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# Oscillation of Second Order Nonlinear Neutral Advanced Functional Difference Equations

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**Abstract**

In this paper, we establish some sufficient conditions for the oscillations of all solutions of the second order nonlinear neutral advanced functional difference equations

$$\Delta[r(n)\Delta(x(n)+p(n)x(n+\tau))]+q(n)f(x(n+\sigma))=0; \quad n\geq n_0, \quad (*)$$

where $\sum_{n=n_0}^{\infty}\frac{1}{r(n)}=\infty$ or $\sum_{n=n_0}^{\infty}\frac{1}{r(n)}<\infty$, and $0\leq p(n)\leq p_0<\infty$, $\tau$ is an integer, and $\sigma$ is a positive integer. The results proved here improve some known results in the literature.

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# 1 Introduction

In this paper we consider the following second order nonlinear neutral advanced functional difference equations of the form:

$$\Delta[r(n)\Delta(x(n) + p(n)x(n + \tau))] + q(n)f(x(n + \sigma)) = 0; \quad n \geq n_0, \qquad (1)$$

where $\Delta$ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$. The following conditions are assumed to be hold throughout the paper:

($a$) $\{r(n)\}_{n=n_0}^{\infty}$ is a sequence of positive real numbers;

($b$) $\{p(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegative real numbers with the property that $0 \leq p(n) \leq p_0 < \infty$;

($c$) $\{q(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegative real numbers and $q(n)$ is not identically zero for large values of $n$;

($d$) $\frac{f(u)}{u} \geq k > 0$ for $u \neq 0$, $k$ is a constant;

and

($e$) $\tau$ is an integer and $\sigma$ is a positive integer.

We shall consider the following two cases,

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty \qquad (2)$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} < \infty. \qquad (3)$$

We note that second order neutral functional difference equations have applications in problems dealing with vibrating masses attached to a elastic ball and in some variational problems. In recent years there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions for different classes of difference equations, we refer to the books [1,2,7] and the papers [6,13]. Also the oscillatory of behavior of neutral functional difference equations has been the subject of intensive study, see, for example, [3-5,8-12,14].

In [12], Zhang et al. established that every solution of the equation

$$\Delta[a(n)\Delta(x(n) - px(n - \tau))] + f(n, x(\sigma(n))) = 0; \quad n \geq n_0 \qquad (4)$$

is either oscillatory or eventually

$$|x(n)| \leq p\,|x(n-\tau)|$$

if

$$\sum_{s=n}^{\infty} q(s) \sum_{i=0}^{m} p^i = \infty,$$

where

$$\frac{f(n,u)}{u} \geq q(n) > 0 \quad for \quad u \neq 0.$$

In [3], Budincevic established that every solution of the equation

$$\Delta[a(n)\Delta(x(n) + px(n-n_0))] + q(n)f(x(n-m_0)) = 0 \tag{5}$$

is either oscillatory or else $x(n) \to 0$ as $n \to \infty$.

In [7], Murugesan et al. established that every solution of the equation (1) is oscillatory if $\tau > 0$, $0 \leq p(n) \leq p_0 < \infty$ and atleast one of the first order advanced difference inequalities

$$\Delta w(n) - \frac{1}{1+p_0} Q_1(n) w(n+\sigma) \geq 0; \tag{6}$$

$$\Delta w(n) - \frac{1}{1+p_0} Q_2(n) w(n+\sigma) \geq 0 \tag{7}$$

has no positive solution.

In this paper our aim is to obtain sufficient conditions for oscillation of all solutions of the equation (1) under the condition

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} < \infty.$$

Let $n_0$ be a fixed nonnegative integer. By a solution of (1) we mean a nontrival real sequence $\{x(n)\}$ which is defined for $n \geq n_0$ and satisfies the equation (1) for $n \geq n_0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if for every positive integer $N > 0$, there exists an $n \geq N$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In the sequel, for the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large $n$.

