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Regularity for Minimizers to Anisotropic Integrals Functions with Nonstandard Growth

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Abstract

In this paper we deal with the problem

$$u \in C_\psi(\Omega),$$

$$\forall\ \omega \in C_\psi(\Omega), \quad \int_\Omega f(x, Du)dx \leq \int_\Omega f(x, D\omega)dx,$$

where $C_\psi(\Omega) = \{w \in u_* + W_0^{1,(p_i)}(\Omega)\ such\ that\ x \to f(x, Dw) \in L^1(\Omega),\ w \geq \psi,\ a.e.\ \Omega\}$. We consider a minimizer $u: \ \Omega \subset R^n \to R$ among all functions that agree on the boundary $\partial\Omega$ with some fixed boundary value u_*. And we assume that the function $\theta = max\{u_*, \psi\}$ makes the density $f(x, Du)$ more integrable under the obstacle problem and we prove that the minimizer u enjoy higher integrability.

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1 Introduction

Throughout this paper Ω will stands for a bounded domain in R^n, $n \geq 2$. For $p_1, \cdots, p_n \in (1, +\infty)$, we let

$$\bar{p}: \frac{1}{\bar{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \quad p_i' = \frac{p_i}{p_i - 1} \quad and \quad p_m = \max_{1 \leq i \leq n}\{p_i\}$$

be the harmonic mean of $p_1, \cdots, p_n$, the Hölder conjugate of p_i, and the maximum value of $p_1, \cdots, p_n$, respectively. In this paper we assume $\bar{p} < n$ and we introduce the Sobolev exponent $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$. The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$, $n \geq 1$ is defined by

$$W^{1,(p_i)}(\Omega) = \{v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \ for\ every\ i = 1, \cdots, n\},$$

and $W_0^{1,(p_i)}(\Omega)$ is denoted to be the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(\Omega)$.

We consider the variational integral

$$\int_\Omega f(x, Du)dx \tag{1.1}$$

where the Ω is a open subset of R^n with $n \geq 2$, $u : \Omega \to R$ and $f(x, z) : \Omega \times R^n \to R$ is measurable with respect x and continuous with respect z. In[1], Leonetti and Petricca considered isotropic minimizers $u \in W^{1,p}(\Omega)$ of the integral functional (1.1), and assume p growth for below: there exist constants $p \in (1, n)$ and $\nu_1 \in (0, +\infty)$, there exists a function $g_1 : \Omega \to [0, +\infty)$ such that

$$\nu_1 |z|^p - g_1(x) \leq f(x, z) \tag{1.2}$$

for almost every $x \in \Omega$ and for all $z \in R^n$. In anisotropic case, $u \in W^{1,(p_i)}(\Omega)$ of the integral functional (1.1), there exist constants $p_i \in (1, +\infty)$ for every $i \in \{1, 2, \cdots, n\}$ and $\nu_2 \in (0, +\infty)$, there exists a function $g_2 : \Omega \to [0, +\infty)$ such that

$$\nu_2 \sum_{i=1}^{n} |z_i|^{p_i} - g_2(x) \leq f(x, z) \tag{1.3}$$

for almost every $x \in \Omega$ and for all $z \in R^n$. The proof is a straightforward modification of the proof of Theorem 1.1 in [1].

In this paper, we continue to consider the anisotropic integral functionals (1.1), and the the density $f(x, z)$ satisfy the following growth condition: there exist constants $p_i \in (1, +\infty)$ for every $i \in \{1, 2, \cdots, n\}$ and $\nu \in (0, +\infty)$, there exists a function $g : \Omega \to [0, +\infty)$ such that

$$\nu \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |z_j|^{p_j} \right)^{\frac{p_i - 2}{p_i}} |z_i|^2 - g(x) \leq f(x, z) \tag{1.4}$$

for almost evert $x \in \Omega$ and for all $z \in R^n$. We fix a boundary datum $u_* \in W^{1,(p_i)}(\Omega)$ and

$$x \to f(x, Du_*) \in L^1(\Omega). \tag{1.5}$$

Let $\psi \in W^{1,(p_i)}(\Omega)$ be any function in Ω with values in $R \cup \{\pm\infty\}$, such that $\theta = max\{u_*, \psi\} \in W^{1,(p_i)}(\Omega)$ and

$$x \to f(x, D\theta) \in L^1(\Omega). \tag{1.6}$$

The set of competing functions for the variational integral (1.1) is

$$C_\psi(\Omega) = \{w \in u_* + W_0^{1,(p_i)}(\Omega) \; such\, that\; x \to f(x, Dw) \in L^1(\Omega),\; w \geq \psi,\; a.e.\, \Omega\},$$

the function ψ is an obstacle.

