(x)\|_p=0$,
(*4*) $\lim\limits_{k\to\infty}\varphi_k(x)=0,\ \forall x\in\Omega\setminus E$,

A closed set $E\subset\mathbf{R}^n$ has zero r-capacity if every compact subset of E has zero p-capacity.

Notice that for $p=n-\varepsilon$, $0<\varepsilon<n-1$, a closed set $E\subset\mathbf{R}^n$ of Hausdorff dimension $\dim_H(E)<\varepsilon$ has zero p-capacity.

Definition 1.2 [1−2] *Let $E\subset\mathbf{R}^n$ be a compact subset of zero Hausdorff measure of n-dimension in $\mathbb{R}^n$. A peak function defined in E is a function $\rho(x)\in C^\infty(\mathbf{R}^n\setminus E)$ for which $\lim\limits_{x\to a}\rho(x)=\infty$, whenever $\alpha\in E$.*

Next is the main result in this present paper.

Theorem 1.3 *Let Ω be a bounded domain of $\mathbb{R}^n$, $E\subset\mathbb{R}^n$ be a compact subset of zero Hausdorff measure of n-dimension in $\mathbb{R}^n$. Let $\psi_1,\psi_2\in L^\infty(\Omega)$. If $u\in W^{1,p}_{loc}(\Omega\setminus E)$ is a weak solution of elliptic variational inequality (1.1) such that the peak function, defined in E as that above, satisfies $\rho(x)\in W^{1,n}_{loc}(\Omega)$, then u extends to Ω as a weak solution of the elliptic variational inequality (1.1) in the whole domain Ω. In particular, it belongs to the Sobolev class $W^{1,p}_{loc}(\Omega)$.*

2 Proof of Theorem 1.3

Proof First, we want to prove that $u \in W^{1,p}_{loc}(\Omega)$. Let $\rho(x)$ be a peak function defined in E, we define a sequence $\{\rho_k(x)\}$ of Lipschitz functions as follows,

$$\rho_k(x) = \begin{cases} 1, & \text{if } \rho(x) \geq k+1; \\ \rho(x) - k, & \text{if } k \leq \rho(x) \leq k+1; \\ 0, & \text{if } \rho(x) \leq k. \end{cases} \tag{2.1}$$

Each of these functions is equal to 1 in its own neighborhood of E. Moreover, $\lim_{k\to\infty} \rho_k(x) = 0$ for all $x \notin E$ and we have

$$\nabla\rho_k = \begin{cases} 0, & \text{if } \rho(x) \geq k+1; \\ \nabla\rho(x), & \text{if } k \leq \rho(x) \leq k+1; \\ 0, & \text{if } \rho(x) \leq k. \end{cases} \tag{2.2}$$

So we observe that $\nabla\rho_k$ is supported on the set $\Omega_k = \{x \in \Omega : k \leq \rho(x) \leq k+1\}$.

For fixed $\varphi \in C_0^\infty(\Omega)$, consider the sequence $\{\eta_k\}$ of Lipschitz functions supported in $\Omega \setminus E$ and given by

$$\eta_k(x) = [1 - \rho_k(x)]\varphi(x), \tag{2.3}$$

By (2.1),

$$\eta_k(x) = \begin{cases} 0, & \text{if } \rho(x) \geq k+1; \\ [k+1-\rho(x)]\varphi(x), & \text{if } k \leq \rho(x) \leq k+1; \\ \varphi(x), & \text{if } \rho(x) \leq k. \end{cases} \tag{2.4}$$

It is easy to know that the sequence $\{\eta_k\}$ of Lipschitz functions supported in $\Omega \setminus E$, and

$$\nabla\eta_k = -\varphi\nabla\rho_k + [1 - \rho_k(x)]\nabla\varphi. \tag{2.5}$$

Since $u \in W^{1,p}_{loc}(\Omega \setminus E)$, the formula of integration by parts holds for any Lipschitz functions sequence $\{\eta_k(x)\}$ supported in $\Omega \setminus E$, i.e.

$$\int_{\Omega\setminus E} (\eta_k \nabla u) dx = -\int_{\Omega\setminus E} (u\nabla\eta_k) dx, \quad \forall \eta_k \in C_0^\infty(\Omega \setminus E). \tag{2.6}$$

Let $\eta_k = (1 - \rho_k)\varphi$ for all $\varphi \in C_0^\infty(\Omega)$, then

$$\int_\Omega ((1-\rho_k)\varphi\nabla u) dx = -\int_\Omega ((1-\rho_k)u\nabla\varphi) dx + \int_\Omega (\varphi u \nabla\rho_k) dx. \tag{2.7}$$

Since $|\nabla\rho_k| \le |\nabla\rho|$, $\lim_{k\to\infty} |\nabla\rho_k| = 0$, a.e., then $\int_\Omega (\varphi u \nabla\rho_k)dx \to 0$. Then by (2.7),

$$\int_\Omega (\varphi\nabla u)dx = -\int_\Omega (u\nabla\varphi)dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{2.8}$$

Hence $u \in W_{loc}^{1,p}(\Omega)$.