# 2 Main Results

In this section, we establish some new oscillation criteria for (1). For the sake of convenience, we define the following notations

$$\begin{aligned} Q(n): &= \min\{q(n), q(n+\tau)\}, \\ (\Delta\rho(n))_+ &= \max\{0, \Delta\rho(n)\}, \\ R(n) &= \sum_{s=n_0}^{n-1} \frac{1}{r(s)}, \end{aligned}$$

and

$$\delta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

**Theorem 2.1** *Assume that (2) holds and $\tau > 0$. Moreover suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ k\rho(s)Q(s) - \frac{((1+p_0)r(s)(\Delta\rho(s))_+)^2}{4\rho(s)} \right] = \infty. \tag{8}$$

*Then every solution of (1) is oscillatory.*

**Proof.** Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$ for all $n \geq n_1$. Define

$$z(n) = x(n) + p(n)x(n+\tau). \tag{9}$$

Then $z(n) > 0$ for all $n \geq n_1$. From (1), we have

$$\Delta(r(n)\Delta z(n)) \leq -kq(n)x(n+\sigma) \leq 0; \quad n \geq n_1. \tag{10}$$

Therefore the sequence $\{r(n)\Delta z(n)\}$ is nonincreasing. We claim that

$$\Delta z(n) > 0 \quad for \quad n \geq n_1. \tag{11}$$

If not, there exists an integer $n_2 \geq n_1$ such that $\Delta z(n_2) < 0$. Then from (10) we obtain

$$r(n)\Delta z(n) \leq r(n_2)\Delta z(n_2), \quad n \geq n_2.$$

Here

$$z(n) \leq z(n_2) + r(n_2)\Delta z(n_2) \sum_{s=n_2}^{n-1} \frac{1}{r(s)}.$$

Letting $n \to \infty$, we get $z(n) \to -\infty$. This contradiction proves that $\Delta z(n) > 0$ for $n \geq n_1$. On the otherhand, using the definition of $z(n)$ and applying (1), for all sufficiently large values of $n$,

$$\begin{aligned}\Delta[r(n)\Delta z(n)] + kq(n)x(n+\sigma) + p_0\Delta[r(n+\tau)\Delta z(n+\tau)] \\ +p_0 kq(n+\tau)x(n+\sigma+\tau) \leq 0\end{aligned}$$

$$\Delta[r(n)\Delta z(n)] + p_0\Delta[r(n+\tau)\Delta z(n+\tau)] + kQ(n)z(n+\sigma) \leq 0.$$

Since $\Delta z(n) > 0$, we have $z(n+\sigma) \geq z(n)$ and hence

$$\Delta[r(n)\Delta z(n)] + p_0\Delta[r(n+\tau)\Delta z(n+\tau)] + kQ(n)z(n) \leq 0. \tag{12}$$

Define

$$w(n) = \rho(n)\frac{r(n)\Delta z(n)}{z(n)}, n \geq n_1. \tag{13}$$

Clearly $w(n) > 0$. From (9), we have

$$r(n+1)\Delta z(n+1) \leq r(n)\Delta z(n).$$

From (13), we have

$$\begin{aligned}\Delta w(n) \quad &= \quad \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)} - \frac{\rho(n)r(n+1)\Delta z(n+1)}{z(n)z(n+1)}\Delta z(n) \\ &\quad + \frac{r(n+1)\Delta z(n+1)}{z(n+1)}\Delta\rho(n) \\ &\leq \quad \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)} - \frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)} + \frac{w(n+1)}{\rho(n+1)}\Delta\rho(n) \\ &\leq \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n)} - \frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)} + \frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1).\end{aligned} \tag{14}$$

Similarly, we define

$$v(n) = \rho(n)\frac{r(n+\tau)\Delta z(n+\tau)}{z(n)}, \quad n \geq n_1. \tag{15}$$