Consider the following problem:

$$u \in C_\psi(\Omega), \tag{1.7}$$

$$\forall\, w \in C_\psi(\Omega), \quad \int_\Omega f(x, Du)dx \leq \int_\Omega f(x, Dw)dx. \tag{1.8}$$

In this paper we deal with regularity of minimizers, [5,6]. Now we ask the following question: if $\theta = max\{u_*, \psi\}$ makes $f(x, D\theta)$ more integrable than (1.6) requires, does the minimizer u enjoy higher integrability? The answer is positive and in this paper we prove the following:

Theorem 1.1 *Let $\sigma > 1$. Assume that $g \in L^\sigma(\Omega)$, $\theta = max\{u_*, \psi\}$ such that $x \to f(x, D\theta) \in L^\sigma(\Omega)$. If $u \in C_\psi(\Omega)$ minimizers the variational integral (1.1) under (1.7), then*
(i) If $\sigma < \frac{n}{\bar{p}}$, then $u - \theta \in L_{weak}^{\frac{n\bar{p}\sigma}{n-\bar{p}\sigma}}(\Omega)$,
(ii) If $\sigma = \frac{n}{\bar{p}}$, then there exists $\alpha > 0$ such that $e^{\alpha|u-\theta|} \in L^1(\Omega)$,
(iii) If $\sigma > \frac{n}{\bar{p}}$, then $u - \theta \in L^\infty(\Omega)$.
Note that $\frac{n\bar{p}\sigma}{n-\bar{p}\sigma} > \frac{n\bar{p}}{n-\bar{p}}$.

Remark 1.1 *We should compare (1.4) with (1.3). Note that for $z_i \in R^n$, $i = 1, 2, \cdots, n$,*

$$|z_i|^2 = (|z_i|^{p_i})^{\frac{2}{p_i}} \leq \left(\sum_{j=2}^n |z_j|^{p_j}\right)^{\frac{2}{p_i}},$$

thus

$$\sum_{i=1}^n \left(\sum_{j=2}^n |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} |z_i|^2 \leq n \left(\sum_{j=2}^n |z_j|^{p_j}\right).$$

This means, up to a constant n, the left hand side of (1.4) is smaller than or equals to the left hand side of (1.3). Thus (1.4) is weaker than (1.3).

Consider a special case, when

$$p_i \geq 2, \; for\; all\; i = 1, 2, \cdots, n, \tag{1.9}$$

we get

$$|z_i|^{p_i-2} = (|z_i|^{p_i})^{\frac{p_1-2}{p_i}} \leq \left(\sum_{j=1}^n |z_j|^{p_j}\right)^{\frac{p_1-2}{p_i}}.$$

This means that (1.4) implies (1.3) in case of (1.9) holds true.

Remark 1.2 *The main feature of this paper lies in the case when*

$$1 < p_i < 2, \ for \ all \ i = 1, 2, \cdots, n. \tag{1.10}$$

In this case,

$$|z_i|^{p_i-2} = (|z_i|^{p_i})^{\frac{p_i-2}{p_i}} \geq \left(\sum_{j=1}^{n} |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}},$$

thus

$$\sum_{i=1}^{n} |z_i|^{p_i} \geq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |z_j|^{p_j}\right)^{\frac{p_i-2}{p_i}} |z_i|^2.$$

This means in the case of (1.10), the condition in the left hand side of (1.4) is weaker than the one in the left hand side of (1.3).

2 Proof of the Main Theorem

We will write c to denote positive constants, possibly different depending on the data $\nu, n, \varepsilon, c(\varepsilon), p_1, p_2, \cdots, p_n$. In order to prove Theorems 1.1, we need a preliminary lemma. The lemma can be found in [2].