Next we verify that u solves the elliptic variational inequality (1.1) in Ω, i.e.

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla(v-u)\rangle dx \ge 0, \quad \forall v \in \mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega). \tag{2.9}$$

SInce $u \in W_{loc}^{1,p}(\Omega\setminus E)$ is a weak solution of elliptic variational inequality (1.1) in $\Omega\setminus E$, then

$$\int_{\Omega\setminus E} \langle \mathcal{A}(x,\nabla u), \nabla(\widetilde{v}-u)\rangle dx \ge 0, \quad \forall \widetilde{v} \in \mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega\setminus E). \tag{2.10}$$

Let

$$\widetilde{v} = u + (1-\rho_k)(v-u), \quad \forall v \in \mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega). \tag{2.11}$$

Since

$$\widetilde{v} - \theta = (u-\theta) + (1-\rho_k)(v-u) \in W_0^{1,p}(\Omega\setminus E), \tag{2.12}$$

$$\begin{aligned}
\widetilde{v} - \psi_1 &= (u-\psi_1) + (1-\rho_k)(v-u) \\
&= (u-\psi_1) + (1-\rho_k)[(v-\psi_1) - (u-\psi_1)] \\
&= (u-\psi_1) + (1-\rho_k)(v-\psi_1) - (1-\rho_k)(u-\psi_1) \\
&= \rho_k(u-\psi_1) + (1-\rho_k)(v-\psi_1) \\
&\ge 0,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\widetilde{v} - \psi_2 &= (u-\psi_2) + (1-\rho_k)(v-u) \\
&= (u-\psi_2) + (1-\rho_k)[(v-\psi_2) - (u-\psi_2)] \\
&= (u-\psi_2) + (1-\rho_k)(v-\psi_2) - (1-\rho_k)(u-\psi_2) \\
&= \rho_k(u-\psi_2) + (1-\rho_k)(v-\psi_2) \\
&\le 0,
\end{aligned} \tag{2.14}$$

then $\widetilde{v}$ in (2.10) can be choosen as (2.11), then (2.10) becomes

$$\int_\Omega (1-\rho_k)\langle \mathcal{A}(x,\nabla u), \nabla(v-u)\rangle dx \ge \int_\Omega \langle \mathcal{A}(x,\nabla u), (v-u)\nabla\rho_k\rangle dx. \tag{2.15}$$

Now we estimate the integral in the right-hand side of (2.15). Noticing that $\psi_1, \psi_2 \in L^\infty(\Omega)$, $|v-u| \le \psi_2 - \psi_1$, $d\rho_k$ is supported in $\Omega_k = \{x \in \Omega : k \le \rho(x) \le k+1\}$, $|d\rho_k| \le |d\rho|$. By condition (i) and the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_\Omega \langle \mathcal{A}(x, du), (v-u)\nabla\rho_k\rangle dx\right| \\
\le\ & \alpha \int_{\Omega_k} |v-u||\nabla u|^{p-1}|\nabla\rho_k| dx \\
\le\ & \alpha\|\psi_2 - \psi_1\|_\infty \left(\int_{\Omega_k} |\nabla u|^p dx\right)^{1-\frac{1}{p}} \left(\int_{\Omega_k} |\nabla\rho|^p dx\right)^{\frac{1}{p}} \\
\le\ & \alpha\|\psi_2 - \psi_1\|_\infty |\Omega_k|^{\frac{1}{p}-\frac{1}{n}} \left(\int_{\Omega_k} |\nabla u|^p dx\right)^{1-\frac{1}{p}} \left(\int_{\Omega_k} |\nabla\rho|^n dx\right)^{\frac{1}{n}} \quad (2.16)
\end{aligned}$$

Since $\psi_1, \psi_2 \in L^\infty(\Omega)$, $u \in W^{1,p}_{loc}(\Omega)$, $\rho \in W^{1,n}_{loc}(\Omega)$, and $|\Omega_k| \to 0$ when $k \to \infty$, then we conclude that the integrals in the above inequality converge to zero. Then (2.17) becomes

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla(v-u)\rangle dx \ge 0, \quad \forall v \in \mathcal{K}^{\theta,p}_{\psi_1,\psi_2}(\Omega). \qquad (2.17)$$

This completes the proof of Theorem 1.3.

□

ACKNOWLEDGEMENTS. The research is supported by Scientific Research Foundation of North China University of Science and Technology(z201219) and National Science Foundation of Hebei Province (A2013209278).

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Received: January 16, 2016