From (9) we have $r(n+\tau+1)\Delta z(n+\tau+1) \leq r(n)\Delta z(n)$.
Using this in (15), we have

$$\begin{aligned}\Delta v(n) &\leq \rho(n)\frac{\Delta(r(n+\tau)\Delta z(n+\tau))}{z(n)} - \frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)} + \frac{\Delta\rho(n)}{\rho(n+1)}v(n+1)\\ &\leq \rho(n)\frac{\Delta(r(n+\tau)\Delta z(n+\tau))}{z(n)} - \frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)} + \frac{(\Delta\rho(n))_+}{\rho(n+1)}v(n+1). \quad (16)\end{aligned}$$

It follows from (14) and (16) that

$$\begin{aligned}\Delta w(n) + p_0\Delta v(n) \leq \rho(n)\frac{\Delta[r(n)\Delta z(n)]}{z(n)} + p_0\rho(n)\frac{\Delta[r(n+\tau)\Delta z(n+\tau)]}{z(n)}\\ -\frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)} + \frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1) - p_0\frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)}\\ +p_0\frac{(\Delta\rho(n))_+}{\rho(n+1)}v(n+1).\end{aligned}$$

In view of (12)and the above ineqality, we obatin

$$\begin{aligned}\Delta w(n) + p_0\Delta v(n) &\leq -k\rho(n)Q(n) - \frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n)} + \frac{(\Delta\rho(n))_+}{\rho(n+1)}w(n+1)\\ &\quad -p_0\frac{\rho(n)v^2(n+1)}{\rho^2(n+1)r(n)} + p_0\frac{(\Delta\rho(n))_+}{\rho(n+1)}v(n+1)\\ &\leq -k\rho(n)Q(n) + \frac{(1+p_0)r(n)(\Delta(\rho(n)_+)^2}{4\rho(n)}.\end{aligned}$$

Summing the above inequality from $n$, to $n-1$, we have

$$w(n)+p_0v(n) \leq w(n_1)+p_0v(n_1) - \sum_{s=n_1}^{n-1}\left[k\rho(s)Q(s) - \frac{(1+p_0)r(s)((\Delta\rho(s))_+)^2}{4\rho(s)}\right],$$

which follows that

$$\sum_{s=n_1}^{n-1}\left[k\rho(s)Q(s) - \frac{(1+p_0)r(s)((\Delta\rho(s))_+)^2}{4\rho(s)}\right] \leq w(n_1) + p_0v(n_1),$$

which contradicts (8). This completes the proof. Choosing $\rho(n) = R(n+\sigma)$. By Theorem 2.1 we have the following results.

**Corollary 2.2** *Assume that (2) holds and $\tau > 0$. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[kR(s+\sigma)Q(s) - \frac{(1+p_0)}{4r(s)R(s+\sigma)}\right] = \infty, \quad (17)$$

*then every solution of (1) is oscillatory.*

**Corollary 2.3** *Assume that (2) holds and $\tau > 0$. If*

$$\liminf_{n\to\infty} \frac{1}{\ln R(n+\sigma)} \sum_{s=n_0}^{n-1} R(s+\sigma)Q(s) > \frac{1+p_0}{4k}, \tag{18}$$

*then every solution of (1) is oscillatory.*

**Proof.** From (18), there exists a $\epsilon > 0$ such that for all large $n$,

$$\frac{1}{\ln R(n+\sigma)} \sum_{s=n_0}^{n-1} R(s+\sigma)Q(s) > \frac{1+p_0}{4k} + \epsilon,$$

it follows that

$$\sum_{s=n_0}^{n-1} R(s+\sigma)Q(s) - \left(\frac{1+p_0}{4k}\right) \ln R(s+\sigma) \geq \epsilon \ln R(n+\sigma),$$

that is

$$\sum_{s=n_0}^{n-1} \left[ R(s+\sigma)Q(s) - \frac{1+p_0}{4kr(s)R(s+\sigma)} \right] \geq \epsilon \ln R(n+\sigma) + \left(\frac{1+p_0}{4k}\right) \ln R(n_0+\sigma). \tag{19}$$

Now, it is obvious that (19) implies (17) and the assertion of Corollary 2.3. follows from Corollary 2.2.