Lemma 2.1 *Let $\omega \in W_0^{1,(p_i)}(\Omega)$, and let $M > 0$, $\gamma > 0$, and $k_0 \geq 0$. Let for every $k > k_0$,*

$$\int_{\{|\omega| \geq k\}} \left\{\sum_{i=1}^{n} |D_i\omega|^{p_i}\right\} dx \leq M[meas\{|\omega| \geq k\}]^{\frac{\gamma\bar{p}}{\bar{p}^*}}. \tag{2.1}$$

Then the following asserting hold:
(i) If $\gamma < 1$, then $\omega \in L_{weak}^{\frac{\bar{p}^*}{1-\gamma}}(\Omega)$,
(ii) If $\gamma = 1$, then there exists $\alpha > 0$ such that $e^{\alpha|\omega|} \in L^1(\Omega)$,
(iii) If $\gamma > 1$, then $\omega \in L^\infty(\Omega)$.

We want to use Lemma 2.1 with $\omega = u - \theta$. Then we get

$$\begin{aligned}
& \int_{\{|u-\theta| \geq k\}} \sum_{i=1}^{n} |D_iu - D_i\theta|^{p_i} dx \\
\leq & \int_{\{|u-\theta| \geq k\}} \sum_{i=1}^{n} |D_iu + D_i\theta|^{p_i} dx \\
\leq & \int_{\{|u-\theta| \geq k\}} \sum_{i=1}^{n} [2^{p_i}(|D_iu|^{p_i} + |D_i\theta|^{p_i})] \, dx \\
\leq & \ 2^{p_m} \int_{\{|u-\theta| \geq k\}} \sum_{i=1}^{n} |D_iu|^{p_i} dx + 2^{p_m} \int_{\{|u-\theta| \geq k\}} \sum_{i=1}^{n} |D_i\theta|^{p_i} dx.
\end{aligned} \tag{2.2}$$

We distinguish between two cases.
Case 1,$p_i \geq 2$. In this case,

$$|D_i u|^{p_i} = (|D_i u|^{p_i})^{\frac{p_i-2}{p_i}} |D_i u|^2 \leq \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2. \qquad (2.3)$$

Integrating this inequality with respect to x , we get

$$\int_{\{|u-\theta|\geq k\}} |D_i u|^{p_i} dx \leq \int_{\{|u-\theta|\geq k\}} \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx. \qquad (2.4)$$

Case 2, $1 < p_i < 2$. Young inequality yields

$$\begin{aligned} & \int_{\{|u-\theta|\geq k\}} |D_i u|^{p_i} dx \\ = & \int_{\{|u-\theta|\geq k\}} \left[\left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{2}} |D_i u|^{p_i} \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{2-p_i}{2}}\right] dx \\ \leq & \; c(\varepsilon) \int_{\{|u-\theta|\geq k\}} \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx + \varepsilon \int_{\{|u-\theta|\geq k\}} \sum_{j=1}^{n} |D_j u|^{p_j} dx. \end{aligned} \qquad (2.5)$$

It is no loss of generality to assume $n\varepsilon < 1$ and $c(\varepsilon) \geq 1$. Thus in both cases, (2.5) holds true. Therefore,

$$\begin{aligned} \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i u|^{p_i} dx \leq & \; c(\varepsilon) \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx \\ & + n\varepsilon \int_{\{|u-\theta|\geq k\}} \sum_{j=1}^{n} |D_j u|^{p_j} dx. \end{aligned} \qquad (2.6)$$

Since $n\varepsilon < 1$, the last term in the right hand side of (2.6) is absorbed by the left hand side. Thus we have

$$\int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i u|^{p_i} dx \leq c \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |D_j u|^{p_j}\right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx. \qquad (2.7)$$

From (2.2) and (2.7), then we apply the p_i growth from below in (1.4) and we have

$$\begin{aligned}
&\int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i u - D_i\theta|^{p_i} dx \\
\leq\ & 2^{p_m} c \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |D_j u|^{p_j} \right)^{\frac{p_i-2}{p_i}} |D_i u|^2 dx + 2^{p_m} \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i\theta|^{p_i} dx \\
\leq\ & 2^{p_m} \frac{c}{\nu} \int_{\{|u-\theta|\geq k\}} f(x, Du) dx + 2^{p_m} \frac{c}{\nu} \int_{\{|u-\theta|\geq k\}} g(x) dx \\
& + 2^{p_m} \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i\theta|^{p_i} dx.
\end{aligned} \tag{2.8}$$

In order to control $\int f(x, Du)$ we need the minimality of u, we define the test function v,

$$v = \theta + T_k(u-\theta) = \begin{cases} \theta + k, & u - \theta \geq k; \\ u, & |u-\theta| < k; \\ \theta - k, & u - \theta \leq -k, \end{cases} \tag{2.9}$$

where $k \in (0, +\infty)$.