**Corollary 2.4** *Assume that (2) holds and $\tau > 0$. If*

$$\liminf_{n\to\infty} [Q(n)R^2(n+\sigma)r(n)] > \frac{1+p_0}{4k}, \tag{20}$$

*then every solution of (1) is oscillatory.*

**Proof.** It is easy to verify that (20) yields the existence $\epsilon > 0$ such that for all large $n$,

$$Q(n)R^2(n+\sigma)r(n) \geq \frac{1+p_0}{4k} + \epsilon.$$

Multiplying by $\frac{1}{R(n+\sigma)r(n)}$ on both sides of the above inequality, we have

$$R(s+\sigma)Q(s) - \frac{1+p_0}{4kr(s)R(s+\sigma)} \geq \frac{\epsilon}{R(s+\sigma)r(s)}$$

which implies that (17) holds. Therefore by Corollary 2.2. every solution of (1) is oscillatory.

Next, choosing $\rho(n) = n$. By Theorem 2.1 we have the following result.

**Corollary 2.5** *Assume that (2) holds and $\tau > 0$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ ksQ(s) - \frac{(1+p_0)r(s)}{4s} \right] = \infty, \tag{21}$$

*then every solution of (1) is oscillatory.*

**Theorem 2.6** *Assume that (3) holds $\tau > 0$, and $\sigma \le \tau$. Suppose also that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (8) holds, and*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ k\delta(s+1)Q(s-\tau) - \frac{(1+p_0)\delta(s)}{4r(s)\delta^2(s+1)} \right] = \infty. \tag{22}$$

*Then every solution of (1) is oscillatory.*

**Proof.** Assume that contrary. Without loss of generality; we may suppose that $\{x(n)\}$ is an eventually positive solution of (1) Then there exists an integer $n_1 \ge n_0$ such that $x(n+\sigma-\tau) > 0$ for all $n \ge n_1$. From (1), $\{r(n)\Delta z(n)\}$ is nonincreasing eventually, where $z(n)$ is defined by (9). Consequently, it is easy to conclude that there exists two possible cases of the sign of $\Delta z(n)$, that is , $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually. If $\Delta z(n) > 0$ eventually, then we are back of the case of Theorem 2.1, and we can get a contradiction to (8). If $\Delta z(n) < 0$, $n \ge n_2 \ge n_1$ then we define the sequence $\{v(n)\}$ by

$$v(n) = \frac{r(n)\Delta z(n)}{z(n)}, \quad n \ge n_3 = n_2 + \tau. \tag{23}$$

Clearly $v(n) < 0$. Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we get

$$r(s)\Delta z(s) \le r(n)\Delta z(n), \quad s \ge n \ge n_3.$$

Dividing the above by $r(s)$ and summing it from $n$ to $l-1$, we obtain

$$z(l) \le z(n) + r(n)\Delta z(n) \sum_{s=n}^{l-1} \frac{1}{r(s)}, \quad l \ge n \ge n_3.$$

Letting $l \to \infty$ in the above inequality, we see that

$$0 \le z(n) + r(n)\Delta z(n)\delta(n), \quad n \ge n_3.$$

Therefore,

$$\frac{r(n)\Delta z(n)}{z(n)}\delta(n) \ge -1, \quad n \ge n_3.$$

From (23), we have

$$-1 \le v(n)\delta(n) \le 0, \quad n \ge n_3. \tag{24}$$

Similarly, we introduce the sequence $\{w(n)\}$ where

$$w(n) = \frac{r(n-\tau)\Delta z(n-\tau)}{z(n)}, \quad n \geq n_3. \tag{25}$$

Obviously $w(n) < 0$. Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we have $r(n-\tau)\Delta z(n-\tau) \geq r(n)\Delta z(n)$. Then $w(n) \geq v(n)$. From (24), we have

$$-1 \leq w(n)\delta(n) \leq 0, \quad n \geq n_3. \tag{26}$$

From (24) and (25), we have

$$\Delta v(n) = \frac{\Delta(r(n)\Delta z(n)}{z(n)} - \frac{v^2(n)}{r(n)}, \tag{27}$$

and

$$\Delta w(n) = \frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n)} - \frac{w^2(n)}{r(n)}. \tag{28}$$

In view of (27) and (28), we obtain,

$$\begin{aligned}\Delta w(n) + p_0\Delta v(n) \leq{}& \frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n)} \\ &+p_0\frac{\Delta(r(n)\Delta z(n))}{\Delta z(n)} - \frac{w^2(n)}{r(n)} - \frac{v^2(n)}{r(n)}.\end{aligned} \tag{29}$$

On the otherhand, proceed as in the proof of Theorem 2.1, we have

$$\Delta(r(n-\tau)\Delta z(n-\tau)) + p_0\Delta(r(n)\Delta z(n)) \leq -kQ(n-\tau)z(n).$$

Using the above inequality in (29), we have

$$\Delta w(n) + p_0\Delta v(n) \leq -kQ(n-\tau) - \frac{w^2(n)}{r(n)} - \frac{v^2(n)}{r(n)}. \tag{30}$$

Multiplying (30) by $\delta(n+1)$ and summing from $n_3$ to $n-1$, we have

$$\begin{aligned}\delta(n)w(n) \;&-\; \delta(n_3+1)w(n_3) + \sum_{s=n_3}^{n-1}\frac{w(s)}{r(s)} + \sum_{s=n_3}^{n-1}\frac{w^2(s)}{r(s)}\delta(s+1) \\ &+\; p_0\delta(n)v(n) - p_0\delta(n_3+1)v(n_3) + p_0\sum_{s=n_3}^{n-1}\frac{v(s)}{r(s)} \\ &+\; p_0\sum_{s=n_3}^{n-1}\frac{v^2(s)}{r(s)}\delta(s+1) + k\sum_{s=n_3}^{n-1}Q(s-\tau)\delta(s+1) \leq 0.\end{aligned}$$

From the above inequality, we have

$$\begin{aligned}\delta(n)w(n) &- \delta(n_3+1)w(n_3) + p_0\delta(n)v(n) - p_0\delta(n_3+1)v(n_3) \\ &+ k\sum_{s=n_3}^{n-1}\delta(s+1)Q(s-\tau) - \frac{(1+p_0)}{4}\sum_{s=n_3}^{n-1}\frac{\delta(s)}{r(s)\delta^2(s+1)} \le 0.\end{aligned}$$

Thus, it follows from the above inequality that

$$\begin{aligned}\delta(n)w(n) &+ p_0\delta(n)v(n) + \sum_{s=n_3}^{n-1}\left[k\delta(s+1)Q(s-\tau) - \frac{(1+p_0)\delta(s)}{4r(s)\delta^2(s+1)}\right] \\ &\le \delta(n_3+1)w(n_3) + p_0\delta(n_3+1)v(n_3).\end{aligned}$$

By (24) and (26), we get a contradiction with (22) and this completes the proof.

**Corollary 2.7** *Assume that (3) holds, $\tau > 0$ and $\sigma \le \tau$. Furthermore that one of conditions (17), (18), (20) and (21) holds and one has (22). Then every solution of (1) is oscillatory.*

**Theorem 2.8** *Assume that (3) holds, $\tau > 0$, and $\sigma \le \tau$. Further more suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (8) holds, and*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\delta^2(s+1)Q(s-\tau) = \infty. \tag{31}$$