For $u \in C_\psi(\Omega)$, we have to show that $v \in C_\psi(\Omega)$. In fact, it is obvious that $v \in W^{1,(p_i)}(\Omega)$. In order to prove $v \in u_* + W_0^{1,(p_i)}(\Omega)$, we notice that $u = u_* \leq \psi$ *a.e.* on $\partial\Omega$, thus $\theta = u_* = u$ *a.e.* on $\partial\Omega$, this implies $T_k(u-\theta) = 0$ on $\partial\Omega$, thus $v - u_* = v - \theta = T_k(v-\theta) = 0$ on $\partial\Omega$. In order to prove $f(x, Dv) \in L^1(\Omega)$, we notice that $Dv = Du$ on $\{|u-\theta| < k\}$ and $Dv = D\theta$ on $\{|u-\theta| \geq k\}$, thus $f(x, Dv) \in L^1(\Omega)$ is guaranteed by $f(x, Du) \in L^1(\Omega)$ and (1.6), and to prove $v \geq \psi$ *a.e.*, we notice that the first case of (2.9), $v = \theta + k \geq \theta \geq \psi$, in the second case of (2.9), $u \geq \psi$, and in the last case of (2.9) $v = \theta - k \geq u \geq \psi$.

We can use minimality (1.8):

$$\begin{aligned}
&\int_{\{|u-\theta|<k\}} f(x, Du) dx + \int_{\{|u-\theta|\geq k\}} f(x, Du) dx = \int_\Omega f(x, Du) dx \\
\leq\ & \int_\Omega f(x, Dv) dx = \int_{\{|u-\theta|<k\}} f(x, Du) dx + \int_{\{|u-\theta|\geq k\}} f(x, D\theta) dx.
\end{aligned} \tag{2.10}$$

Since u and θ have finite energy, all the integral functionals are finite; then we can drop $\int_{\{|u-\theta|\leq k\}} f(x, Du) dx$ from both sides and we get

$$\int_{\{|u-\theta|\geq k\}} f(x, Du) dx \leq \int_{\{|u-\theta|\geq k\}} f(x, D\theta) dx. \tag{2.11}$$

This inequality can be used in (2.8), and we obtain

$$
\begin{aligned}
& \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i u - D_i \theta|^{p_i} dx \\
\leq\ & 2^{p_m} \frac{c}{\nu} \int_{\{|u-\theta|\geq k\}} f(x, D\theta) dx + 2^{p_m} \frac{c}{\nu} \int_{\{|u-\theta|\geq k\}} g(x) dx \\
& +2^{p_m} \int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i \theta|^{p_i} dx \\
=\ & \int_{\{|u-\theta|\geq k\}} H(x) dx,
\end{aligned}
\tag{2.12}
$$

where

$$
H(x) = \frac{2^{p_m} c}{\nu} f(x, D\theta) + \frac{2^{p_m} c}{\nu} g(x) + 2^{p_m} \sum_{i=1}^{n} |D_i \theta|^{p_i}. \tag{2.13}
$$

The assumption on $D\theta$, $g(x)$, and f guarantee that

$$
H(x) \in L^{\sigma}(\Omega). \tag{2.14}
$$

Then, using Hölder inequality, we can obtain

$$
\int_{\{|u-\theta|\geq k\}} H(x) dx \leq \left(\int_{\Omega} H^{\sigma} dx \right)^{\frac{1}{\sigma}} |\{|u-\theta| \geq k\}|^{\frac{\sigma-1}{\sigma}}, \tag{2.15}
$$

we insert this inequality into (2.12) and we get

$$
\int_{\{|u-\theta|\geq k\}} \sum_{i=1}^{n} |D_i u - D_i \theta|^{p_i} dx \leq \|H\|_{L^{\sigma}(\Omega)} |\{|u-\theta| \geq k\}|^{\frac{\sigma-1}{\sigma}}. \tag{2.16}
$$

Now

$$
\frac{\sigma - 1}{\sigma} = \frac{1 - \frac{1}{\sigma}}{1 - \frac{\bar{p}}{n}} \frac{\bar{p}}{\bar{p}^*} \tag{2.17}
$$

and we can apply Lemma 2.1 with $\gamma = \frac{1-\frac{1}{\sigma}}{1-\frac{\bar{p}}{n}}$. We complete the proof of Theorem 1.1.

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