*then every solution of (1) is oscillatory.*

**Proof.** Assume the contrary. Without loss of generality, we assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \ge n_0$ such that $x(n-\tau) > 0$ for all $n \ge n_1$. By (1), $\{r(n)\Delta z(n)\}$ is nonincreasing, eventually where $z(n)$ is defined by (9). Consequently, it is easy to conclude that there exist two possible cases of the sign of $\{\Delta z(n)\}$, that is $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually. If $\Delta z(n) < 0$, $n \ge n_2 \ge n_1$, then we define $w(n)$ and $v(n)$ as in Theorem 2.6. Then proceed as in the proof of Theorem 2.6, we obtain (24), (26) and (30). Multiplying (30) by $\delta^2(n+1)$, and summing from $n_3$ to $n-1$, where $n_3 \ge n_2+\tau$, we get

$$\begin{aligned}\delta^2(n)w(n) &- \delta^2(n_3+1)w(n_3+1) + 2\sum_{s=n_3}^{n-1}\frac{w(s)\delta(s+1)}{r(s+1)} + p_0\delta^2(n)v(n) \\ &- p_0\delta^2(n_3+1)v(n_3) + 2\sum_{s=n_3}^{n-1}\frac{v(s)\delta(s+1)}{r(s+1)} + \sum_{s=n_3}^{n-1}\frac{w^2(s)\delta^2(s+1)}{r(s)}\end{aligned}$$

$$+ \quad p_0 \sum_{s=n_3}^{n-1} \frac{v^2(s)}{r(s)} \delta^2(s+1) + k \sum_{s=n_3}^{n-1} Q(s-\tau)\delta^2(s+1) \leq 0. \qquad (32)$$

It follows from (3) and (24) that

$$\begin{aligned} \left| \sum_{s=n_3}^{\infty} \frac{w(s)\delta(s+1)}{r(s+1)} \right| &\leq \sum_{s=n_3}^{\infty} \frac{|w(s)\delta(s+1)|}{r(s+1)} \leq \sum_{s=n_3}^{\infty} \frac{1}{r(s+1)} < \infty, \\ \sum_{s=n_3}^{n-1} \frac{w^2(s)\delta^2(s+1)}{r(s)} &\leq \sum_{s=n_3}^{\infty} \frac{1}{r(s)} < \infty. \end{aligned}$$

In view of (26), we have

$$\begin{aligned} \left| \sum_{s=n_3}^{\infty} \frac{v(s)\delta(s+1)}{r(s+1)} \right| &\leq \sum_{s=n_3}^{\infty} \frac{|v(s)\delta(s+1)|}{r(s+1)} \leq \sum_{s=n_3}^{\infty} \frac{1}{r(s+1)} < \infty, \\ \sum_{s=n_3}^{\infty} \frac{v^2(s)\delta^2(s+1)}{r(s)} &\leq \sum_{s=n_3}^{\infty} \frac{1}{r(s)} < \infty. \end{aligned}$$

From (32), we get

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \delta^2(s+1)Q(s-\tau) < \infty,$$

which is a contradiction with (31). This completes the proof.

**Corollary 2.9** *Assume that (3) holds, $\tau > 0$ and $\sigma \leq \tau$. Suppose also that one of conditions (17), (18), (20) and (21) holds, and one has (31). Then every solution of (1) is oscillatory.*

**Theorem 2.10** *Assume that (2) holds and $\tau < 0$. Moreover suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ k\rho(s)Q(s) - \frac{(1+p_0)r(s+\tau)(\Delta\rho(s)_+)^2}{4\rho(s)} \right] = \infty. \qquad (33)$$

*Then every solution of (1) is oscillatory.*

**Proof.** Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \geq n_0$ such that $x(n-\tau) > 0$ for all $n \geq n_1$. Define $z(n)$ by (9). Similar to the proof of Theorem 2.1, there exists an integer $n_2 \geq n_1$ such that $\Delta z(n+\tau) > 0$ for $n \geq n_2$ and

$$\Delta[r(n)\Delta z(n)] + p_0[r(n+\tau)\Delta z(n+\tau)] + kQ(n)z(n+\tau) \leq 0 \quad forn \geq n_2. \quad (34)$$

Define the sequence $\{w(n)\}$ by

$$w(n) = \rho(n)\frac{r(n)\Delta z(n)}{z(n+\tau)}, \quad n \geq n_2. \tag{35}$$

Then $w(n) > 0$. By (10) we get

$$r(n+1)\Delta z(n+1) \leq r(n+\tau)\Delta z(n+\tau),$$

and

$$\begin{aligned}
\Delta w(n) &\leq \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n+\tau)} - \frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n+\tau)} + \frac{w(n+1)}{\rho(n+1)}\Delta\rho(n) \\
&\leq \rho(n)\frac{\Delta(r(n)\Delta z(n))}{z(n+\tau)} - \frac{\rho(n)w^2(n+1)}{\rho^2(n+1)r(n+\tau)} + \frac{w(n+1)}{\rho(n+1)}(\Delta\rho(n))_+
\end{aligned} \tag{36}$$

Again define the sequence $\{v(n)\}$ by

$$v(n) = \rho(n)\frac{r(n+\tau)\Delta z(n+\tau)}{z(n+\tau)}, \quad n \geq n_2. \tag{37}$$

The rest of the proof is similar to that of Theorem 2.1 and so is omitted. This completes the proof.

Choosing $\rho(n) = R(n+\tau)$. By Theorem 2.10 we have the following oscillation criteria.

**Corollary 2.11** *Assume that (2) holds and $\tau < 0$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[kR(s+\tau)Q(s) - \frac{1+p_0}{4r(s+\tau)R(s+\tau)}\right] = \infty, \tag{38}$$

*then every solution of (1) is oscillatory.*

**Corollary 2.12** *Assume that (2) holds and $\tau < 0$. If*

$$\liminf_{n\to\infty} \frac{1}{\ln R(n+\tau)} \sum_{s=n_0}^{n-1} R(s+\tau)Q(s) > \frac{1+p_0}{4k}, \tag{39}$$

*then every solution of (1) is oscillatory.*

**Proof.** By Corollary 2.11, the proof is similar to that of Corollary 2.3, we omit the details.

**Corollary 2.13** *Assume that (2) holds and $\tau < 0$. If*

$$\liminf_{n\to\infty} [Q(n)R^2(n+\tau)r(n+\tau)] > \frac{1+p_0}{4k}, \tag{40}$$

*then every solution of (1) is oscillatory.*

**Proof.** By Corollary 2.11, the proof is similar to that of Corollary 2.4, and so the proof is omitted.

Next, we choosing $\rho(n) = n$. From Theorem 2.4, we have the following result.

**Corollary 2.14** *Assume that (2) holds and $\tau < 0$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ ksQ(s) - \frac{(1+p_0)r(s+\tau)}{4s} \right] = \infty, \tag{41}$$

*then every solution of (1) is oscillatory.*

Next, we will give the following results under the case when (3) and $\sigma \le \tau$.

**Theorem 2.15** *Assume that (3) holds, $\tau < 0$, and $\sigma \le -\tau$. Further, suppose that there exists a positive real valued sequence $\{\rho(n)\}_{n=n_0}^{\infty}$ such that (33) holds. Suppose also that one of the following holds:*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left[ k\delta(s+1)Q(s+\tau) - \frac{(1+p_0)\delta(s)}{4r(s)\delta^2(s+1)} \right] = \infty, \tag{42}$$

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \delta^2(s+1)Q(s+\tau) = \infty. \tag{43}$$

*Then every solution of (1) is oscillatory.*

**Proof.** Assume the contrary. Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Then there exists an integer $n_1 \ge n_0$ such that $x(n) > 0$ for all $n \ge n_1$. From (1), $\{r(n)\Delta z(n)\}$ is nonincreasing eventually where $z(n)$ is defined by (9). Consequently it is easy to conclude that there exist two possible cases of sign of $\{\Delta z(n)\}$. That is $\Delta z(n) > 0$ or $\Delta z(n) < 0$ eventually. If $\Delta z(n) > 0$ eventually, then we are back of the case of Theorem 2.10 and we can get a contradiction to (33). If $\Delta z(n) < 0$ eventually, then there exists an integer $n_2 \ge n_1$ such that $\Delta z(n) < 0$ for all $n \ge n_2$.

Define the sequence $\{w(n)\}$ and $\{v(n)\}$ as follows:

$$w(n) = \frac{r(n+\tau)\Delta z(n+\tau)}{z(n)} \quad for \quad n \ge n_3 = n_2 + 2\tau$$

and

$$v(n) = \frac{r(n+2\tau)\Delta z(n+2\tau)}{z(n)} \quad for \quad n \ge n_3 = n_2 + 2\tau.$$

The rest of the proof can be proceed as in Theorem 2.6 of Theorem 2.8. We can obtain a contradiction to (42) or (43) respectively. The proof is complete.

# 3 Examples

**Example 3.1** *Consider the second order neutral advanced difference equation*

$$\Delta[(n+\tau)\Delta(x(n)+p(n)x(n+\tau))]+\frac{\lambda}{n}f(x(n+\sigma))=0, \quad n=1,2,3,.... \tag{44}$$

*where* $r(n)=n+\tau$, $\tau$ *and* $\sigma$ *are positive integers,* $\rho(n)=n+\tau$, $f(x)=x(1+x^2)$, $0\le p(n)\le p_0<\infty$, $q(n)=\frac{\lambda}{n}$ *and* $\lambda>0$. *Let* $k=1$. *Then*

$$\limsup_{n\to\infty}\sum_{s=1}^{n}k\rho(s)Q(s)-\frac{(1+p_0)r(s)((\Delta\rho(s))_+)^2}{4\rho(s)}$$

$$=\limsup_{n\to\infty}\sum_{s=1}^{n}\left[\lambda-\frac{(1+p_0)}{4}\right]=\infty,$$

*for* $\lambda>\frac{1+p_0}{4}$. *Hence, by Theorem 2.1, every solution of (44) is oscillatory for* $\lambda>\frac{1+p_0}{4}$.

**Example 3.2** *Consider the following second order neutral advanced difference equation*

$$\Delta[(n-\tau)\Delta(x(n)+p(n)x(n+\tau))]+q(n)f(x(n+\sigma))=0, \quad n\ge 1, \tag{45}$$

*where* $r(n)=n-\tau$, $q(n)=\frac{\lambda}{n}$, $0\le p(n)\le p_0<\infty$, $\tau$ *is a negative integer,* $\sigma$ *is a positive integer, and* $f(x)=x(1+x^2)$ *and* $\lambda>0$. *It is easy to see that*

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\left[ksQ(s)-\frac{(1+p_0)r(s+\tau)}{4s}\right]$$

$$=\limsup_{n\to\infty}\sum_{s=1}^{n}\left[\lambda-\frac{1+p_0}{4}\right]=\infty,$$

*for* $\lambda>\frac{1+p_0}{4}$. *Hence, by Corollary 2.14, every solution of (45) is oscillatory.*

**Example 3.3** *Consider the following second order neutral advanced difference equation*

$$\Delta[e^n(x(n)+p(n)x(n+2))]+e^{2n}f(x(n+1))=0, \quad n\ge 1, \tag{46}$$

*where* $0\le p(n)\le p_0<\infty$, $f(x)=x(1+x^2)$, $r(n)=e^n$, $q(n)=e^{2n}$. *It is easy to see that*

$$\sum_{n=1}^{\infty}\frac{1}{r(n)}<\infty.$$

*Choose $k = 1$. Then*

$$\limsup_{n\to\infty} \sum_{s=1}^{n} \delta^2(s+1)Q(s-2)$$

$$\begin{aligned} &= \limsup_{n\to\infty} \sum_{s=1}^{n} \frac{1}{e^{2n}(e-1)^2} e^{2(n-2)} \\ &= \infty. \end{aligned}$$

*Also*

$$\liminf_{n\to\infty}[Q(n)R^2(n+1)r(n)] = \liminf_{n\to\infty}\left[e^{2n}\left(\frac{e^n-1}{e^n(e-1)}\right)^2 e^n\right] > \frac{1+p_0}{4}$$

*Then by Corollary 2.9 every solution of (46) is oscillatory.*

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